ON TOPOLOGICAL EQUIVALENCE OF LINEAR FLOWS WITH APPLICATIONS TO BILINEAR CONTROL SYSTEMS

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ABSTRACT. This paper classifies continuous linear flows using concepts and techniques from topological dynamics. Specifically, the concepts of equivalence and conjugacy are adapted to flows on vector bundles, and the Lyapunov decomposition is characterized using the induced flows on the Grassmann and the flag bundles. These results are then applied to bilinear control systems, for which their behavior in $\mathbb{R}^d$, on the projective space $\mathbb{P}^{d-1}$, and on the Grassmannians is characterized.

1. INTRODUCTION

This paper classifies continuous linear flows on vector bundles with compact metric base space, presenting a generalization of the classification of linear autonomous differential equations based on topological conjugacies. We refer to the monograph of Cong [10] which includes an exposition of equivalences and normal forms for nonautonomous linear differential equations (emphasizing results based on the ergodic theory). For linear autonomous equations, it is a classical theorem (see [13]) that topological conjugacies of the corresponding flows in $\mathbb{R}^d$ give only a rough classification, since all exponentially stable equations are equivalent. In [1], the authors have presented a classification and theory of normal forms for linear differential equations $\dot{x} = Ax$ related to the exponential growth rates and the corresponding decomposition of $\mathbb{R}^d$ into subspaces of equal exponential growth rates. These are the Lyapunov spaces given by sums of generalized eigenspaces corresponding to eigenvalues of A with equal real parts. The purpose of this paper is to develop a similar theory for general linear flows. One of our main motivations comes from bilinear control systems, which

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are analyzed in the final section of this paper. Note that also many linear parameter-varying and linear switching systems can be considered as linear flows (see [18]).

Spaces of equal exponential behavior are of interest, since they form the basis of results on invariant manifolds and Grobman–Hartman type theorems. For nonautonomous problems or linear flows, there are several concepts generalizing the real parts of eigenvalues, in particular, the Sacker–Sell spectrum based on exponential dichotomies and the Morse spectrum based on the exponential growth behavior of chains and the related subbundle decomposition [3, 5, 14, 15]. Here we follow the latter approach, since it is based on topological dynamics and is also well suited for control systems. Thus, the main goal of this paper is to classify linear flows according to their (exponential) subbundle decompositions that were first studied by Selgrade.

In Sec. 2, we introduce concepts of equivalence and conjugacy for linear flows. Section 3 studies topological equivalence in vector bundles. It turns out that, just as in the matrix case, this concept characterizes the stable and unstable bundles of hyperbolic linear flows. Section 4 introduces the spectrum, the Lyapunov index, and the short Lyapunov index of linear flows. Section 5 characterizes linear flows with the same short Lyapunov index via a graph constructed of the induced flows on the Grassmann bundles. Finally, Sec. 6 presents an application to the classification of bilinear control systems. Here some more specific information can be obtained due to the specific nature of control flows.

2. Conjugacy and equivalence

In this section, we present dynamical concepts of "equivalence" and "conjugacy" that are adequate for linear flows on vector bundles and, more generally, for skew product flows.

Recall that flows (topological dynamical systems) on a metric space X are given by a continuous mapping $\Phi : \mathbb{R} \times X \to X$ with $\Phi(0, x) = x$ and $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$ for all $s, t \in \mathbb{R}$ and $x \in X$. One defines the topological conjugacy and equivalence as follows (see, e.g., [7, 17]).

Definition 2.1. Let $\Psi_i : \mathbb{R} \times X_i \to X_i$, be topological dynamical systems defined on metric spaces X_i, $i = 1, 2$. We say that Ψ_1 and Ψ_2 are

(i) *conjugate* if there exists a homeomorphism $h : X_1 \to X_2$ such that $h(\Psi_1(t, x)) = \Psi_2(t, h(x))$ for all $x \in X_1$ and $t \in \mathbb{R}$;

(ii) *equivalent* if there exists a homeomorphism $h : X_1 \to X_2$ and, for each $x \in X_1$, a strictly increasing and continuous time parametrization mapping $\tau_x : \mathbb{R} \to \mathbb{R}$ such that $h(\Psi_1(t, x)) = \Psi_2(\tau_x(t), h(x))$ for all $x \in X_1$.

Here we are interested in flows on vector bundles of product form $\pi : \mathcal{V} = B \times H \to B$, where B is a metric space, H is a finite-dimensional Hilbert

space, and π is the projection onto the first component. Usually, we take $H = \mathbb{R}^d$ with the Euclidean inner product.

Remark 2.2. For general vector bundles, one only requires that locally they are the product of an open subset of the metric space B by H (see [9] or [5, Appendix B]). We refrain from writing down the proofs in the general case, since this is not relevant for our intended applications. However, the general case would require only minor modifications that are technically somewhat involved.

We always assume that the base space B is compact. For $b \in B$, the set $\mathcal{V}_b = \pi^{-1}(b)$ is called the fiber over the base point b. A linear flow Φ on a vector bundle $\pi : \mathcal{V} \to B$ is a flow Φ on $\mathcal{V}$ which has the form

$$\Phi(t, b, x) = (\theta(t, b), \varphi(t, b, x)), \tag{1}$$

where $\theta(t, b)$ is a flow on the base space B (corresponding to the transport of the fibers) and $\varphi : \mathbb{R} \times B \times H \to B \times H$ is linear in x, i.e., for all $\alpha \in \mathbb{R}$, $x_1, x_2 \in H$, and $b \in B$, we have

$$\varphi(t, b, \alpha(x_1 + x_2)) = \alpha\varphi(t, b, x_1) + \alpha\varphi(t, b, x_2).$$

Thus, a linear flow preserves the fibers and is linear in each fiber. Where notationally convenient, we write instead of $\Phi(t, v)$ either $\Phi_t(v)$ or $\Phi(t)v$ with $v = (b, x) \in \mathcal{V}$.

We define adequate concepts of conjugacy which preserve the fiber structure in a slightly more general setting of skew product flows $\Phi : \mathbb{R} \times X \times Y \to X \times Y$ on metric spaces X and Y, which have the form

$$\Phi(t, x, y) = (\theta(t, x), \varphi(t, x, y)),$$

where θ and φ are as above, but omitting the linearity requirement. For these flows, the adequate concepts of conjugacy and equivalence preserve the skew product structure.

Definition 2.3. For $i = 1, 2$, let X_i and Y_i be metric spaces and let $\Phi_i : \mathbb{R} \times X_i \times Y_i \to X_i \times Y_i$, $\Phi_i = (\theta_i, \varphi_i)$ be skew product flows. We say that Φ_1 and Φ_2 are

(i) *skew conjugate* if there exists a skew homeomorphism

$$h = (f, g) : X_1 \times Y_1 \to X_2 \times Y_2$$

such that

$$h(\Phi_1(t, x, y)) = \Phi_2(t, h(x, y)),$$

i.e., $f : X_1 \to X_2$ and $g : X_1 \times Y_1 \to Y_2$ with

$$f(\theta_1(t, x)) = \theta_2(t, f(x)) \text{ for all } (t, x) \in \mathbb{R} \times X_1,$$

$$g(\theta_1(t, x), \varphi_1(t, x, y)) = \varphi_2(t, f(x), g(x, y)) \text{ for all } (t, x, y) \in \mathbb{R} \times X_1 \times Y_1;$$

(ii) *skew equivalent* if there exists a homeomorphism
$$h = (f, g) : X_1 \times Y_1 \to X_2 \times Y_2$$
as above that maps trajectories of Φ_1 onto trajectories of Φ_2, preserving the orientation, but possibly with a time shift. In other words, for each $(x, y) \in X_1 \times Y_1$ there exists a continuous, strictly increasing time parametrization $\tau_{x,y} : \mathbb{R} \to \mathbb{R}$ such that
$$h(\Phi_1(t, x, y)) = \Phi_2(\tau_{x,y}(t), h(x, y));$$

(iii) *base conjugate* if the base flows are conjugate, i.e., there exists a homeomorphism $f : B_1 \to B_2$ such that $f(\theta_1(t, b)) = \theta_2(t, f(b))$ for all $(t, b) \in \mathbb{R} \times B_1$, and analogously for the base equivalence.

Clearly, the base conjugacy (base equivalence) is a prerequisite for skew conjugacy (skew equivalence).

3. TOPOLOGICAL CONJUGATION AND EQUIVALENCE IN VECTOR BUNDLES

This section is devoted to the study of topological conjugacy of linear flows in vector bundles. Just as for matrices, i.e., for linear differential equations of the form $\dot{x}(t) = Ax(t)$, $A \in \mathrm{gl}(d, \mathbb{R})$, the key point is to show that any two exponentially stable (or unstable) linear flows are topologically conjugate. The proofs in this section are modelled for the matrix case (see, e.g., [13, proof of Theorem IV.5.1 and p. 113]).

Let Φ be a linear flow on a vector bundle $\pi : \mathcal{V} \to B$ with compact base space B.

Lemma 3.1. *Suppose that for some norm $\|\cdot\|$ on $\mathcal{V}$ there are $a > 0$ and $C > 0$ such that*
$$\|\Phi(t, v)\| \le C\, e^{-at} \|v\| \quad \text{for all } t \ge 0.$$
Then for all $\alpha < a$ there exists $\tau = \tau(\alpha) > 0$ such that for all $v \in \mathcal{V}$ and all $t \ge \tau$
$$\|\Phi(t, v)\| \le e^{-\alpha t} \|v\|.$$

Proof. Let $\alpha < a$. Then there exists $\tau = \tau(\alpha) > 0$ such that for all $t \ge \tau$
$$C < e^{t(a-\alpha)}$$
and hence for all $v \in \mathcal{V}$ with $\|v\| = 1$
$$\|\Phi(t, v)\| \le Ce^{-at} < e^{t(a-\alpha)}e^{-at} = e^{-at}.$$
This implies for all $v \in \mathcal{V}$ and all $t \ge \tau$
$$\|\Phi(t, v)\| = \left\| \Phi\left(t, \frac{v}{\|v\|}\right) \right\| \|v\| \le e^{-\alpha t} \|v\|.$$
The lemma is proved. $\square$

We proceed to the existence of an adapted norm.

Proposition 3.2. *Let $a \in \mathbb{R}$ and suppose that for some (and hence for every) norm $\|\cdot\|$ on $\mathcal{V}$ there exists $C > 0$ such that*

$$\|\Phi_t v\| \leq C\ e^{-at}\ \|v\|\ \text{ for all } t \geq 0.$$

Then for every $\alpha < a$ there exists a norm $\|\cdot\|_b^*$ depending continuously on $b \in B$ such that

$$\|\Phi_t v\|_{b \cdot t}^* \leq e^{-\alpha t}\ \|v\|_b^*\ \text{ for all } t \geq 0,$$

where we use the short form $b \cdot t := \theta(t, b)$.

Proof. Since all norms on $\mathcal{V}$ are equivalent, it does not matter which norm is used in the assumption. Let $\alpha < a$, choose $\tau = \tau(\alpha) > 0$ according to Lemma 3.1 and define a norm on the vector bundle as follows:

$$\|v\|^* = \int_0^\tau e^{\alpha s}\ \|\Phi(s, v)\|\ ds.$$

Furthermore, for all $t > 0$ we write

$$t = n\tau + T \text{ with } 0 \leq T < \tau.$$

Then

$$\|\Phi(t, v)\|^* = \int_0^\tau e^{\alpha s}\ \|\Phi(s, \Phi(t, v))\|\ ds = \int_0^\tau e^{\alpha s}\ \|\Phi(s + t, v)\|\ ds$$

$$= \int_0^\tau e^{\alpha s}\ \|\Phi(n\tau + T + s, v)\|\ ds$$

$$= \int_0^{\tau - T} e^{\alpha s}\ \|\Phi(n\tau, \Phi(T + s, v))\|\ ds$$

$$+ \int_{\tau - T}^\tau e^{\alpha s}\ \|\Phi((n + 1)\tau, \Phi(T - \tau + s, v))\|\ ds$$

$$= \int_T^\tau e^{\alpha(s - T)}\ \|\Phi(n\tau, \Phi(s, v))\|\ ds$$

$$+ \int_0^T e^{\alpha(s - T + \tau)}\ \|\Phi((n + 1)\tau, \Phi(s, v))\|\ ds,$$

using the time transformations $s := T + s$ and $s := T - \tau + s$, respectively. Observe that, by the choice of τ, one has for all $w \in \mathcal{V}$ and all $n = 0, 1, \ldots$

$$\|\Phi(n\tau, w)\| \leq e^{-\alpha n \tau}\ \|w\|.$$

Hence one finds

$$\|\Phi(t,v)\|^* \leq \int_T^\tau e^{\alpha(s-T)} e^{-\alpha n\tau} \|\Phi(s,v)\| \, ds$$

$$+ \int_0^T e^{\alpha(s-T+\tau)} e^{-\alpha(n+1)\tau} \|\Phi(s,v)\| \, ds$$

$$= \int_T^\tau e^{\alpha(s-T-n\tau)} \|\Phi(s,v)\| \, ds$$

$$+ \int_0^T e^{\alpha(s-T+\tau-(n+1)\tau)} \|\Phi(s,v)\| \, ds$$

$$= e^{-\alpha t} \int_0^\tau e^{\alpha s} \|\Phi(s,v)\| \, ds = e^{-\alpha t} \|v\|^* .$$

The proposition is proved. □

This proposition shows that an exponentially stable linear flow $\Phi \colon \mathbb{R} \times \mathcal{V} \longrightarrow \mathcal{V}$ admits an adapted norm on the vector bundle, with respect to which the orbits decrease uniformly. As in the matrix case, this is a key tool in characterizing skew equivalent flows.

Theorem 3.3. *Let Φ and Ψ be linear flows on vector bundles $\mathcal{V} = B \times \mathbb{R}^d \to B$ and $\mathcal{W} = C \times \mathbb{R}^d \to C$, respectively, with compact bases. If the flows are base equivalent and both are exponentially stable, then they are skew equivalent.*

Proof. By Proposition 3.2, there exist $\alpha, \beta > 0$ and adapted norms $\|\cdot\|_\Phi$ and $\|\cdot\|_\Psi$ such that for all v and all $t \geq 0$

$$\|\Phi(t,v)\|_\Phi \leq e^{-\alpha t} \|v\|_\Phi \quad \text{and} \quad \|\Psi(t,v)\|_\Psi \leq e^{-\beta t} \|v\|_\Psi .$$

Running times backward, we obtain for $t \leq 0$

$$\|\Phi(t,v)\|_\Phi^* \geq e^{\alpha|t|} \|v\|_\Phi \quad \text{and} \quad \|\Psi(t,v)\|_\Psi \geq e^{\beta|t|} \|v\|_\Psi .$$

Using the above estimates, we see that for each $v \neq 0$, i.e., not in the zero section Z, the trajectory $\Phi(t,v)$ crosses the unit sphere bundle $\mathcal{S}_\Phi = \{w \in \mathcal{V}, \|w\|_\Phi = 1\}$ exactly once, and each trajectory $\Psi(t,v)$ crosses the unit sphere bundle $\mathcal{S}_\Psi = \{w \in \mathcal{W}, \|w\|_\Psi = 1\}$ exactly once.

First, we define a homeomorphism h_0 from $\mathcal{S}_\Phi$ to $\mathcal{S}_\Psi$. Denote the base equivalence by $g : B \to C$ and define for $v = (b,x) \in \mathcal{S}_\Phi$

$$h_0(b,x) = \left(g(b), \frac{x}{\|x\|_\Psi} \right) ;$$

here $\|x\|_\Psi$ denotes the adapted norm of $(g(b), x) \in C \times \mathbb{R}^d$. Note that the inverse of h_0 exists and is given by

$$h_0^{-1}(c, y) = \left(g^{-1}(c), \frac{y}{\|y\|_\Phi}\right), \quad w = (c, y) \in \mathcal{S}_\Psi.$$

Let $Z := B \times \{0\}$ denote the zero section of $\mathcal{V}$. To extend h_0 to all of $\mathcal{V}$, we define $\tau(v)$ for $v \in \mathcal{V} \backslash Z$ to be the time with

$$\|\Phi(\tau(v), v)\|_\Phi = 1.$$

This time depends continuously on $v \in \mathcal{V} \backslash Z$. It is immediate that

$$\tau(\Phi(t, v)) = \tau(v) - t. \tag{2}$$

Now we define a homeomorphism $h : \mathcal{V} \to \mathcal{W}$ as follows:

$$h(v) = \begin{cases} \Psi(-\tau(v), h_0(\Phi(\tau(v), v))) & \text{for } v \notin Z, \\ (g(b), 0) & \text{for } v \in Z. \end{cases}$$

Then h is a conjugacy: first, observe that h maps fibers into fibers by the base equivalence of Φ and Ψ. Furthermore, the conjugation property follows using (2) from

$$\begin{aligned} h(\Phi(t, v)) &= \Psi(-\tau(\Phi(t, v)), h_0(\Phi(\tau(\Phi(t, v)), \Phi(t, v)))) \\ &= \Psi(-[\tau(v) - t], h_0(\Phi(\tau(v) - t), \Phi(t, v)) \\ &= \Psi(t, \Psi(-\tau(v), h_0(\Phi(\tau(v), v)) \\ &= \Psi(t, h(v)). \end{aligned}$$

Since τ and the flows Φ and Ψ are continuous, it follows that h is continuous at points $v \notin Z$. To verify the continuity at the zero section Z, note that if v_j converges to an element of Z, then $\tau_j = \tau(v_j)$ goes to the negative infinity. By setting

$$w_j = h_0(\Phi(\tau_j, v_j))$$

we have that $|w_j|_\Psi = 1$. Thus, by the definition of h and stability of Ψ,

$$\|h(v_j)\|_\Psi = \|\Psi(-\tau_j, w_j)\|_\Psi \leq e^{-\beta|\tau_j|}$$

must go to zero. Therefore, $h(v_j)$ converges to $0 = h(0)$. This proves the continuity at the zero.

To show that h is injective, we take v and w such that $h(v) = h(w)$. If v is in the zero section, then $0 = h(v) = h(w)$ and, therefore, $v = w$, since both are in the same fiber. Now we assume that v is not in the zero section. Then $h(w) = h(v)$ and hence w is not in the zero section. If we set $\tau = \tau(v)$, we find

$$h(\Phi(\tau, v)) = \Psi(\tau, h(v)) = \Psi(\tau, h(w)) = h(\Psi(\tau, w)).$$

This shows that $h(\Phi(\tau, w)) = h(\Phi(\tau, v)) \in \mathcal{S}_\Psi$ (since $\Phi(\tau, v) \in \mathcal{S}_\Phi$) and, therefore, $\Phi(\tau, w) \in \mathcal{S}_\Phi$ and $\tau(w) = \tau(v) = \tau$.

Since $h_0(\Phi(\tau, v)) = h(\Phi(\tau, v)) = h(\Phi(\tau, w)) = h_0(\Phi(\tau, w))$ and h_0 is injective, we have $\Phi(\tau, v) = \Phi(\tau, w)$ and, therefore, $v = w$. Thus, h is injective in all cases.

Reversing the roles of Φ and Ψ in the above arguments, we obtain that h^{-1} exists (and, therefore, h is surjective) and is continuous. This completes the proof. $\square$

Corollary 3.4. *Let Φ and Ψ be linear flows on vector bundles with compact bases.*

(i) *The flows are skew equivalent if they are base equivalent, and both flows are exponentially unstable.*

(ii) *Suppose that both flows are hyperbolic, i.e., the vector bundles can be written as Whitney sums of exponentially stable and unstable subbundles. Then they are skew conjugate iff they are base conjugate and the dimensions of their stable (and unstable) subbundles coincide.*

Proof. Item (i) of this corollary is proved via time reversal. Skew conjugacy in item (ii) follows by piecing together the stable and the unstable parts of a flow, just as in the matrix case. Conversely, the base conjugacy follows trivially and the dimension condition follows by the domain invariance theorem (see [16]), since a conjugacy maps fibers $\{u\} \times \mathbb{R}^{l^+}$ of the stable bundle onto fibers of the stable bundle. $\square$

This corollary shows that the topological conjugacy in vector bundles gives a very rough classification of linear flows in terms of stable and unstable subbundles. Smooth conjugacies, however, result in a very fine classification: recall the situation for linear ordinary differential equations $\dot{x} = Ax$ and $\dot{y} = By$ with the linear flows φ and ψ in $\mathbb{R}^d$, respectively. The systems φ and ψ in $\mathbb{R}^d$ are C^k-conjugate (for $k \geq 1$) iff they are linearly conjugate iff the matrices A and B are similar. The corresponding result for linear flows is given in the following proposition.

Definition 3.5. Let Φ and Ψ be linear flows on vector bundles $\mathcal{V} = B \times \mathbb{R}^d \to B$ and $\mathcal{W} = C \times \mathbb{R}^d \to C$ respectively, with compact bases. We say that $h = (f, g) : B \times \mathbb{R}^d \to C \times \mathbb{R}^d$ is a C^k-conjugacy $(k \geq 1)$ between Φ and Ψ if h is a skew conjugacy and for all $b \in B$ the mappings $g(b, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ are C^k-diffeomorphisms.

Proposition 3.6. *Let $\Phi = (\theta, \varphi)$ and $\Psi = (\vartheta, \psi)$ be linear flows on vector bundles $\mathcal{V} = B \times \mathbb{R}^d \to B$ and $\mathcal{W} = C \times \mathbb{R}^d \to C$ respectively, with compact bases. If Φ and Ψ are C^1-conjugate via $h = (f, g)$, then they are linearly conjugate in the following sense:*

$$\varphi(t, \cdot, b) = [D_x g(\theta_t b, 0)]^{-1} \circ \psi(t, \cdot, f(b)) \circ D_x g(b, 0). \qquad (3)$$

Proof. We will use the following notation:

$$\Phi_t(b, x) = (\theta_t b, \varphi(t, x, b)),$$
$$\Psi_t(c, y) = (\vartheta_t c, \psi(t, y, c)),$$
$$h(b, x) = (f(b), g(b, x)).$$

The conjugation property yields

$$h \circ \Phi_t(b, x) = h(\theta_t b, \varphi(t, x, b)) = (f(\theta_t b), g(\theta_t b, \varphi(t, x, b)))$$
$$= (\vartheta_t f(b), \psi(t, g(b, x), f(b))),$$

and hence

$$\varphi(t, x, b) = g^{-1}(\theta_t b, \psi(t, g(b, x), f(b))). \tag{4}$$

Differentiation of (4) with respect to x yields

$$\varphi(t, \cdot, b) = D_x g^{-1}(\theta_t b, \psi(t, g(b, x), f(b))) \circ D_x \psi(t, g(b, x), f(b)) \circ D_x g(b, x). \tag{5}$$

The zero section $B \times \{0\}$ is invariant with respect to the flow, hence evaluating (5) at the zero section yields with $\psi(t, g(b, 0), f(b)) = g(\theta_t b, \varphi(t, 0, b)) = g(\theta_t b, 0)$ and $D_x g^{-1}(\theta_t b, g(\theta_t b, 0)) = [D_x g(\theta_t b, g(\theta_t b, 0))]^{-1} = [D_x g(\theta_t b, 0)]^{-1}$ for all $t \in \mathbb{R}$ the result

$$\varphi(t, \cdot, b) = [D_x g(\theta_t b, 0)]^{-1} \circ \psi(t, \cdot, f(b)) \circ D_x g(b, 0). \tag{6}$$

The proposition is proved. $\square$

Remark 3.7. In the case of linear differential equations, linear conjugations preserve the eigenvalues and Jordan structure of the matrices. If the conjugating skew homeomorphism $h = (f, g)$ of two linear flows is linear, i.e., $g : B \times \mathbb{R}^d \to \mathbb{R}^d$ is linear in the second argument, the flows are called cohomologous. According to Proposition 3.6, this holds if two linear flows are C^1-conjugate. Cohomologous flows preserve the Morse spectrum (see below for the definition) and the associated subbundle decomposition (see [5, Proposition 5.4.4] for the case of identical base flows, but the proof is easily extended to conjugate base flows using uniform continuity of f).

Corollary 3.4 and Proposition 3.6 show what type of classification can be achieved using topological and C^k-conjugacies of linear flows. As in the matrix case, neither of them results in a dynamic characterization of flows whose subbundles have the same exponential behavior. In the next two sections, we make this precise and provide a characterization using conjugacies of the induced flows on the Grassmann bundles.

4. Spectrum and Lyapunov index of linear flows

This section recalls the (Morse) spectrum of a linear flow and introduces its Lyapunov index.

Recall the following notation from topological dynamics (see, e.g., [6] or [5, Appendix B]). For a flow Φ on a compact metric space Y, a compact subset $K \subset Y$ is said to be *isolated invariant* if it is invariant and there exists a neighborhood N of K, i.e., a set N with $K \subset \text{int } N$ such that $\Phi(t, x) \in N$ for all $t \in \mathbb{R}$ implies $x \in K$. Denote the α- and ω-limit set from a point $x \in Y$ by $\alpha(x)$ and $\omega(x)$, respectively. A *Morse decomposition* is a finite collection $\{\mathcal{M}_i, \ i = 1, \ldots, n\}$ of nonempty, pairwise disjoint, and isolated compact invariant sets such that

(i) for all $x \in Y$ one has $\omega(x), \alpha(x) \subset \bigcup\limits_{i=1}^{n} \mathcal{M}_i$, and

(ii) suppose that there exist $\mathcal{M}_{j_0}, \mathcal{M}_{j_1}, \ldots, \mathcal{M}_{j_l}$ and $x_1, \ldots, x_l \in Y \setminus \bigcup\limits_{i=1}^{n} \mathcal{M}_i$ such that $\alpha(x_i) \subset \mathcal{M}_{j_{i-1}}$ and $\omega(x_i) \subset \mathcal{M}_{j_i}$ for $i = 1, \ldots, l$; then it follows that $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_l}$.

The elements of a Morse decomposition are called *Morse sets*. Observe that $\mathcal{M}_i \preceq \mathcal{M}_j$ if $\alpha(x) \subset \mathcal{M}_i$ and $\omega(x) \subset \mathcal{M}_j$ for some x defines an order on the Morse sets. A Morse decomposition is finer than another one if each element of the second is contained in an element of the first. If a finest Morse decomposition exists, its elements are maximal chain transitive sets, i.e., maximal sets that have the property that for all elements x, y and all $\varepsilon, T > 0$ there exists an (ε, T)-chain from x to y given by $n \in \mathbb{N}$, $T_0, \ldots, T_{n-1} \geq T$, and $x_0 = x, \ldots, x_n = y$ such that $d(\Phi(T_i, x_i), x_{i+1}) < \varepsilon$ for $i = 0, \ldots, n - 1$.

The following theorem goes back to Selgrade [15] and provides a decomposition via chain transitivity properties in the projective bundle.

Theorem 4.1 (Selgrade). *Let Φ be a linear flow on a vector bundle $\pi :$ $\mathcal{V} \to B$ with a chain transitive flow on the compact base space B. Then the chain recurrent set of the induced flow $\mathbb{P}\Phi$ on the projective bundle $\mathbb{P}\mathcal{V}$ has finitely many linearly ordered components $\{\mathcal{M}_1, \ldots, \mathcal{M}_l\}$, and $1 \leq l \leq d :=$ $\dim \mathcal{V}_b, \ b \in B$. These $\mathcal{M}_i$ provide the finest Morse decomposition. Every maximal chain transitive set $\mathcal{M}_i$ defines an invariant subbundle of $\mathcal{V}$ via*

$$\mathcal{V}_i = \mathbb{P}^{-1}\left(\mathcal{M}_i\right) = \{v \in \mathcal{V}, \ v \notin Z \text{ implies } \mathbb{P}v \in \mathcal{M}_i\},$$

and the following decomposition into a Whitney sum holds:

$$\mathcal{V} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_l.$$

By analogy with the matrix case, we call this subbundle decomposition the *Lyapunov decomposition* of $\mathcal{V}$ with respect to Φ.

With an appropriate concept of exponential growth rates, this decomposition also yields a notion of spectrum. For points $v \in \mathcal{V}$ not in the zero section Z, the Lyapunov exponent (or exponential growth rate of the

corresponding trajectory) is given by

$$\lambda(v) = \limsup_{t \to \infty} \frac{1}{t} \log \|\Phi_t v\|, \tag{7}$$

and the Lyapunov spectrum Σ_{Ly} of the linear flow Φ is the set of all Lyapunov exponents

$$\Sigma_{Ly} = \{\lambda(v), \ v = (b, x) \in \mathcal{V} \text{ with } x \neq 0\}. \tag{8}$$

While the Lyapunov spectrum can be very complicated, the concept of the Morse spectrum [5] yields a simple structure. It is defined via (ε, T)-chains in the projective bundle. Recall that for $\varepsilon, T > 0$ an (ε, T)-chain ζ in $\mathbb{P}\mathcal{V}$ of Φ is given by $n \in \mathbb{N}$, $T_0, \ldots, T_{n-1} \geq T$, and $p_0, \ldots, p_n$ in $\mathbb{P}\mathcal{V}$ with $d(\Phi(T_i, p_i), p_{i+1}) < \varepsilon$ for $i = 0, \ldots, n-1$. Define the finite time exponential growth rate of such a chain ζ (or "chain exponent") by

$$\lambda(\zeta) = \left(\sum_{i=0}^{n-1} T_i\right)^{-1} \sum_{i=0}^{n-1} (\log \|\Phi(T_i, v_i)\| - \log \|v_i\|),$$

where $v_i \in \mathbb{P}^{-1}(p_i)$. For a Lyapunov subbundle $\mathcal{V}_i$ projecting to a maximal chain transitive set $\mathcal{M}_i$ in the projective bundle, the Morse spectrum of $\mathcal{V}_i$ is

$$\Sigma_{Mo}(\mathcal{V}_i) = \Big\{\lambda \in \mathbb{R} : \text{there exist } \varepsilon^k \to 0, \ T^k \to \infty, \text{ and}$$

$$(\varepsilon^k, T^k)\text{-chains } \zeta^k \text{ in } \mathcal{M}_i \text{ such that } \lambda(\zeta^k) \to \lambda \text{ as } k \to \infty\Big\}.$$

The main results on the Morse spectrum are collected in the following theorem.

Theorem 4.2. *Let Φ be a linear flow on a vector bundle $\pi : \mathcal{V} \to B$ with the chain transitive flow on the base space B. Then the Morse spectrum*

$$\Sigma_{Mo}(\Phi) := \bigcup_{i=1}^{l} \Sigma_{Mo}(\mathcal{V}_i)$$

contains the Lyapunov spectrum, and for every i

$$\Sigma_{Mo}(\mathcal{M}_i) = [\kappa^*(\mathcal{V}_i), \kappa(\mathcal{V}_i)],$$

where $\kappa^*(\mathcal{V}_i) = \inf \Sigma_{Mo}(\mathcal{V}_i)$, $\kappa(\mathcal{V}_i) = \sup \Sigma_{Mo}(\mathcal{V}_i)$, $\kappa^*(\mathcal{V}_i) < \kappa^*(\mathcal{V}_j)$, and $\kappa(\mathcal{V}_i) < \kappa(\mathcal{V}_j)$ for $i < j$; the boundary points $\kappa^*(\mathcal{V}_i)$ and $\kappa(\mathcal{V}_i)$ are Lyapunov exponents of Φ.

Further, we show that the flow induced on the projective bundle $\mathbb{P}\mathcal{V} = B \times \mathbb{P}^{d-1}$ allows to recover the subbundle decompositions associated with the (Morse) spectrum of a linear flow.

Theorem 4.3. *For $i = 1, 2$, let Φ_i be linear flows on $\mathcal{V}_i = B_i \times \mathbb{R}^d$ with projective flows $\mathbb{P}\Phi_i$ in $\mathbb{P}\mathcal{V}_i = B_i \times \mathbb{P}^{d-1}$. Assume that the base spaces B_i are compact and chain transitive. Denote the associated Lyapunov decompositions by $\bigoplus_{j=1}^{l} \mathcal{V}_i^j = \mathcal{V}_i$. Let $h = (f, g) : \mathbb{P}\mathcal{V}_1 \to \mathbb{P}\mathcal{V}_2$ be a skew equivalence between $\mathbb{P}\Phi_1$ and $\mathbb{P}\Phi_2$. Then*

- (i) *h maps chain recurrent components of $\mathbb{P}\Phi_1$ onto chain recurrent components of $\mathbb{P}\Phi_2$;*
- (ii) *h preserves the order of the chain recurrent components;*
- (iii) *Σ_1 and Σ_2 have the same number of spectral intervals, and h preserves the order between these intervals;*
- (iv) *h maps the associated bundle decompositions into each other, and the dimensions of the corresponding fibers agree.*

Proof. (i) and (ii): Lemma 4.3 and Proposition 5.2 in [1] prove these facts for flows over the same base space. The same proofs, with minor adjustments, go through for skew equivalences of projected flows.

(iii) follows directly from (ii) and the properties of the Morse spectrum.

(iv) Note that it follows from (i) that $h[\mathbb{P}\mathcal{V}_1^j] = \mathbb{P}\mathcal{V}_2^j$ for all $j = 1, \ldots, l$. In order to see that the dimensions of $\mathcal{V}_1^j$ and $\mathcal{V}_2^j$ coincide, observe that each (projective) fiber $\mathbb{P}\mathcal{V}_1^j(b) \subset \{b\} \times \mathbb{P}^{d-1} \cong \mathbb{P}^{d-1}$ is mapped homeomorphically onto the fiber $\mathbb{P}\mathcal{V}_2^j(f(b)) \subset \{f(b)\} \times \mathbb{P}^{d-1} \cong \mathbb{P}^{d-1}$. The canonical projection $\pi : \mathbb{R}^d \to \mathbb{P}^{d-1}$ is a submersion and hence the fibers are submanifolds of $\mathbb{P}^{d-1}$ (see [16]). Since h is a homeomorphism, the fibers have the same dimension by the domain invariance theorem (see [16]). Hence the linear dimensions of $\mathcal{V}_1^j(b)$ and $\mathcal{V}_2^j(f(b))$ coincide. $\qquad\square$

Remark 4.4. We cannot give a complete characterization of linear flows, for which the projective flows are topologically skew conjugate. Indeed, this question is open even for single matrices, see [1, Remark 5.4]. It is shown there that topological conjugacy of projected flows also preserves certain detail characteristics within the eigenspace decomposition. Theorem 4.3 shows that the existence of a topological skew conjugacy of the projected flows is a much stronger requirement than the existence of a topological skew conjugacy for linear flows (cf. Corollary 3.4).

For each Lyapunov subbundle $\mathcal{V}_i$, the Morse spectrum $\Sigma_{Mo}(\mathcal{V}_i)$ describes the exponential behavior of the solutions $\varphi(\cdot, b, x)$ with $(b, x) \in \mathcal{V}_i$. Hence our interest is in finding dynamical characterizations of the Lyapunov decomposition and the dimensions of the subbundles. For matrices, the Lyapunov forms summarize such a characterization (see [1]). This idea is generalized to linear flows as follows.

Definition 4.5. Consider a linear flow Φ on a vector bundle $\pi : \mathcal{V} \to B$ with the Lyapunov decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_l$. The Lyapunov index

$L(\Phi)$ of Φ is the matrix

$$
\begin{bmatrix} \Lambda_1 & & 0 \\ & \ddots & \\ 0 & & \Lambda_l \end{bmatrix}, \text{ where } \Lambda_i = \begin{bmatrix} \kappa^*(\mathcal{V}_i), \ \kappa(\mathcal{V}_i) & & 0 \\ & \ddots & \\ 0 & & \kappa^*(\mathcal{V}_i), \ \kappa(\mathcal{V}_i) \end{bmatrix},
$$

where the block size of Λ_i is the dimension $\dim \mathcal{V}_i$ of the corresponding subbundle. The blocks are arranged according to the order of the Lyapunov bundles. Two linear flows Φ_i are called Lyapunov equivalent if $L(\Phi_1) = L(\Phi_2)$.

Remark 4.6. The Lyapunov equivalence is an equivalence relation on the set of linear flows with a fixed dimension d (on compact chain transitive base spaces). Each class has a unique Lyapunov index given by l pairs of real numbers $\kappa^*(\mathcal{V}_1) < \cdots < \kappa^*(\mathcal{V}_l)$, $\kappa(\mathcal{V}_1) < \cdots < \kappa(\mathcal{V}_l)$ and l natural numbers $d_i = m(\mathcal{V}_i)$.

Remark 4.7. For matrices, one can find in every Lyapunov equivalence class a unique (diagonal) flow of the form $e^{\Lambda t}$ hence representing a normal form, called the Lyapunov normal form. For general linear flows, we use the matrix above only as a symbol for the corresponding equivalence class of linear flows. In particular, for a given base flow one should not expect that a linear flow of such a form exists.

Following the matrix case in [1], we also give the following definition.

Definition 4.8. The short Lyapunov index $SL(\Phi)$ of a linear flow Φ is given by the vector of the dimensions d_i of the l Lyapunov subbundles (in their natural order): $SL(\Phi) = (l, d_1, \ldots, d_l)$.

Two linear flows Φ_1 and Φ_2 have the same short Lyapunov index if and only if the (ordered) blocks of $L(\Phi_1)$ and $L(\Phi_2)$ have the same dimensions. This form does not contain stability information, since it does not include the actual size of the Lyapunov exponents, only their order. To separate the stable, center, and unstable bundles, one may also introduce the following definitions.

Definition 4.9.   (i) The short zero-Lyapunov index $SL_0(\Phi)$ is given by the vector of the dimensions d_i of the Lyapunov subbundles (in their natural order), and the number of subbundles for which the Morse spectrum is negative, includes the zero, and is positive: $SL_0(\Phi) = (l^-, l^0, l^+, d_1, \ldots, d_k)$, where $l = l^- + l^0 + l^+ \le d$ is the number of Lyapunov subbundles.

(ii) The stability Lyapunov index of Φ is given by the dimensions of the stable, center, and unstable subbundles, i.e., $SL^s(\Phi) = (l^-, l^0, l^+)$.

Clearly, a system is hyperbolic if $l^0 = 0$.

5. Grassmann graphs and finest Morse decompositions

In this section, we provide a characterization of Lyapunov equivalent linear flows using a graph (the Grassmann graph) constructed of the induced flows on the Grassmann bundles. This characterization generalizes the matrix case.

First, we recall some facts on Morse decompositions in Grassmann bundles. We denote by $\mathbb{G}_i$ the Grassmannian of i-dimensional subspaces of $\mathbb{R}^d$ (which can be identified with a subset of the projective space of the exterior product $\Lambda^i \mathbb{R}^d$). The kth flag of $\mathbb{R}^d$ is given by the following k-sequences of subspace inclusions:

$$\mathbb{F}_k = \{F_k = (V_1, \ldots, V_k),\ V_i \subset V_{i+1},\ \text{and}\ \dim V_i = i\ \text{for}\ i = 1, \ldots, k\}.$$

For $k = d$, we obtain the complete flag $\mathbb{F} = \mathbb{F}_d$. For a vector bundle $\mathcal{V} = B \times \mathbb{R}^d$, we obtain bundles of Grassmannians $\mathbb{G}_k \mathcal{V} = B \times \mathbb{G}_k$ and flags $\mathbb{F}_k \mathcal{V} = B \times \mathbb{F}_k$. For simplicity, we denote the flows induced by a linear flow Φ on these bundles also by Φ. We will need the following specific information on the finest Morse decompositions in the Grassmann bundles.

Theorem 5.1. *Let Φ be a linear flow on a vector bundle $\pi : \mathcal{V} \to B$ with the chain transitive compact base space B and dimension d. Let $\mathcal{V}_i$ of dimension d_i, $i = 1, \ldots, l$, be the Lyapunov subbundles. Define for $1 \leq k \leq d$ the index set*

$$I(k) := \{(k_1, k_2, \ldots, k_l),\ k_1 + k_2 + \cdots + k_l = k\ \text{and}\ 0 \leq k_i \leq d_i\}.$$

Then the finest Morse decomposition in the Grassmann bundle $\mathbb{G}_k \mathcal{V} \to B$ exists and is given by the sets

$$\mathcal{N}^k_{k_1, \ldots, k_l} = \mathbb{G}_{k_1} \mathcal{V}_1 \oplus \cdots \oplus \mathbb{G}_{k_l} \mathcal{V}_l,\ (k_1, \ldots, k_l) \in I(k), \tag{9}$$

with the interpretation that on the right-hand side, we have in every fiber $\mathcal{V}_b$ over $b \in B$ the sum of arbitrary k_i-dimensional subspaces of $\mathcal{V}_{i,b}$. In particular, these Morse sets are maximal chain transitive sets of $\mathbb{G}_k \mathcal{V}$.

For the proof of this theorem, we note that [4, Theorem 6] establishes that a finest Morse decomposition exists and that (9) defines a Morse decomposition (but not necessarily the finest one). By [2, Theorem 9.11], the maximal chain transitive sets in $\mathbb{G}_k \mathcal{V}$ are obtained as components of the fixed point set of a diagonal matrix $h(q) = \mathrm{diag}[\lambda_1 \mathrm{id}, \ldots, \lambda_l \mathrm{id}]$, where $\lambda_1 > \cdots > \lambda_l$ and sizes of blocks are as in Definition 4.5. The fixed point set of $h(q)$ in a Grassmannian is obtained by the construction above. The result of [2] was obtained under a restrictive hypothesis on the base space. Recently, Patrao and San Martin removed this restriction and proved the result in a much more general context (see [11, Theorem 7.5]).

Theorem 5.1 allows us to characterize short Lyapunov equivalence using a graph defined via the finest Morse decompositions on the Grassmann bundles. On each $\mathbb{G}_k \mathcal{V}$, $k = 1, \ldots, d$, we use the order $\preceq_k$ given by the

finest Morse decomposition and denote the corresponding directed order graph by G_k. The transitivity edges of G_k are the edges (e_1, e_2) in G_k, for which there exist nodes $n_1, \ldots, n_l$, $l \geq 3$, such that $(n_1 = e_1, \ldots, n_l = e_2)$ is a path in G_k. Then the elementary graph $E(G_k)$ is obtained by removing all transitivity edges.

Definition 5.2. For a linear flow Φ, the Grassmann graph is given by the elementary graphs $E(G_k)$, $k = 1, \ldots, d$, corresponding to the finest Morse decompositions in the Grassmann bundles together with the directed edges from nodes $\mathcal{N}^{k-1}_{k_1, \ldots, k_l}$ in $E(G_{k-1})$ to nodes $\mathcal{N}^k_{j_1, \ldots, j_l}$ in $E(G_k)$ if $k_i \leq j_i$ for all $i = 1, \ldots, d$.

Thus, the nodes of the Grassmann graph are the Morse sets and there exists an edge from a node $\mathcal{N}^{k-1}$ on the level $k-1$ to a node $\mathcal{N}^k$ on the level k if each subbundle in the Whitney decomposition (9) for $\mathcal{N}^{k-1}$ is contained in the corresponding subbundle for $\mathcal{N}^k$. We have removed transitivity edges on each level k in order to simplify the graph.

Remark 5.3. Theorem 5.1 describes an indexing system for the finest Morse decomposition on each Grassmann bundle $\mathbb{G}_k \mathcal{V}$ that corresponds to the parametrization of the short Lyapunov index.

We proceed to discuss how one can recover information about the Lyapunov bundles from the Grassmann graph.

Definition 5.4. Let G be the Grassmann graph of a linear flow. An *increasing path* p in G is a path (of length d) from the level $\mathbb{G}_1 \mathcal{V}$ to the level $\mathbb{G}_d \mathcal{V}$. The *in-order* of a node $n \in G$ is the number of edges that end at n and the *out-order* is the number of edges that begin at n. For an increasing path $p = (n_1, \ldots, n_d)$ in G, we define its *simple length*

$$\mathrm{sl}(p) = \max \{k, \text{ in-order } (n_k) \leq 1\}.$$

For a node n on the level $\mathbb{G}_1 \mathcal{V} = \mathbb{P}^{d-1} \mathcal{V}$, we define its *multiplicity* as

$$\mathrm{mult}(n) = \max\{\mathrm{sl}(p), p \text{ is an increasing path with the initial node } n\}.$$

Lemma 5.5. *Let a linear flow Φ on $\mathcal{V}$ be given. For a Lyapunov bundle $\mathcal{V}_i$, we denote the corresponding Morse set of the flow $\mathbb{P}\Phi$ by $\mathcal{M}_i = \mathbb{P}\mathcal{V}_i \subset \mathbb{P}\mathcal{V}$. Then the multiplicity $\mathrm{mult}(\mathcal{M}_i)$ of $\mathcal{M}_i$ in the Grassmann graph $G(\Phi)$ of Φ is equal to the (linear) dimension $\dim \mathcal{V}_i$.*

Proof. The proof follows directly from Theorem 5.1 and the assumption. $\square$

This lemma says that one can recover dimensions of the Lyapunov bundles from the Grassmann graph. Furthermore, the order of the Lyapunov bundles can be recovered from the order $\preceq$ on the level $\mathbb{G}_1 \mathcal{V}$ of the graph. Hence we can hope to use Grassmann graphs for the characterization of the short Lyapunov index of a linear flow.

Definition 5.6. Let G and G' be finite directed graphs. A mapping $h : G \to G'$ is called a *graph homomorphism* if for all edges (n_1, n_2) in G, $(h(n_1), h(n_2))$ is an edge in G'. Furthermore, h is a *graph isomorphism* if h is bijective and h and h^{-1} are graph homomorphisms.

Theorem 5.7. *Let Φ and Ψ be linear flows on vector bundles of equal dimension and with compact chain transitive base spaces. Then the short Lyapunov indices $SL(\Phi)$ and $SL(\Psi)$ coincide iff the Grassmann graphs $G(\Phi)$ and $G(\Psi)$ are isomorphic.*

Proof. Let the Grassmann graphs $G(\Phi)$ and $G(\Psi)$ be isomorphic.

(i) We show that the subgraphs on each level k are uniquely defined and that they are isomorphic. Then the isomorphism between the graphs also shows that for each k, the edges between the level $k-1$ and level k correspond to each other. The only node with out-order 0 is the unique node n_l on the highest level l. All nodes n, for which there exists an edge (n, n_l), are on the level $l-1$. All nodes n' that are not on the level $l-1$ and for which there exists an edge (n', n) with n on the level $l-1$, are on the level $l-2$, etc. This algorithm stops after l' steps, i.e., after determining the nodes on level l', and all nodes are associated with some level. Then $l - l' + 1 = d$, the dimension of the underlying vector bundles. We re-index the levels so that the smallest level is 1. Since the graphs $G(\Phi)$ and $G(\Psi)$ are isomorphic, it follows that the subgraphs on the same level k are isomorphic. Note that the node corresponding to the Morse set $\mathcal{M}_1$ on $\mathbb{G}_1\mathcal{V}$ is the unique node with in-order 0.

(ii) The length of any increasing path $(n_1, n_2, \ldots, n_d)$ defines the dimension d.

(iii) For each node on the level $\mathbb{G}_1\mathcal{V}$, its multiplicity defines the dimension of the corresponding Lyapunov space.

(i)–(iii) mean that for a linear flow, its short Lyapunov index can be uniquely reconstructed from the Grassmann graph and hence isomorphic Grassmann graphs belong to linear flows with identical short Lyapunov indices. And vice versa, short Lyapunov indices define Grassmann graphs by their construction. □

Theorem 5.7 characterizes the Lyapunov subbundles and their dimensions, i.e., the short Lyapunov form $SL(\Phi)$ for a linear flow Φ. Together with the topological characterization in Corollary 3.4, one also obtains results on the short zero-Lyapunov index $SL_0(\Phi)$, generalizing the situation for linear differential equations $\dot{x} = Ax$. In Sec. 6, we will analyze bilinear control systems in more detail.

6. Applications to bilinear control systems

In the last twenty years, the problem of the classification of control systems allowing state and feedback transformations has been extensively studied. In particular, we mention the approach due to Kang and Krener [8] based on Taylor expansions and more geometric approaches to the equivalence for (nonlinear) control systems that are based on equivalent distributions defined by a system on the tangent bundle. This point of view allows us to redefine the controls (via feedback) and requires that the control range is a linear, unbounded space (see, e.g., the recent survey of Respondek and Tall [12]). This section approaches the classification of bilinear control systems from a topological point of view, as is common in the theory of dynamical systems (see, e.g., [7,17]). Most of the proofs are based on results from the previous sections and in some cases more specific information can be obtained due to the specific nature of bilinear control flows.

We denote the set of $(d \times d)$-matrices with real entries by $\mathrm{gl}(d, \mathbb{R})$.

Definition 6.1. A bilinear control system in $\mathbb{R}^d$ is given by a set of matrices $\{A_0, \ldots, A_m\} \subset \mathrm{gl}(d, \mathbb{R})$ and a control range $U \subset \mathbb{R}^m$, which is assumed to be a compact and convex set with $0 \in \mathrm{int}\, U$:

$$\dot{x}(t) = A(u(t))x(t) = \left[A_0 + \sum_{i=1}^{m} u_i(t)A_i\right] x(t), \tag{10}$$

$$u \in \mathcal{U} := \{u : \mathbb{R} \to U \ \text{ for all } \ t \in \mathbb{R}, \ \text{locally integrable}\}.$$

For all $(u, x) \in \mathcal{U} \times \mathbb{R}^d$, the system has a unique solution $\varphi(t, x, u)$, $t \in \mathbb{R}$, such that $\varphi(0, x, u) = x$. We denote by $\mathbf{B}(d, m, U)$ the set of bilinear control systems $\Sigma = (A_0, \ldots, A_m, U)$ in $\mathbb{R}^d$ with m controls and control range U.

A linear dynamical system (the control flow) associated with a control system in the following way (see [5]):

$$\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d, \quad \Phi(t, u, x) = (\theta(t, u), \varphi(t, x, u)), \tag{11}$$

where we denote the shift in the base by $\theta(t, u(\cdot)) = u(t + \cdot)$. Dynamical system (11) is a linear skew-product flow on the vector bundle $\mathcal{U} \times \mathbb{R}^d$. Continuity of Φ follows if $\mathcal{U} \subset L_\infty(\mathbb{R}, \mathbb{R}^m)$ is endowed with the weak* topology, i.e., the weakest topology such that for all $\psi \in L_1(\mathbb{R}, \mathbb{R}^m)$ the mappings

$$L_\infty(\mathbb{R}, \mathbb{R}^m) \to \mathbb{R}, \quad u \mapsto \int_{\mathbb{R}} u(t)^T \psi(t) dt$$

are continuous. We also note that $\mathcal{U}$ becomes a compact metrizable space; a metric is obtained by choosing a countable dense subset $\{\psi_n\}$ in $L_1(\mathbb{R}, \mathbb{R}^m)$ and defining

$$\mathrm{d}(u, v) = \sum_{n=1}^{\infty} 2^{-n} \frac{\left|\int_{\mathbb{R}} [u(t) - v(t)]^T \psi_n(t) dt\right|}{1 + \left|\int_{\mathbb{R}} [u(t) - v(t)]^T \psi_n(t) dt\right|}.$$

Note that the shift on $\mathcal{U}$ is chain transitive and chain recurrent.

6.1. **Base conjugation for bilinear control systems.** In this section, we analyze when two bilinear control systems are base conjugate, i.e., the corresponding shifts on the control functions are conjugate. Note that the base $\mathcal{U}$ is considered in the weak* topology of L_∞ and hence the continuity of the conjugation mapping implies that the conjugation preserves, in an appropriate way, the duality relation between L_∞ and L_1.

Two subsets U_1 and U_2 of $\mathbb{R}^m$ are called affine isomorphic if there exists an invertible affine mapping H on $\mathbb{R}^m$ with $H[U_1] = U_2$. This means that there exist an invertible matrix $M \in \mathbb{R}^{m \times m}$ and a vector $b \in \mathbb{R}^m$ such that

$$H(x) = Mx + b. \tag{12}$$

Then the inverse is

$$H^{-1}(y) = M^{-1}y - M^{-1}b. \tag{13}$$

Proposition 6.2. *Let $U_1, U_2 \subset \mathbb{R}^m$ be compact and convex, and consider for $i = 1, 2$*

$$\mathcal{U}_i := \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m), \ u(t) \in U_i \text{ for a.a. } t \in \mathbb{R}\},$$

with shifts $\theta_i : \mathbb{R} \times \mathcal{U}_i \longrightarrow \mathcal{U}_i$. If the sets U_1 and U_2 are affine isomorphic, then there exists a homeomorphism $f : \mathcal{U}_1 \to \mathcal{U}_2$ in the weak topology such that*

$$f(\theta_1(t, u)) = \theta_2(t, f(u)) \text{ for all } t \in \mathbb{R}.$$

Proof. Using the affine isomorphism H as in (12) between U_1 and U_2, we define a mapping

$$f : \mathcal{U}_1 \to \mathcal{U}_2, \ u \mapsto (f(u))(s) = (H(u(s)), \ s \in \mathbb{R}.$$

Then the conjugation property

$$f(u(t + \cdot)) = f(u)(t + \cdot)$$

holds. This mapping is continuous, since for $u_n \to u$ in $\mathcal{U}_1$ and every $\psi \in L_1(\mathbb{R}, \mathbb{R}^m)$, we have

$$\int_{\mathbb{R}} [f(u_n)(t) - f(u)(t)]^T \psi(t)dt = \int_{\mathbb{R}} [H(u_n(t)) - H(u(t))]^T \psi(t)dt$$

$$= \int_{\mathbb{R}} [M(u_n(t) - u(t))]^T \psi(t)dt = \int_{\mathbb{R}} [u_n(t) - u(t)] M^T \psi(t)dt.$$

Since $M^T \psi(\cdot) \in L_1$, this converges to zero for $n \to \infty$. The inverse of f is constructed using the inverse (13) of H.   □

As a consequence of this proposition, we obtain that any two shift flows with a scalar control are conjugate.

Corollary 6.3. *Each of the following conditions implies that U_1 and U_2 are affinely isomorphic and hence the corresponding shifts are conjugate*:

(i) *the sets U_1 and U_2 are compact intervals in $\mathbb{R}$ with nonempty interiors*;

(ii) *the sets U_1 and U_2 are convex hulls of $2m$ points of $\mathbb{R}^m$ in the form*

$$U_i = \mathrm{co}(v_i^1, \ldots, v_i^m, -v_i^1, \cdots - v_i^m\}$$

with linearly independent $v_i^1, \ldots, v_i^m$.

Proof. (i) The affine isomorphism is obtained by shifting each interval so that the origin becomes the middle point and then mapping the boundary points to each other.

(ii) Define a linear isomorphism H on the linear basis by $H(v_1^j) = v_2^j$ for $j = 1, \ldots, m$. $\qquad\square$

Corollary 6.4. *Let $\rho > 0$, and consider for a compact and convex set $U \subset \mathbb{R}^m$ the control range $\rho \cdot U$. Then the shifts on $\mathcal{U}$ and $\mathcal{U}^\rho := \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m), u(t) \in \rho U \text{ for all } t \in \mathbb{R}\}$ are conjugate.*

Proof. The proof follows from Proposition 6.2, since $H : u \mapsto \rho u$ is linear. $\qquad\square$

6.2. Topological conjugation and equivalence in $\mathbb{R}^d$ and $\mathbb{P}^{d-1}$.

The results of Sec. 3 are immediately applicable to bilinear control systems. Let $\Sigma_1 = (A_0, \ldots, A_m, U_1)$ and $\Sigma_2 = (B_0, \ldots, B_m, U_2)$ be two bilinear control systems in $\mathbf{B}(d, m, U_i)$ with linear flows $\Phi = (\theta, \varphi)$ and $\Psi = (\vartheta, \psi)$, respectively.

Corollary 6.5. *Consider two bilinear control systems with conjugate base flows.*

(i) *If both flows are exponentially (un)stable, then they are skew conjugate.*

(ii) *If both flows are hyperbolic, i.e., the vector bundles $\mathcal{U}_i \times \mathbb{R}^d$ can be written as the Whitney sums of exponentially stable and unstable subbundles, and if the dimensions of their stable (and unstable) subbundles coincide, then they are skew conjugate.*

The proof follows directly from Corollary 3.4. Corollary 6.5 generalizes the well-known result for hyperbolic matrices to bilinear control systems.

Corollary 6.6. *Let $\Phi = (\theta, \varphi)$ and $\Psi = (\vartheta, \psi)$ be the control flows in $\mathbb{R}^d$ of two bilinear control systems $\Sigma_i \in \mathbf{B}(d, m, U_i)$. If Φ and Ψ are C^1-conjugate via $h = (f, g)$, then all matrices $A(u) = A_0 + \sum_{i=1}^m u_i A_i$ and $B(u) = B_0 + \sum_{i=1}^m u_i B_i$ are linearly conjugate in the sense that, for each constant control $u \in U_1$, there exists an invertible matrix $T(u) \in \mathrm{Gl}(d, \mathbb{R})$ such that*

$$A(u) = T^{-1}(u) B(f(u)) T(u).$$

Proof. According to Proposition 3.6, we have for all $t \in \mathbb{R}$ and $u \in \mathcal{U}_1$:

$$\varphi(t, \cdot, u) = [D_x g(\theta_t u, 0)]^{-1} \circ \psi(t, \cdot, f(u)) \circ D_x g(u, 0). \qquad (14)$$

Note that the constant controls $u(t) \equiv u \in \mathcal{U}_1$ for all $t \in \mathbb{R}$ are fixed points of the shift θ. Hence we obtain for all $u \in U_1$

$$\varphi(t, \cdot, u) = [D_x g(u, 0)]^{-1} \circ \psi(t, \cdot, f(u)) \circ D_x g(u, 0). \qquad (15)$$

The differentiation of (15) with respect to t yields at $t = 0$ the result

$$A(u) = T^{-1}(u) B(f(u)) T(u) \text{ for all } u \in U_1, \qquad (16)$$

where we have set $T(u) := D_x g(u, 0)$. $\square$

Remark 6.7. The proof also shows that for $U_1 = U_2 = U$ with $f = \text{id}$, the relation $A(u) = T^{-1}(u) B(u) T(u)$ holds for all $u \in U$. One obtains, e.g., for $u = 0$, that A_0 and B_0 are similar matrices.

Remark 6.8. Note that linear conjugacy of the matrices in the sense of Eq. (16) does not automatically imply that the flows Φ and Ψ are linearly (and hence C^k) conjugate. This would follow from simultaneous equivalence of the matrices, i.e., the fact that there exists a basis transformation $T \in \text{Gl}(d, \mathbb{R})$ such that $A_i = T^{-1} B_i T$ for $i = 0, \ldots, m$. The result of Corollary 6.6 contains this situation for linear differential equations as a special case by considering $u = 0$.

Next, we discuss how the control system induced on the projective space $\mathbb{P}^{d-1}$ allows us to recover the subbundle decompositions associated with the (Morse) spectrum of a bilinear control system (cf. Sec. 4).

A system $\Sigma \in \mathbf{B}(d, m, U)$ induces a (nonlinear) control system $\mathbb{P}\Sigma$ on the projective space $\mathbb{P}^{d-1}$ as follows:

$$\dot{s}(t) = \mathbb{P}A(u(t), s(t)) = \mathbb{P}A_0(s) + \sum_{i=1}^{m} u_i \mathbb{P}A_i(s), \qquad (17)$$

$$\mathbb{P}A_i(u, s) = (A_i - s^T A_i s \cdot I)s \text{ for all } i = 0, \ldots, m.$$

Here ^{T} denotes the transposition and I is the identity $(d \times d)$-matrix. For all $(u, s) \in \mathcal{U} \times \mathbb{P}^{d-1}$, the system has a unique solution denoted by $\mathbb{P}\varphi(t, s, u)$ for all $t \in \mathbb{R}$ with $\mathbb{P}\varphi(0, s, u) = s$. The associated dynamical system has the form

$$\mathbb{P}\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{P}^{d-1} \to \mathcal{U} \times \mathbb{P}^{d-1}, \quad \mathbb{P}\Phi(t, u, s) = (\theta(t, u), \mathbb{P}\varphi(t, s, u)). \quad (18)$$

The *Morse spectrum* of the system Σ is $\Sigma_{Mo} = \bigcup_{j=1}^{l} \Sigma_{Mo}(\mathcal{E}_j)$, where the $\mathcal{E}_j$ are the maximal chain transitive sets (or Morse sets) of $\mathbb{P}\Phi$. As before, we denote by $\mathcal{V}_j$ the Lyapunov subbundle of Φ associated with the Morse set $\mathcal{E}_j$ for $j = 1, \ldots, l$. Recall that the projections of the maximal chain transitive sets of $\mathbb{P}\Phi$ onto the projective space, i.e., $E_j = \{y \in \mathbb{P}^{d-1}: \text{there}$

exists $u \in \mathcal{U}$ such that $(u, y) \in \mathcal{E}_j\}$, are the chain control sets of system (17) (see [5, Chap. 4]).

Corollary 6.9. *For $i = 1, 2$, let $\Sigma_i \in \mathbf{B}(d, m, U_i)$ be two bilinear control systems with associated flows Φ_i in $\mathcal{U}_i \times \mathbb{R}^d$ and projected flows $\mathbb{P}\Phi_i$ in $\mathcal{U}_i \times \mathbb{P}^{d-1}$. Denote the associated bundle decompositions by $\bigoplus_{j=1}^{l} \mathcal{V}_i^j = \mathcal{U}_i \times \mathbb{R}^d$. Let $h = (f, g) : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathcal{U}_2 \times \mathbb{P}^{d-1}$ be a skew equivalence between $\mathbb{P}\Phi_1$ and $\mathbb{P}\Phi_2$. Then the following assertions hold.*

(i) *Let E be a chain control set for the system Σ_1. Then*

$$\Big\{ s_2 \in \mathbb{P}^{d-1} : \text{there exists } u_2 \in \mathcal{U}_2 \text{ such that } (u_2, s_2) = (f(u_1), g(u_1, s_1))$$

for some $(u_1, s_1) \in \mathcal{U}_1 \times \mathbb{P}^{d-1}$ such that $\mathbb{P}\varphi(t, s_1, u_1) \in E$ for all $t \in \mathbb{R} \Big\}$

is a chain control set of Σ_2, and every chain control set of Σ_2 is of this form.

(ii) *The relation between the chain control sets from* (i) *preserves the order of the chain control sets.*

(iii) *Σ_1 and Σ_2 have the same number of spectral intervals, and h preserves the order between these intervals.*

(iv) *h maps the associated bundle decompositions into each other, and the dimensions of the corresponding fibers agree.*

Proof. (i) and (ii) Theorem 4.3 proves these facts for the maximal chain transitive sets. Then the assertion for the chain control sets follows, since for a chain control set E, the lift

$$\mathcal{E} = \{(u, y) \in \mathcal{U} \times \mathbb{P}^{d-1}, \ \varphi(t, y, u) \in E \text{ for all } t \in \mathbb{R}\}$$

is a maximal chain transitive set.

(iii) and (iv) follow directly from Theorem 4.3. $\square$

6.3. The Lyapunov index of bilinear control systems.

This section characterizes the (short) Lyapunov index of bilinear control systems as introduced in Sec. 5 for general linear flows. The specific structure of bilinear systems allows us to give a more complete description of the (exponential) subbundles and their dimension. Note that it follows from Remark 4.4 that conjugacies of the projective flows do not characterize the Lyapunov index of a bilinear control system, since the requirement $h(\mathbb{P}\Phi_1(t, u, x)) = \mathbb{P}\Phi_2(t, h(u, x))$, i.e., of mapping trajectories into trajectories, is too strong. Hence we employ a concept that relates to mappings of the Morse decompositions of the projective flow (see [1, Theorem 5.5]).

Theorem 6.10. *Consider two bilinear control systems $\Sigma_i \in \mathbf{B}(d, m, U_i)$ which are base conjugate via $f : \mathcal{U}_1 \to \mathcal{U}_2$. Then Σ_1 and Σ_2 have the same short Lyapunov index iff there exists a skew homeomorphism $h = (f, g) : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathcal{U}_2 \times \mathbb{P}^{d-1}$, where $g : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$, that maps the finest*

Morse decomposition of $\mathbb{P}\Phi_1$ *into the finest Morse decomposition of* $\mathbb{P}\Phi_2$, *i.e., h maps Morse sets into Morse sets and preserves their order.*

Proof. Let $h : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathcal{U}_2 \times \mathbb{P}^{d-1}$ be a homeomorphism that maps the finest Morse decomposition of $\mathbb{P}\Phi_1$ into the finest Morse decomposition of $\mathbb{P}\Phi_2$. This means, in particular, that both systems have the same number of spectral intervals, and these are ordered according to their minimal (or maximal) elements. It remains to show that the associated bundle decompositions have the same dimension. This follows exactly as assertion (iv) in Theorem 4.3. Hence the short Lyapunov indices of the systems coincide.

For the converse, we order the Morse sets of $\mathbb{P}\Phi_1$ and $\mathbb{P}\Phi_2$ in their natural order and concentrate on one corresponding pair, say $\mathcal{M}^1$ for $\mathbb{P}\Phi_1$ and $\mathcal{M}^2$ for $\mathbb{P}\Phi_2$. By Theorem 4.1, the lifts $\mathbb{P}^{-1}\mathcal{M}^i$ of $\mathcal{M}^i$ to $\mathcal{U}_i \times \mathbb{R}^d$ are subbundles $\mathcal{V}_j^i$ such that $\mathcal{U}_i \times \mathbb{R}^d = \bigoplus_{j=1}^{l} \mathcal{V}_j^i$, and one can choose, for every $u_1 \in \mathcal{U}_1$ and $u_2 = f(u_1) \in \mathcal{U}_2$, a basis $x_1^i(u_i), \ldots, x_{k_j}^i(u_i) \in \mathbb{R}^d$ such that $\mathcal{V}_j^i(u_i) = \mathrm{span}\{x_1(u_i), \ldots, x_{k_j}(u_i)\}$. Since the subbundles are continuous decompositions of $\mathcal{U}_i \times \mathbb{R}^d$, these choices can be made continuous. We define a family of linear, invertible mappings on $\mathbb{R}^d$ via $T_u(x_k^1(u_1)) = x_k^2(u_2)$, $k = 1, \ldots, d$. The projection $\mathbb{P}T : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathcal{U}_2 \times \mathbb{P}^{d-1}$ is the desired skew homeomorphism. $\qquad\square$

Corollary 6.11. *Consider two bilinear control systems* $\Sigma_i \in \mathbf{B}(d, m, U_i)$ *such that the corresponding flows* Φ_i *are base conjugate and hyperbolic. Then* Σ_1 *and* Σ_2 *have the same short zero-Lyapunov index iff their linear flows* Φ_i *in* $\mathcal{U} \times \mathbb{R}^d$ *are skew conjugate, and there exists a skew homeomorphism* $h : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathcal{U}_2 \times \mathbb{P}^{d-1}$ *preserving the finest Morse decompositions of the projected flows.*

Proof. By Theorem 6.10, the flows have the same short Lyapunov index iff a homeomorphism h as above exists. Additionally, Corollary 6.5 shows that the dimension of the stable subbundles is fixed (and hence the short zero-Lyapunov index is defined) iff the linear flows are conjugate. $\qquad\square$

Finally, we mention the application of Theorem 5.7 on Grassmann graphs to bilinear control systems.

Corollary 6.12. *Let* $\Sigma_1 \in \mathbf{B}(d, m, U_1)$ *and* $\Sigma_2 \in \mathbf{B}(d, m, U_2)$ *be bilinear control systems. Then the short Lyapunov indices* $SL(\Sigma_1)$ *and* $SL(\Sigma_2)$ *coincide iff the Grassmann graphs of* Φ_1 *and* Φ_2 *are isomorphic.*

6.4. **Families of bilinear control systems.** For bilinear control systems with compact control range, it is of great interest to study the change in system behavior under varying control range, specifically controllability, stability and stabilization, and bifurcation phenomena. The theory developed

in this paper further illuminates the properties of these families of control systems.

Consider the family of bilinear control systems

$$\Sigma^\rho \in \mathbf{B}(d, m, U^\rho) \text{ with } U^\rho = \rho \cdot U, \quad \rho \geq 0, \tag{19}$$

where $U \subset \mathbb{R}^m$ is convex, compact, and such that $0 \in \operatorname{int} U$. Thus, the sets of admissible controls are

$$\mathcal{U}^\rho = \{u : \mathbb{R} \to U^\rho, \text{ locally integrable}\}.$$

For $\rho \in [0, \infty)$, the objects related to $(19)^\rho$ can be denoted by a superscript ρ. The systems Σ^ρ induce (nonlinear) control systems $\mathbb{P}\Sigma^\rho$ on the projective space $\mathbb{P}^{d-1}$ as in (17) with control range U^ρ. For the corresponding chain control sets E_j^ρ, we define the mappings

$$E_j : [0, \infty] \to \mathcal{C}(\mathbb{P}^{d-1}), \ \rho \mapsto E_j^\rho, \ j = 1, \dots, l, \tag{20}$$

where l is the number of different real parts of eigenvalues of A_0, and $\mathcal{C}(\mathbb{P}^{d-1})$ is the set of compact subsets of $\mathbb{P}^{d-1}$ with the Hausdorff metric (see [5]). Note that the mappings $E_j(\cdot)$ are increasing in ρ for all $j = 1, \dots, l$. The mappings of the subbundle decompositions corresponding to (20) and their dimensions are given by

$$\mathcal{V}_j : [0, \infty) \to \mathcal{L}(\mathcal{U} \times \mathbb{R}^d), \ \rho \mapsto \mathcal{V}_j^\rho, \ j = 1, \dots, l, \tag{21}$$
$$m_j : [0, \infty) \to \{0, \dots, d\}, \ \rho \mapsto \dim(\mathcal{V}_j^\rho) =: m_j^\rho, \ j = 1, \dots, l,$$

where $\mathcal{L}(\mathcal{U} \times \mathbb{R}^d)$ is the space of linear subbundles of $\mathcal{U} \times \mathbb{R}^d$. Note that the mappings $m_j(\cdot)$ are piecewise constant, increasing, having at most $d-1$ points of discontinuity. The exact number of discontinuities depends on the number l of different real parts of eigenvalues of A_0 and on the successive mergers of the chain control sets $E_j(\rho)$ as ρ increases. Theorem 6.10 implies the following characterization of the family $(19)^\rho$ of bilinear control systems in terms of the short Lyapunov indices.

Theorem 6.13. *Let $\Sigma^\rho \in \mathbf{B}(d, m, U^\rho)$ be a family (19) of bilinear control systems depending on the parameter $\rho \geq 0$. Then the following statements are equivalent for two systems Σ^{ρ_1} and Σ^{ρ_2}:*

(i) *Σ^{ρ_1} and Σ^{ρ_2} have the same short Lyapunov indices*

$$SL(\Sigma^{\rho_1}) = SL(\Sigma^{\rho_2});$$

(ii) *ρ_1 and ρ_2 are in the same (constant) interval of $m_j(\cdot)$ for all $j = 1, \dots, l$;*

(iii) *there exists a skew homeomorphism $h : \mathcal{U}^{\rho_1} \times \mathbb{P}^{d-1} \to \mathcal{U}^{\rho_2} \times \mathbb{P}^{d-1}$ mapping the finest Morse decomposition of $\mathbb{P}\Phi^{\rho_1}$ into the finest Morse decomposition of $\mathbb{P}\Phi^{\rho_2}$;*

(iv) *the Grassmann graphs of Σ^{ρ_1} and Σ^{ρ_2} are isomorphic.*

Proof. (i) $\Rightarrow$ (ii) If for one $j \in \{1, \ldots, l\}$ we have $m_j(\rho_1) \neq m_j(\rho_2)$, then the subbundle decompositions $\bigoplus \mathcal{V}_j^{\rho_1}$ and $\bigoplus \mathcal{V}_j^{\rho_1}$ do not have the same dimensions and hence the short Lyapunov indices differ.

(ii) $\Rightarrow$ (i) follows directly from the definition of the short Lyapunov index.

(i) $\Leftrightarrow$ (iii) First, note that U^{ρ_1} and U^{ρ_2} are linearly isomorphic via $H :$ $U^{\rho_1} \to U^{\rho_2}$, $H(u) = \dfrac{\rho_2}{\rho_1} u$, and hence by Proposition 6.2 the shifts on the sets $\mathcal{U}^{\rho_1}$ and $\mathcal{U}^{\rho_2}$ of admissible control functions are conjugate. Now the result follows from Theorem 6.10.

(i) $\Leftrightarrow$ (iv) follows from Corollary 6.12. $\qquad\square$

If we add the hyperbolicity, a characterization in terms of the short zero-Lyapunov indices and conjugacies is obtained.

Corollary 6.14. *Let $\Sigma^{\rho} \in \mathbf{B}(d, m, U^{\rho})$ be a family* (19) *of bilinear control systems depending on the parameter $\rho \geq 0$. Then for two systems Σ^{ρ_1} and Σ^{ρ_2} with hyperbolic linear flows Φ^{ρ_1} and Φ^{ρ_2}, respectively, the following statements are equivalent*:

(i) *Σ^{ρ_1} and Σ^{ρ_2} have the same short zero-Lyapunov indices $SL_0(\Sigma^{\rho_1}) = SL_0(\Sigma^{\rho_2})$;*

(ii) *ρ_1 and ρ_2 are in the same* (*constant*) *interval of $m_j(\cdot)$ for all $j = 1, \ldots, l$, and the dimensions of the stable subbundles coincide;*

(iii) *the linear flows Φ^{ρ_1} and Φ^{ρ_2} are skew conjugate, and there exists a skew homeomorphism $h : \mathcal{U}^{\rho_1} \times \mathbb{P}^{d-1} \to \mathcal{U}^{\rho_2} \times \mathbb{P}^{d-1}$ mapping the finest Morse decomposition of $\mathbb{P}\Phi^{\rho_1}$ into the finest Morse decomposition of $\mathbb{P}\Phi^{\rho_2}$.*

Proof. (i) $\Leftrightarrow$ (iii) This follows from Corollary 6.11.

(i) $\Leftrightarrow$ (ii) By Theorem 6.13, the short Lyapunov indices coincide if ρ_1 and ρ_2 are in the same interval. Since the short zero-Lyapunov index contains additional information only on the dimension of the stable subbundle, the assertion follows. $\qquad\square$

Much more can be said about the discontinuity points of the mappings $m_j(\cdot)$ (see [5, Chap. 7]).

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