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Singular linear systems on Lie groups; equivalence

Victor Ayala [a,*], Philippe Jouan [b]

[a] Instituto de Alta Investigación, Universidad de Tarapacá, Sede Iquique Chile and Universidad Católica del Norte, Antofagasta, Chile
[b] Lab. R. Salem, CNRS UMR 6085, Université de Rouen, avenue de l'université BP 12, 76801 Saint-Étienne-du-Rouvray, France

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ABSTRACT

In this paper, we define different kinds of singular systems on Lie groups and we analyze some of them. Furthermore, we state sufficient conditions for a general nonlinear singular system defined on a manifold to be equivalent by diffeomorphism to one of these models. Some examples are computed.

1. Introduction

The purpose of this paper is to design models of nonlinear singular systems by means of the so-called linear systems on Lie groups, which are actually nonlinear, despite their name that comes from their similarity with linear systems in $\mathbb{R}^n$.

More accurately a control-affine system on a Lie group is said to be linear if its drift is an infinitesimal automorphism (see Bourbaki [1]), called linear vector field in a geometric control context, and the controlled fields are right (or left)-invariant. Thanks to the Equivalence Theorem of [2] these systems appear as models for a large class of nonlinear systems: the finite ones, that is the systems whose generated Lie algebra is finite dimensional. To be more precise the Equivalence Theorem states that (under some technical assumptions) a finite system is equivalent by diffeomorphism to a linear system on a Lie group or a homogeneous space.

It is worthwhile to notice that in order to state the Equivalence Theorem in its full generality it is necessary to extend the definition of linear systems to homogeneous spaces.

On the other hand linear and nonlinear singular equations and control systems have attracted a lot of interest under different names: differential–algebraic equations (DAE) or systems, descriptors, degenerate systems. Good accounts of the linear theory can be found in the books [3,4]. The more recent book [5] deals also with nonlinear DAE, analyzed through linearization (for this approach see also [6,7]) and contains a detailed bibliography. Our approach is more geometric (see for instance [8–11]) but opposite to what happens for general nonlinear differential–algebraic systems defined on manifolds, the Lie structure allows us to get a natural splitting into the differential and algebraic parts, which is moreover global. The counterpart is that we deal with systems that generate a finite dimensional Lie algebra only.

Our goal to design models of nonlinear singular systems is reached in three steps:

1. The first one consists in defining models of singular linear systems on a Lie group G, that is systems of the kind
$$E_g.\dot{g} = \mathcal{X}_g + Y_g + \sum_{j=1}^{m} u_j Y_g^j$$
where $\mathcal{X}$ is a linear vector field, Y and the Y^js are right-invariant, and E_g is a noninvertible linear map defined on each tangent space T_gG. They were introduced in [12] in the case where E is a derivation of the Lie algebra $\mathfrak{g}$ of G and $E_g = TR_g.E.TR_{g^{-1}}$.
2. In a second step we should analyze these models. This analysis is not essential to state an equivalence theorem but it validates the interest of the models. They would be worthless if their analysis was not possible.
3. To finish we have to state and prove the equivalence of finite singular systems with our models after having checked that can be extended to homogeneous spaces.

The first step is realized in Section 3, after having recalled the basic definitions in Section 2. It is natural to require that the linear mapping E_g that acts on the tangent space T_gG be related to the Lie structure. This leads to $E_g = TR_g.E.TR_{g^{-1}}$ where E is a derivation or a noninvertible morphism of the Lie algebra $\mathfrak{g}$ of G. Another possibility is to define a singular linear system using a

* Corresponding author.
E-mail addresses: vayala@uta.cl (V. Ayala), philippe.jouan@univ-rouen.fr (P. Jouan).

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noninvertible Lie group morphism θ, that is to consider systems of the kind

$$T_g\theta.\dot{g} = \mathcal{X}_{\theta(g)} + Y_{\theta(g)} + \sum_{j=1}^{m} u_j Y^j_{\theta(g)}.$$

These systems are quite different from the previous ones, in particular the usual existence and uniqueness of the solutions does not hold.

Section 4 is devoted to the analysis of singular systems defined by Lie algebra morphisms. It is shown that under some additional (but natural) assumptions these systems can be analyzed, that is decomposed into a nonsingular part and an algebraic one. As well the additional assumptions than the analysis itself are close to the usual ones for singular linear systems in $\mathbb{R}^n$. The analysis of singular systems defined by a derivation has been previously made in [13] and the one of singular systems defined by Lie group morphisms is postponed to Section 7.

The third step is done in Section 6 where the equivalence theorem is stated. Its proof, that makes use of the Equivalence Theorem of [2] recalled in the Appendix, follows the technical Section 5 where it is shown that the models defined by derivation or morphism can be extended to homogeneous spaces.

Section 7 deals with the group morphism case. As explained above the reasons to deal with these systems in a different part are that on the one hand their analysis is very easy but on the other one the proof of the equivalence Theorem is quite different for them. They have also the drawback that neither the existence nor the uniqueness of the solutions are guaranteed.

Two examples are presented in Section 8. The first one is a decomposition into the horizontal and the vertical part of a morphism model on the 2-dimensional affine group. In the second example an algebraic system on the Heisenberg group is solved.

2. Basic definitions and notations

In this section the definition of linear vector fields and some of their properties are recalled. More details can be found in [2] (see also [14]).

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra (the set of right-invariant vector fields, identified with the tangent space at the identity).

The right (resp. left) translation by $g \in G$ is denoted by R_g (resp. L_g) and its differential at the point h by T_hR_g, or by TR_g if no confusion can happen (resp. T_hL_g or TL_g).

A vector field on G is said to be *linear if its flow is a one parameter group of automorphisms*. Notice that a linear vector field is consequently analytic and complete.

The following characterization will be useful in the sequel.

Characterization of linear vector fields

Let $\mathcal{X}$ be a vector field on a connected Lie group G. The following conditions are equivalent:

1. *$\mathcal{X}$ is linear;*
2. *$\mathcal{X}$ belongs to the normalizer of $\mathfrak{g}$ in the algebra $V^\omega(G)$ of analytic vector fields of G, that is*

$$\forall Y \in \mathfrak{g} \qquad [\mathcal{X}, Y] \in \mathfrak{g}, \tag{1}$$

   and verifies $\mathcal{X}(e) = 0$;
3. *$\mathcal{X}$ verifies*

$$\forall g, g' \in G \qquad \mathcal{X}_{gg'} = TL_g.\mathcal{X}_{g'} + TR_{g'}.\mathcal{X}_g \tag{2}$$

According to (1) one can associate to a given linear vector field $\mathcal{X}$ the derivation D of $\mathfrak{g}$ defined by:

$$\forall Y \in \mathfrak{g} \qquad DY = -[\mathcal{X}, Y],$$

that is $D = -\mathrm{ad}(\mathcal{X})$. The minus sign in this definition comes from the formula $[Ax, b] = -Ab$ in $\mathbb{R}^n$. It also enables to avoid a minus sign in the formula:

$$\forall Y \in \mathfrak{g}, \quad \forall t \in \mathbb{R} \qquad \varphi_t(\exp Y) = \exp(e^{tD}Y),$$

where $(\varphi_t)_{t\in\mathbb{R}}$ stands for the flow of $\mathcal{X}$.

An *affine vector field* is an element of the normalizer $\mathfrak{N}$ of $\mathfrak{g}$ in $V^\omega(G)$, that is

$$\mathfrak{N} = \mathrm{norm}_{V^\omega(G)}\mathfrak{g} = \{L \in V^\omega(G);\ \forall Y \in \mathfrak{g}, \quad [L, Y] \in \mathfrak{g}\},$$

so that an affine vector field is linear if and only if it vanishes at the identity.

It can be shown (see [2]) that an affine vector field can be uniquely decomposed into a sum $L = \mathcal{X} + Y$ where $\mathcal{X}$ is linear and Y right-invariant.

Let $\mathcal{X}$ be a linear vector field and F its translation to the tangent space at the identity, that is $F_g = TR_{g^{-1}}.\mathcal{X}_g$ for all $g \in G$. The following formulas are computed in [15].

1. The differential of F at the point g is:

$$T_gF = (D + \mathrm{ad}(F_g)) \circ TR_{g^{-1}}. \tag{3}$$

2. For $g = \exp(tY)$ one has

$$F(\exp tY) = \sum_{k=1}^{+\infty} (-1)^{k-1}\frac{t^k}{k!}\mathrm{ad}^{k-1}(Y)DY. \tag{4}$$

Let us consider now the nonlinear control system on a manifold M:

$$\dot{x} = \frac{d}{dt}x = f(x) + \sum_{j=1}^{m} u_j g_j(x),$$

where f and the g_j's are smooth vector fields, and $u = (u_1, \dots, u_m)$ belongs to $\mathbb{R}^m$. Let $\mathcal{L}$ be the Lie algebra of vector fields generated by f and the g_js. The **rank of the system** at a point x is the dimension of $\{\xi(x);\ \xi \in \mathcal{L}\}$. The system satisfies the **rank condition** if this rank is equal to the dimension of M at all points, that is if $\{\xi(x);\ \xi \in \mathcal{L}\} = T_xM$ for all $x \in M$.

The **admissible inputs** are the locally essentially bounded functions from $[0, +\infty[$ to $\mathbb{R}^m$.

3. Different types of singular systems on Lie groups

Let us consider a linear system on a Lie group G. It has the following form:

$$\dot{g} = \mathcal{X}_g + Y_g + \sum_{j=1}^{m} u_j Y^j_g, \tag{5}$$

where $\mathcal{X}$ is a linear field on G and Y and the Y^j's are right-invariant. The drift vector field is here the affine vector field $\mathcal{X} + Y$ and the system is right-invariant if $\mathcal{X} = 0$.

This system becomes a singular one if some noninvertible mapping is applied to $\dot{g}$. In the paper [12] the authors consider a derivation E of the Lie algebra $\mathfrak{g}$ of G (identified with the tangent space T_eG at the identity) and define the singular system as

$$E_g.\dot{g} = \mathcal{X}_g + Y_g + \sum_{j=1}^{m} u_j Y^j_g, \tag{6}$$

where $E_g = TR_g \circ E \circ TR_{g^{-1}}$.

Another possibility is to replace the derivation E by a noninvertible Lie algebra morphism P, that is to consider the model

$$P_g.\dot{g} = \mathcal{X}_g + Y_g + \sum_{j=1}^{m} u_j Y^j_g, \tag{7}$$

where $P_g = TR_g \circ P \circ TR_{g^{-1}}$. This model, referred to as the morphism one, is shortly analyzed in the next section.

Remarks.

1. A natural generalization of the previous models would consist of replacing E by any noninvertible linear operator A of $\mathfrak{g}$. We would obtain

$$A_g.\dot{g} = \mathcal{X}_g + Y_g + \sum_{j=1}^{m} u_j Y_g^j$$

where $A_g = TR_g \circ A \circ TR_{g^{-1}}$. This singular controlled equation is well defined but it has several drawbacks, the main one being the difficulty to analyze this model due to the lack of nice decomposition of $\mathfrak{g}$. The necessity of such a decomposition will appear clearly in the next section.

2. Another temptation is to define a singular model close to the morphism one, but through a Lie group morphism θ. More accurately, let us define a singular linear system in the following way:

$$T_g\theta.\dot{g} = \mathcal{X}_{\theta(g)} + Y_{\theta(g)} + \sum_{j=1}^{m} u_j Y^j_{\theta(g)}, \tag{8}$$

where θ is a noninvertible morphism of the Lie group G. This system has the advantage to be easy to analyze but has an important default, the lack of uniqueness of the trajectories. It is nevertheless considered in Section 7.

4. A short analysis of the morphism model

The title of this section is due to the fact that the analysis of the morphism model is not made here in its full generality, which would be beyond the scope of this paper. It is however sufficient to show the interest of the model and to enlight the difficulties that can be encountered.

4.1. Horizontal and vertical decomposition

Notation: The complexification of the Lie algebra $\mathfrak{g}$ (resp. of a subspace $\mathfrak{h}$ of $\mathfrak{g}$) is denoted $\mathfrak{g}^{\mathbb{C}}$ (resp. $\mathfrak{h}^{\mathbb{C}}$). Let

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \neq 0} \mathfrak{g}_\lambda$$

be the decomposition of $\mathfrak{g}$ according to the generalized eigenspaces of the Lie algebra morphism P (whenever the eigenvalue λ is not real $\mathfrak{g}_\lambda$ is the real part of $\mathfrak{g}^{\mathbb{C}}_\lambda \oplus \mathfrak{g}^{\mathbb{C}}_{\overline{\lambda}}$).

Lemma 1. *Let $\mathfrak{v} = \mathfrak{g}_0$ and $\mathfrak{h} = \bigoplus_{\lambda\neq 0}\mathfrak{g}_\lambda$. Then $\mathfrak{v}$ is an ideal, $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, and they verify $\mathfrak{v} \oplus \mathfrak{h} = \mathfrak{g}$.*

Proof.

1. Let us first prove that for any eigenvalues α and β of P hold the inclusion

$$[\mathfrak{g}^{\mathbb{C}}_\alpha, \mathfrak{g}^{\mathbb{C}}_\beta] \subset \mathfrak{g}^{\mathbb{C}}_{\alpha\beta},$$

with the usual convention that $\mathfrak{g}^{\mathbb{C}}_{\alpha\beta}$ is equal to $\{0\}$ if $\alpha\beta$ is not an eigenvalue of P. Let $X_1, \dots, X_l$ (resp. $Y_1, \dots, Y_k$) be a basis of $\mathfrak{g}^{\mathbb{C}}_\alpha$ (resp. of $\mathfrak{g}^{\mathbb{C}}_\beta$) such that $PX_i = \alpha X_i + \epsilon_{i-1}X_{i-1}$ where $\epsilon_{i-1} = 0{,}1$ and with the convention $X_i = 0$ if $i \leq 0$ (resp. $PY_j = \beta Y_j + \eta_{j-1}Y_{j-1}$ where $\eta_{j-1} = 0{,}1$ and with the convention $Y_j = 0$ if $J \leq 0$). Then

$$\begin{aligned}(P - \alpha\beta I)[X_i, Y_j] &= [\alpha X_i + \epsilon_{i-1}X_{i-1}, \beta Y_j + \eta_{j-1}Y_{j-1}] \\ &\quad -\alpha\beta[X_i, Y_j] \\ &= \alpha\eta_{j-1}[X_i, Y_{j-1}] + \beta\epsilon_{i-1}[X_{i-1}, Y_j] \\ &\quad +\epsilon_{i-1}\eta_{j-1}[X_{i-1}, Y_{j-1}].\end{aligned}$$

From that equality it is clear that $(P - \alpha\beta I)[X_1, Y_1] = 0$.
Let us assume that there exists a positive integer m such that $(P - \alpha\beta I)^m[X_i, Y_j] = 0$ as soon as $i + j \leq r$ where $r \geq 2$, and let i, j such that $i + j = r + 1$. Then

$$\begin{aligned}&(P - \alpha\beta I)^{m+1}[X_i, Y_j] = \\ &\quad (P - \alpha\beta I)^m(\alpha\eta_{j-1}[X_i, Y_{j-1}] + \beta\epsilon_{i-1}[X_{i-1}, Y_j] \\ &\quad + \epsilon_{i-1}\eta_{j-1}[X_{i-1}, Y_{j-1}]) = 0,\end{aligned}$$

which proves the result.

2. The previous complex inclusion implies that for any eigenvalues α and β:

$$\begin{aligned}[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] &= [(\mathfrak{g}^{\mathbb{C}}_\alpha + \mathfrak{g}^{\mathbb{C}}_{\overline{\alpha}}) \cap \mathfrak{g}, (\mathfrak{g}^{\mathbb{C}}_\beta + \mathfrak{g}^{\mathbb{C}}_{\overline{\beta}}) \cap \mathfrak{g}] \\ &\subset [\mathfrak{g}^{\mathbb{C}}_\alpha + \mathfrak{g}^{\mathbb{C}}_{\overline{\alpha}}, \mathfrak{g}^{\mathbb{C}}_\beta + \mathfrak{g}^{\mathbb{C}}_{\overline{\beta}}] \cap \mathfrak{g} \subset \mathfrak{g}_{\alpha\beta} + \mathfrak{g}_{\alpha\overline{\beta}}.\end{aligned}$$

3. The conclusion of the lemma is an immediate consequence of this inclusion. ■

According to Lemma 1 we can define two connected subgroups of G, denoted by V and H, respectively generated by the Lie algebras $\mathfrak{v}$ and $\mathfrak{h}$. The subgroup V is normal in G but not the subgroup H in general.

Some difficulties

In order to go further some decomposition of the group G with respect to V and H is needed. It is at least necessary to be able to define the quotient G/V, hence that V be a closed subgroup of G. But this condition is not guaranteed "à priori". For instance if G is the 2-dimensional torus $\mathbb{T}^2$, and P is a rank 1 linear map whose kernel has an irrational slope, then the subgroup V is dense in $\mathbb{T}^2$. Moreover the decomposition $g = hv$ where $g \in G$, $h \in H$ and $v \in V$ need not be unique in general.

However the subgroups V and H are closed, and G is equal to the semidirect product $V \rtimes H$ as soon as G is simply connected, as proved in [1] (Chap. III, §6, n^{o}6) (with notations opposite to Bourbaki's ones). In that case the decomposition $g = hv$ exists and is unique for all $g \in G$.

On the other hand, in order to decompose the system as in the next Proposition 1, it is necessary to project the linear vector field $\mathcal{X}$ to the quotients G/V and G/H, that is on H and V. Since V and H are connected, and according to [2], this is possible if and only if the Lie algebras $\mathfrak{v}$ and $\mathfrak{h}$ are invariant under the derivation D associated to the linear field $\mathcal{X}$. In that case $\mathcal{X}$ is tangent to V and H.

Consequently we make in this section the following assumptions:

1. The group G is simply connected hence isomorphic to the semidirect product $V \rtimes H$.
2. The Lie algebras $\mathfrak{h}$ and $\mathfrak{v}$ are D-invariant.

Under these assumptions we can define two projected systems, the first one (Σ_H) on H, and the second one (Σ_V) on V, by:

$$(\Sigma_H): \qquad P_h.\dot{h} = \mathcal{X}_h + Y_h^H + \sum_{j=1}^{m} u_j Y_h^{j,H}, \tag{9}$$

$$(\Sigma_V): \qquad P_v.\dot{v} = \mathcal{X}_v + Y_v^V + \sum_{j=1}^{m} u_j Y_v^{j,V}, \tag{10}$$

where if $Z \in \mathfrak{g}$ then Z^H (resp. Z^V) stands for the projection of Z onto $\mathfrak{h}$ (resp. onto $\mathfrak{v}$).

Proposition 1. *Let $u(t)$ be an admissible input, let $g(t)$ be an absolutely continuous curve in G and let $g(t) = h(t)v(t)$ be its decomposition into the horizontal part $h(t)$ and the vertical one $v(t)$.*

Then $g(t)$ is a solution of the singular system (7) if and only if the pair $(h(t), v(t))$ is solution of the system:

$$\begin{cases} P_h.\dot{h} = \mathcal{X}_h + Y_h^H + \sum_{j=1}^m u_j Y_h^{j,H} \\ P_{hvh^{-1}}.(TL_h.TR_{h^{-1}}).\dot{v} = TL_h.TR_{h^{-1}}.\mathcal{X}_v + Y^V_{hvh^{-1}} + \sum_{j=1}^m u_j Y^{j,V}_{hvh^{-1}}. \end{cases}$$

Remark. The first of these equations is (Σ_H), but since it depends on $h(t)$, the second one is different from (Σ_V) though it evolves on V. We will later assume that the elements of V and H commute. The second equation will be exactly (Σ_V) in that case.

Proof. Let $g \in G$ and let $g = hv$ be its decomposition in $h \in H$ and $v \in V$.

1. According to Formula (2) the linear vector field at g can be decomposed as:

   $$\mathcal{X}_g = \mathcal{X}_{hv} = TR_v.\mathcal{X}_h + TL_h.\mathcal{X}_v = TR_v.\mathcal{X}_h + TR_h.(TL_h.TR_{h^{-1}}.\mathcal{X}_v).$$

   Under the assumption that $\mathfrak{h}$ and $\mathfrak{v}$ are D-invariant, $TR_v.\mathcal{X}_h$ belongs to $TR_v(T_hH)$ and $TR_h.(TL_h.TR_{h^{-1}}.\mathcal{X}_v)$ belongs to $TR_h(TR_{hvh^{-1}}\mathfrak{v}) = TR_h(T_{hvh^{-1}}V)$. The second claim deserves to be proved: let $X \in \mathfrak{v}$, then

   $$TL_h.TR_{h^{-1}}.X_v = TR_{hvh^{-1}}.(\mathrm{Ad}(h)).X$$

   If $h = \exp(Z)$, with $Z \in \mathfrak{h}$, then $\mathrm{Ad}(h)X = \mathrm{Ad}(\exp(Z))X = e^{\mathrm{ad}(Z)}X$ belongs to $\mathfrak{v}$ because $\mathfrak{v}$ is an ideal. Since H is connected $\mathrm{Ad}(h).X \in \mathfrak{v}$ for all $h \in H$. Moreover hvh^{-1} belongs to the normal subgroup V.
2. Let $Y \in \mathfrak{g}$. It can be written as $Y = Y^H + Y^V$ where $Y^H \in \mathfrak{h}$ and $Y^V \in \mathfrak{v}$. Then

   $$Y_g = TR_{hv}(Y^H + Y^V) = TR_v.TR_h.Y^H + TR_h.TR_{hvh^{-1}}.Y^V = TR_v.Y_h^H + TR_h.Y^V_{hvh^{-1}}.$$

   Again the first term $TR_v.Y_h^H$ belongs to $TR_v(T_hH)$ and the second one $TR_h.Y^V_{hvh^{-1}}$ to $TR_h(TR_{hvh^{-1}}\mathfrak{v}) = TR_h(T_{hvh^{-1}}V)$.
3. Let us now consider the restricting map P. At the point g it is:

   $$P_g = TR_g.P.TR_{g^{-1}} = TR_v.TR_h.P.TR_{h^{-1}}.TR_{v^{-1}}.$$

   Consequently

   $$\begin{aligned} P_g.\dot{g} &= TR_v.TR_h.P.TR_{h^{-1}}.TR_{v^{-1}}(TR_v.\dot{h} + TL_h.\dot{v}) \\ &= TR_v.P_h.\dot{h} + TR_h.TR_{hvh^{-1}}.P.TR_{hv^{-1}h^{-1}}.(TL_h.TR_{h^{-1}})\dot{v} \\ &= TR_v.P_h.\dot{h} + TR_h.P_{hvh^{-1}}.(TL_h.TR_{h^{-1}})\dot{v}, \end{aligned}$$

   where as previously the first term $TR_v.P_h.\dot{h}$ belongs to $TR_v(T_hH)$ and the second one $TR_h.P_{hvh^{-1}}.(TL_h.TR_{h^{-1}})\dot{v}$ to $TR_h(TR_{hvh^{-1}}\mathfrak{v}) = TR_h(T_{hvh^{-1}}V)$.

Let $g(t) = h(t)v(t)$, where $h(t)$ (resp. $v(t)$) is an absolutely continuous curve in H (resp. in V). The equality $P_g.\dot{g} = \mathcal{X}_g + Y_g + \sum_{j=1}^m u_j Y_g^j$ writes (almost everywhere)

$$\begin{aligned} &TR_v.P_h.\dot{h} + TR_h.P_{hvh^{-1}}.(TL_h.TR_{h^{-1}}).\dot{v} \\ &\quad = TR_v.\mathcal{X}_h + TR_h.(TL_h.TR_{h^{-1}}.\mathcal{X}_v) + TR_v.Y_h^H + TR_h.Y^V_{hvh^{-1}} \\ &\qquad + \sum_{j=1}^m u_j(TR_v.Y_h^{j,H} + TR_h.Y^{j,V}_{hvh^{-1}}), \end{aligned}$$

and is satisfied if and only if

$$\begin{cases} TR_v.P_h.\dot{h} = TR_v.\mathcal{X}_h + TR_v.Y^H + \sum_{j=1}^m u_j TR_v.Y_h^{j,H} \\ TR_h.P_{hvh^{-1}}.(TL_h.TR_{h^{-1}}).\dot{v} = TR_h.(TL_h.TR_{h^{-1}}.\mathcal{X}_v) + TR_h.Y^V_{hvh^{-1}} \\ \qquad + \sum_{j=1}^m u_j TR_h.Y^{j,V}_{hvh^{-1}}, \end{cases}$$

that is if and only if

$$\begin{cases} P_h.\dot{h} = \mathcal{X}_h + Y_h^H + \sum_{j=1}^m u_j Y_h^{j,H} \\ P_{hvh^{-1}}.(TL_h.TR_{h^{-1}}).\dot{v} = TL_h.TR_{h^{-1}}.\mathcal{X}_v + Y^V_{hvh^{-1}} + \sum_{j=1}^m u_j Y^{j,V}_{hvh^{-1}}. \end{cases} \quad \blacksquare$$

4.2. Analysis of the horizontal system

Let us denote by Q the inverse of the restriction of P to $\mathfrak{h}$. The horizontal system (9):

$$(\Sigma_H): \qquad P_h.\dot{h} = \mathcal{X}_h + Y_h^H + \sum_{j=1}^m u_j Y_h^{j,H},$$

is equivalent to (with $Q_h = TR_h \circ Q \circ TR_{h^{-1}}$):

$$\dot{h} = Q_h.\mathcal{X}_h + Q_h.Y_h^H + \sum_{j=1}^m u_j Q_h.Y_h^{j,H}. \tag{11}$$

This is a well defined nonlinear system on H, but not a linear one in general. The vector fields

$$h \longmapsto Q_h.Y_h^{j,H} = TR_h.Q.TR_{h^{-1}}.(TR_h.Y^{j,H}) = TR_h.(Q.Y^{j,H})$$

are clearly right-invariant, but the vector field $h \longmapsto Q_h.\mathcal{X}_h$ is not linear in general.

Indeed, let us denote this vector field by $Q\mathcal{X}$. Since $Q\mathcal{X}(e) = 0$ its differential $T_e(Q\mathcal{X})$ at e is well defined and for any $Y \in \mathfrak{g}$ we have $[Y, Q\mathcal{X}](e) = T_e(Q\mathcal{X}).Y$.

Thanks to Formula (4) recalled in Section 2 a standard computation left to the reader shows that $T_e(Q\mathcal{X}) = QD$. If $Q\mathcal{X}$ was linear, then QD would be a derivation, but

$$\begin{aligned} QD[X,Y] &= [QDX, QY] + [QX, QDY] \\ &\neq [QDX, Y] + [X, QDY] \qquad \text{in general.} \end{aligned}$$

4.3. Analysis of the vertical system

The analysis of the vertical system (10) is much more difficult and here we restrict ourselves to the case where the elements of H and V commute. Then the vertical system becomes

$$(\Sigma_V): \qquad P_v.\dot{v} = \mathcal{X}_v + Y_v^V + \sum_{j=1}^m u_j Y_v^{j,V}.$$

It is analyzed under some additional assumptions. Nevertheless the computation of the solution that is exhibited below shows the difficulties related to the nonlinearity of the equations.

The restriction of P to $\mathfrak{v}$ is nilpotent, we will denote it by N in order to emphasize this fundamental feature, and we will make the following assumptions:

(i) The index of N is 2 (that is $N \neq 0$ and $N^2 = 0$).
(ii) The derivation D associated to $\mathcal{X}$ is invertible (which implies that $\mathfrak{v}$ is nilpotent).
(iii) There exists an invertible endomorphism $\widetilde{D}$ of $\mathfrak{g}$ such that $ND = \widetilde{D}N$.

Remark about the assumptions (*ii*) **and** (*iii*). *Consider the singular linear system in* $\mathbb{R}^n$*:*

$$N\dot{x} = Ax + Bu(t),$$

where N is nilpotent. In [4] *the classical algorithm for solving such a system is presented in the case where* $A = I$*. Actually it can be very easily extended to* $A \neq I$ *if A is invertible and if there exists another invertible matrix* $\widetilde{A}$ *such that* $NA = \widetilde{A}N$*. These assumptions are exactly* (*ii*) *and* (*iii*)*. Moreover the invertibility of A guarantees the uniqueness of the solutions.*

Finally our main restriction is about the index of N.

Let $v(t)$ be a solution of $N_v.\dot{v} = \mathcal{X}_v + Y_v(t)$, where $Y_v(t)$ stands for $Y_v^V + \sum_{j=1}^m u_j(t)Y_v^{j,V}$, for some admissible input $u(t)$. We can multiply both sides by N_v, and since $N_v^2 = 0$ we get $0 = N_v(\mathcal{X}_v + Y_v(t))$, or after translation:

$$0 = NF(v) + NY(t) \tag{12}$$

(Recall that $F(v) = TR_{v^{-1}}.\mathcal{X}_v$). Assuming the control $u(t)$ to be derivable, and according to Formula (3) of Section 2, we get by differentiation:

$$\begin{aligned}0 &= N(D + \mathrm{ad}(F_v))TR_{v^{-1}}.\dot{v} + N.\dot{Y}(t)\\ &= \widetilde{D}.N.TR_{v^{-1}}.\dot{v} + [N.F_v, N.TR_{v^{-1}}.\dot{v}] + N.\dot{Y}(t)\\ &= \widetilde{D}.TR_{v^{-1}}.N_v.\dot{v} + [N.F_v, TR_{v^{-1}}.N_v.\dot{v}] + N.\dot{Y}(t)\\ &= \widetilde{D}.TR_{v^{-1}}(\mathcal{X}_v + Y_v(t)) + [N.F_v, TR_{v^{-1}}(\mathcal{X}_v + Y_v(t))] + N.\dot{Y}(t)\\ &= (\widetilde{D} + \mathrm{ad}(N.F_v))(F_v + Y(t)) + N.\dot{Y}(t).\end{aligned}$$

Let

$$\Theta(v, t) = (\widetilde{D} + \mathrm{ad}(N.F_v))(F_v + Y(t)) + N.\dot{Y}(t).$$

If $v = \exp(sZ)$, then $\frac{d}{ds|_{s=0}}F(\exp(sZ)) = DZ$, and taking into account $F(\exp(0Z)) = 0$, we get:

$$\frac{d}{ds|_{s=0}}\Theta(\exp(sZ), t) = (\widetilde{D} - \mathrm{ad}(Y(t))N)(DZ).$$

Consequently the differential of Θ with respect to v at the point e is equal to:

$$\frac{\partial}{\partial v}\Theta(e, t) = (\widetilde{D} - \mathrm{ad}(Y(t))N) \circ D.$$

Since D and $\widetilde{D}$ are invertible, this differential is invertible for small $Y(t)$. In particular if $Y = 0$ then $Y(t) = \sum_{j=1}^m u_j(t)Y^{j,V}$ is small as soon as $u(t)$ is small.

Conclusion. *If* $Y(t)$ *is small enough, then there is a neighborhood of e in which* $v(t)$ *is uniquely determined by the nondifferential implicit equation:*

$$\Theta(v, t) = (\widetilde{D} + \mathrm{ad}(N.F_v))(F_v + Y(t)) + N\dot{Y}(t) = 0. \tag{13}$$

Initial conditions. The previous formulas show also that not all points v of V can be initial conditions. Actually let B be the image of $Y^V + \sum_j u_j Y^{j,V}$ and let K be the kernel of N. The equality (12), $0 = NF(v) + NY(t)$, is equivalent to $F(v) + Y(t) \in K$ and implies $F(v) \in B + K$.

Finally an initial condition v_0 should belong to $F^{-1}(B + K)$.

Conclusion. The computations done in this section show that under natural and usual assumptions the singular systems under consideration can be analyzed, that is decomposed into a horizontal system and a vertical one (Proposition 1), and that the vertical system, which is actually algebraic, can be solved. The solution is presented here for the index $N = 2$, but can certainly be extended to any value of N (with some technical difficulties). These computations can be applied to practical cases as shown by two examples of Section 8. This validates the interest of the Lie algebra morphism model (the derivation model was analyzed in [13]).

5. Extension to homogeneous spaces

The purpose of this section is to extend the previous definitions of singular systems to homogeneous spaces G/A (the set of left cosets of A), where A is a closed subgroup of G. Recall that according to the Equivalence Theorem the systems whose generated Lie algebra is finite dimensional are equivalent by diffeomorphism to linear systems on a Lie group *or homogeneous spaces*. This is due to the possible difference between the dimension of the state space and the one of the generated Lie algebra, and makes the extension to homogeneous spaces absolutely necessary.

Important remark. *When considering a homogeneous space G/A we can always assume that G is simply connected and that the tangent mapping* Π_* *to the projection* Π *from G onto G/A is a Lie algebra isomorphism, in other words that no projection of a nonzero right-invariant vector field vanishes (see* [2] *for details). This assumption is implicit in the sequel.*

From [2] we know necessary and sufficient conditions for the possibility to project the full linear system (5) onto G/A. Indeed right-invariant vector fields can always be projected and it is shown in [2] that a linear vector field $\mathcal{X}$ can be projected on G/A if and only if the subgroup A is invariant under the flow of $\mathcal{X}$. If the subgroup A is connected this condition is equivalent to the invariance of the Lie algebra $\mathfrak{a}$ of A for the derivation D associated to $\mathcal{X}$.

To this condition we should add another one about the restricting map. Of course we should assume that the Lie algebra $\mathfrak{a}$ of A is invariant under the derivation E in the derivation case (resp. for the morphism P in the morphism case), but this condition is not sufficient in general.

Let Π stand for the projection of G onto G/A. The kernel of its differential $T_g\Pi$ at the point g is equal to the left translation $T_eL_g\mathfrak{a}$ of $\mathfrak{a}$. Let E be any endomorphism of $\mathfrak{g}$ and $E_g = TR_g \circ E \circ TR_{g^{-1}}$. In order to correctly define an induced linear mapping on the tangent space to G/A at the point gA it is necessary (and sufficient) that $T_eL_g\mathfrak{a}$ be invariant under E_g. This condition is equivalent to:

$$\forall g \in G \qquad \mathfrak{a} \text{ is invariant under } \quad \mathrm{Ad}(g^{-1}) \circ E \circ \mathrm{Ad}(g) \tag{14}$$

By differentiation this condition can be transformed into a condition at the Lie algebra level, but both in the case of a derivation and in that of a morphism these conditions are rather unnatural and will not be stated here.

Let us rather consider the fact that E (resp. P) acts on $\mathfrak{g}$, viewed as the set of right-invariant vector fields, as a derivation (resp. as a morphism) by:

$$g \in G \longmapsto E_gX_g = (EX)_g \quad (\text{resp. } P_gX_g = (PX)_g) \qquad \text{for any } X \in \mathfrak{g}.$$

Let $\overline{E}$ be a smooth mapping on $T(G/A)$ that preserves the fibers, is linear on each of them, but is not invertible on at least one fiber. In other words $\overline{E}$ is a morphism of the vector bundle $T(G/A)$ that induces the identity on G/A and whose restriction to $T_{gA}G/A$ is not invertible for all gA. If $\overline{E}$ acts on $\Pi_*\mathfrak{g}$ as a derivation (resp. as a morphism), then $\Pi_*\mathfrak{g}$ is invariant and $\overline{E}$ can be lifted to a derivation or a morphism E of $\mathfrak{g}$ defined by $E = (\Pi_*)^{-1} \circ \overline{E} \circ \Pi_*$. It is clear that at each point gA, the linear mapping $\overline{E}_{gA}$ is induced by E_g.

In summary a singular linear system on a homogeneous space G/A will be defined by:

$$\overline{E}_p.\dot{p} = L(p) + \sum_{j=1}^m u_jY^j \tag{15}$$

where L is an affine vector field on G/A (the projection of $\mathcal{X} + Y$), the Y^js are projections of right-invariant vector fields, and the restricting map $\overline{E}$ is defined as above.

6. Equivalence

In this section we are interested in finding conditions for a singular control system on a connected manifold M to be equivalent by diffeomorphism to a singular control system on a Lie group or a homogeneous space. For this purpose we will use the equivalence theorem of [2], which is recalled in the appendix.

But before trying to state such conditions we should define what a nonlinear singular control system on a manifold M is. The natural way is to start from a general control-affine system on M referred to as *the full control system on M*:

$$\dot{x} = \frac{d}{dt}x = f(x) + \sum_{j=1}^{m} u_j g_j(x), \tag{16}$$

where f and the g_j's are smooth vector fields on M, and $u = (u_1, \dots, u_m)$ belongs to $\mathbb{R}^m$. In order to obtain a singular system we should apply some restricting mapping to $\frac{d}{dt}x$.

For this purpose let $\mathcal{E}$ be a morphism of the vector bundle TM that induces the identity on M and whose restriction $\mathcal{E}_x$ to T_xM is not invertible for all x. We get the model

$$\mathcal{E}_x.\dot{x} = f(x) + \sum_{j=1}^{m} u_j g_j(x). \tag{17}$$

Some necessary conditions

1. The Lie algebra generated by a system (6) or (7) is always finite dimensional, since it is included in $\mathbb{R}(\mathcal{X} + Y) \oplus \mathfrak{g}$. A necessary condition is consequently that the Lie algebra generated by f and the g_j's be finite dimensional.
2. In order to apply the equivalence theorem, it is also necessary that the full linear system satisfies the rank condition.
3. Let us consider the vector bundle morphism $\mathcal{E}$. It acts on the vector fields of M in a standard way: if X is a smooth vector field on M, then $\mathcal{E}.X$ is the vector field defined by $(\mathcal{E}.X)_x = \mathcal{E}_x.X_x$. It is therefore meaningful to check if $\mathcal{E}$ acts as a derivation, or as a morphism, on some algebra of vector fields of M.

Notations. The Lie algebra generated by f and the g_j's will be denoted by $\mathcal{L}$. We will also have to deal with the so-called zero-time ideal of $\mathcal{L}$, that is the ideal $\mathcal{L}_0$ of $\mathcal{L}$ generated by the g_j's.

Theorem 1. *The system (17) is assumed to satisfy the following conditions:*

1. *The vector fields $f, g_1, \dots, g_m$ are complete.*
2. *The Lie algebra $\mathcal{L}$ generated by f and the g_j's is finite dimensional and satisfies the rank condition, i.e. its rank is equal to $n = \dim(M)$ at all points.*
3. *The vector bundle morphism $\mathcal{E}$ acts on $\mathcal{L}_0$ if rank $(\mathcal{L}_0) = n$ (resp. on $\mathcal{L}$ if rank $(\mathcal{L}_0) = n-1$) as a derivation (resp. as a morphism).*

Then the system is equivalent by diffeomorphism to a singular system of the kind (6) (resp. of the kind (7)) on a Lie group or to the projection of such a system on a homogeneous space.

If the first assumption is not satisfied, the result holds locally.

Proof. The first two conditions imply that the rank of $\mathcal{L}_0$ is constant, equal to the rank of $\mathcal{L}$ that is $\dim(M) = n$, or to $n - 1$ (see [2]).

Let G be the simply connected Lie group whose Lie algebra $\mathfrak{g}$ is isomorphic to $\mathcal{L}_0$ if rank $(\mathcal{L}_0) = n$ and to $\mathcal{L}$ if rank $(\mathcal{L}_0) = n - 1$. According to the equivalence theorem of [2] the manifold M is diffeomorphic to a homogeneous space G/A of G and, if we denote this diffeomorphism by Φ, we have:

1. If rank $(\mathcal{L}_0) = n$ then the tangent mapping Φ_* is an isomorphism from $\mathcal{L}_0$ onto $\mathfrak{g}$ and the full system (16) is equivalent to a linear one on G/A through the diffeomorphism Φ.
2. If rank $(\mathcal{L}_0) = n - 1$ then the tangent mapping Φ_* is an isomorphism from $\mathcal{L}$ onto $\mathfrak{g}$ and the full system (16) is equivalent to an invariant one on G/A through the diffeomorphism Φ.

In both cases we can define E on $T(G/A)$ by $E = \Phi_* \circ \mathcal{E} \circ (\Phi_*)^{-1}$. Since Φ_* is an isomorphism from $\mathcal{L}_0$ or $\mathcal{L}$ onto $\Pi_*\mathfrak{g}$ it is clear that E is a derivation of $\Pi_*\mathfrak{g}$ if $\mathcal{E}$ is a derivation, and a morphism of $\Pi_*\mathfrak{g}$ if $\mathcal{E}$ is a morphism.

Let $L = \Phi_* f$ and $Y_j = \Phi_* g_j$ for $j = 1, \dots, m$. According to Section 5 the system (17) is diffeomorphic to the singular linear system on G/A:

$$E_p.\dot{p} = L(p) + \sum_{j=1}^{m} u_j Y_j. \quad \blacksquare$$

Corollary 1. *Under the same assumptions, if moreover M is simply connected and* $\dim(\mathcal{L}_0) = \dim(M)$ *(resp.* $\dim(\mathcal{L}) = \dim(M)$*), then the system (16) is diffeomorphic to a singular linear system (resp. a singular right-invariant system) on a Lie group.*

If M is not simply connected, then the same statement holds locally.

Proof. The assumptions imply $\dim(M) = \dim(G)$. The subgroup A is consequently discrete, hence reduced to the identity because of the simple connectedness of M. ■

7. The group morphism case

We consider in this section the singular system (8) defined through a Lie group morphism θ, that is:

$$T_g\theta.\dot{g} = \mathcal{X}_{\theta(g)} + Y_{\theta(g)} + \sum_{j=1}^{m} u_j Y^j_{\theta(g)} \tag{18}$$

where θ is a noninvertible morphism of the Lie group G.

As explained in the introduction these systems are rather different from the previous ones. Their analysis is very easy (but they have the drawback that neither the existence nor the uniqueness of the solutions are guaranteed) and the proof of the equivalence Theorem is quite different for them.

Since $T_g\theta.\dot{g} = \frac{d}{dt}\theta(g)$ we see that if $x(t)$ is a trajectory of (18) then $\theta(x(t))$ is a trajectory, contained in the image of θ, of the nonsingular linear system:

$$\frac{d}{dt}\dot{g} = \mathcal{X}_g + Y_g + \sum_{j=1}^{m} u_j Y^j_g. \tag{19}$$

Conversely let $y(t)$ be such a trajectory. Then any absolutely continuous curve $x(t)$ such that $\theta(x(t)) = y(t)$ is solution of (18). It is clear that for a given initial point x_0 the solution is not unique.

The defaults of this system appear here clearly.

1. The singular system (18) has solutions only if the system (19) has solutions contained in the image of θ.
2. In that case the solutions of (18) are not unique.

Remark. As well as in the other cases such a system can obviously be defined on a homogeneous space.

Let us consider again the full nonlinear system (16) defined on a manifold M. Let $\mathcal{T}$ be a smooth mapping from M to M which is not a diffeomorphism. We can define the singular system

$$T_x\mathcal{T}.\dot{x} = f(\mathcal{T}(x)) + \sum_{j=1}^{m} u_j g_j(\mathcal{T}(x)). \tag{20}$$

which is clearly a candidate for equivalence with a system defined by Eq. (18) on a Lie group, or its projection on a homogeneous space.

If the mapping $\mathcal{T}$ is equivalent to the projection on G/A of a Lie group morphism, then its tangent mapping should apply elements of $\mathcal{L}_0$ (resp. $\mathcal{L}$) to elements of the same algebra $\mathcal{L}_0$ (resp. $\mathcal{L}$) (the notations are the ones of the previous section). However $\mathcal{T}$ is assumed to not be a diffeomorphism so that it cannot be asserted that vector fields are related to the g_j's (and to f if necessary) through $\mathcal{T}$, and we have to make the following assumption.

Assumption $\mathcal{A}$. Each g_j (and also f if rank $(\mathcal{L}_0) = n-1$) is related by $\mathcal{T}$ to a vector field on M denoted by $\mathcal{T}_*g_j$ (resp. by $\mathcal{T}_*f$) that belongs to $\mathcal{L}_0$ (to $\mathcal{L}$ if rank $(\mathcal{L}_0) = n-1$).

If we denote by $\exp(tg_j)$ the flow of g_j this means that $\mathcal{T}(\exp(tg_j)(x))$ is a trajectory of $\mathcal{T}_*g_j$ for all $x \in M$.

Theorem 2. *The system (20) is assumed to satisfy the following conditions:*

1. *The vector fields $f, g_1, \ldots, g_m$ are complete.*
2. *The Lie algebra $\mathcal{L}$ generated by f and the g_j's is finite dimensional and satisfies the rank condition, i.e. its rank is equal to $n = \dim(M)$ at all points.*
3. *Assumption $\mathcal{A}$ is satisfied.*

Then the system is equivalent by diffeomorphism to a singular system of the kind (18) on a Lie group or a homogeneous space.

If the first assumption is not satisfied, the result holds locally.

Proof. The beginning of the proof is the same as the one of Theorem 1.

Then the first thing to notice is that Assumption $\mathcal{A}$ extends to all vector fields of $\mathcal{L}_0$ (resp. of $\mathcal{L}$ if rank $(\mathcal{L}_0) = n-1$), that is for all $X \in \mathcal{L}_0$ there exists a $\mathcal{T}$-related vector field $\mathcal{T}_*X$ on M and this vector field belongs to $\mathcal{L}_0$ (resp. to $\mathcal{L}$); the mapping $\mathcal{T}_*$ is moreover a Lie algebra morphism of $\mathcal{L}$ (resp. $\mathcal{L}_0$) (see [16]), Theorem (7.9)).

We also know that $\mathcal{L}_0$ (resp. $\mathcal{L}$) is isomorphic to the Lie algebra $\mathfrak{g}$ of the group G, via a Lie algebra isomorphism Ψ that verifies $\Phi_* = \Pi_* \circ \Psi$ where Π stands for the projection from G onto G/A (recall that Π_* is a Lie algebra isomorphism). Consequently we can define on $\mathfrak{g}$ the morphism $P = \Psi \circ \mathcal{T}_* \circ \Psi^{-1}$. Since G is simply connected there exists a group morphism θ on G such that $T_e\theta = P$. Notice that we do not know "à priori" if the subgroup A is θ-invariant.

Let us denote by $\Theta = \Phi \circ \mathcal{T} \circ \Phi^{-1}$ the image of $\mathcal{T}$ by Φ. It is a smooth mapping from G/A to G/A and we are left to show that for all $g \in G$:

$$\theta(g)A = \Theta(gA).$$

It is enough to show this equality for $g = \exp(Z)$ and for all $Z \in \mathfrak{g}$. Since $\mathcal{L}_0$ (resp. $\mathcal{L}$) is isomorphic to $\mathfrak{g}$ let $Z = \Psi(\zeta)$, and let $x_0 \in M$ the point that verifies $\Phi(x_0) = A$. We get:

$$\begin{aligned}\theta(\exp(Z))A &= \exp(P.Z)A = \exp(\Psi.\mathcal{T}_*\zeta)A = \exp(\Phi_*\mathcal{T}_*\zeta)(A)\\ &= \Phi(\exp(\mathcal{T}_*\zeta)(x_0)) = \Phi \circ \mathcal{T}(\exp(\zeta)(x_0))\\ &= \Theta(\Phi(\exp(\zeta)(x_0))) = \Theta(\exp(Z)A).\end{aligned}$$

This shows that Θ is the mapping induced by θ onto G/A (hence that A is θ-invariant) and proves that the system (20) is diffeomorphic to a singular system of the kind (18) on G/A. ■

8. Examples

8.1. Example 1

In this example we follow the reference [17]. Let G be the connected component of the identity element in the 2-dimensional affine group:

$$G = Aff_+(2) = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : \ x > 0 \text{ and } y \in \mathbb{R} \right\}.$$

Its Lie algebra $\mathfrak{g} = \mathfrak{aff}(2)$ is solvable with basis

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad \text{The bracket rule is} \quad [X, Y] = Y.$$

The Lie algebra is here identified with the set of left-invariant vector fields.

In the basis (X, Y) a derivation writes $D = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$ and the corresponding linear vector field $\mathcal{X}$ at $g \in G$ is given by $\mathcal{X}_g = \begin{pmatrix} 0 & a(x-1)+by \\ 0 & 0 \end{pmatrix}$.

Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. It is a $\mathfrak{g}$-morphism that decomposes $\mathfrak{g}$ into $\mathfrak{v} = \text{Span}\,\{Y\}$ and $\mathfrak{h} = \text{Span}\,\{X\}$. In order to apply the results of Section 4 it is necessary that $\mathfrak{v}$ and $\mathfrak{h}$ be D-invariant. We consequently choose $a = 0$, and for simplicity $b = 1$.

With the controlled linear field $B = \alpha X + \beta Y$ we get the singular system:

$$\Sigma_G : \qquad P_g(\dot{g}) = \mathcal{X}_g + uY_g, \qquad \text{where } g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \equiv (x, y).$$

An easy computation shows that the tangent mapping to the left translation by $g = (x, y)$ is the multiplication by x, so that $P_g = P$ for all g and the system becomes in coordinates:

$$\begin{cases} \dot{x} &= u\alpha x \\ 0 &= y + u\beta x \end{cases}$$

This is an algebraic–differential system. It is clear that $x(t)$ is (uniquely) determined by the first equation and $y(t)$ by the second one. Moreover it can be shown that the system is controllable in any time $T > 0$ as soon as $\alpha, \beta \neq 0$.

Remark. In the previous example the restriction of D to $\mathfrak{v}$ is invertible. Up to a Lie algebra automorphism the only other possibility of noninvertible $\mathfrak{g}$-morphism is $P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. This morphism is nilpotent hence $\mathfrak{v} = \mathfrak{g}$ and the restriction of D to $\mathfrak{v}$ cannot be invertible, since no derivation of $\mathfrak{g}$ is invertible. This results in the loss of uniqueness of the solutions. Indeed for general derivation and controlled vector field, we get the system:

$$\begin{cases} 0 &= u\alpha x \\ \dot{x} &= a(x-1) + by + u\beta x \end{cases}$$

The first equation implies $u\alpha = 0$, and the second one is underdetermined, whether α is either zero or not.

8.2. Example 2

We provide here an example of the computation of the vertical system as described in Section 4.3.

The Heisenberg group is herein denoted by V, and (X, Y, Z) stands for the usual basis of its Lie algebra $\mathfrak{v}$ (hence $[X, Y] = Z$). Consider the derivation D and the morphism N defined in this basis by:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ e & 0 & 2 \end{pmatrix} \quad \text{and } N = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where c and e are arbitrary constants. It is easily checked that D is an invertible derivation and N a nilpotent morphism of the Lie algebra $\mathfrak{v}$.

According to [2] the linear vector field associated to D is

$$\mathcal{X} = x\frac{\partial}{\partial x} + (cx + y)\frac{\partial}{\partial y} + (ex + 2z + \frac{1}{2}cx^2)\frac{\partial}{\partial z}.$$

Let us assume that the drift of the system is equal to $\mathcal{X}$ and that the trace on V of the controlled vector fields is reduced to $Y^{1,V} = X$. The vertical system is then:

$$N\dot{v} = \mathcal{X}_v + uX.$$

There are many choices of a derivation $\widetilde{D}$ of $\mathfrak{v}$ that satisfies $ND = \widetilde{D}N$, for instance $\widetilde{D} = D$. We choose the simplest one:

$$\widetilde{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In order to apply the results of Section 4.3 we need first compute $F_v = TR_{v^{-1}}.\mathcal{X}_v$ for all $v \in V$. The product in V being

$$(x', y', z') * (x, y, z) = (x + x', y + y', z + z' + xy'),$$

we get easily:

$$TR_{v^{-1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -x & 1 \end{pmatrix} \quad \text{and } F_v = TR_{v^{-1}}.\mathcal{X}_v = \begin{pmatrix} x \\ cx + y \\ ex + 2z - \frac{1}{2}cx^2 - xy \end{pmatrix}.$$

From this we obtain:

$$NF_v = \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}, \quad \mathrm{ad}(NF_v) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -x & 0 & 0 \end{pmatrix}, \quad \text{and } \widetilde{D} + \mathrm{ad}(NF_v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x & 0 & 2 \end{pmatrix}.$$

With the notations of Section 4.3, and for any derivable input u we have here $Y(t) = (u, 0, 0)^T$, $N\dot{Y}(t) = (0, \dot{u}, 0)^T$, and

$$\theta(v, t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x & 0 & 2 \end{pmatrix} \begin{pmatrix} x + u \\ cx + y \\ ex + 2z - \frac{1}{2}cx^2 - xy \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{u} \\ 0 \end{pmatrix} = \begin{pmatrix} x + u \\ cx + y + \dot{u} \\ -x(x + u) + 2(ex + 2z - \frac{1}{2}cx^2 - xy) \end{pmatrix},$$

so that the implicit equation $\theta(v, t) = 0$ is globally equivalent to the algebraic relations:

$$\begin{cases} x(t) &= -u(t) \\ y(t) &= cu(t) - \dot{u} \\ z(t) &= \frac{1}{2}eu(t) + \frac{1}{2}u(t)\dot{u}(t) - \frac{1}{4}cu^2(t) \end{cases}$$

Notice that $x(t)$, $y(t)$ and $z(t)$ are completely determined by these relations, and that D is here invertible.

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Appendix

Recall that an affine vector field on a Lie group G is obtained by adding a right-invariant vector field Y to a linear one $\mathcal{X}$.

Let H be a closed subgroup of G. The projection of an affine vector field $\mathcal{X} + Y$ onto the homogeneous space G/H (manifold of left cosets of H) exists if and only if the subgroup H is $\mathcal{X}$-invariant. In that case it will be referred to as *an affine vector field on the homogeneous space* G/H (see [2] for an intrinsic characterization).

The definition of a linear system is generalized in the following way: a linear system on a Lie group or a homogeneous space is defined as

$$\dot{x} = L(x) + \sum_{j=1}^{m} u_j Y_j(x)$$

where L is an affine vector field and the Y_j's are right-invariant if the state space is a Lie group, and projections of right-invariant vector fields if the state space is a homogeneous space.

Let us consider the following smooth system, defined on a connected manifold M:

$$(S) \qquad \dot{x} = f(x) + \sum_{j=1}^{m} u_j g_j(x)$$

Equivalence Theorem ([2]). *We assume the family* $\{f, g_1, \ldots, g_m\}$ *to be transitive. Then the system* (S) *is diffeomorphic to a linear system on a Lie group or a homogeneous space if and only if the vector fields* $f, g_1, \ldots, g_m$ *are complete and generate a finite dimensional Lie algebra.*

More accurately, let G *(resp.* G_0*) be the connected and simply connected Lie group whose Lie algebra is* $\mathcal{L}$ *(resp.* $\mathcal{L}_0$*). Under the previous conditions the rank of* $\mathcal{L}_0$ *is constant, equal to* $\dim(M)$ *or* $\dim(M) - 1$*, and:*

(i) *if rank* $(\mathcal{L}_0) = \dim(M)$*, in particular if there exists one point* $p_0 \in M$ *such that* $f(p_0) = 0$*, then* S *is diffeomorphic to a linear system on a homogeneous space* G_0/H *of* G_0*;*
(ii) *if rank* $(\mathcal{L}_0) = \dim(M) - 1$*, then* S *is diffeomorphic to an invariant system on a homogeneous space* G/H *of* G*.*

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