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Review

Geometric Structures Generated by the Same Dynamics. Recent Results and Challenges

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Abstract: The main contribution of this review is to show some relevant relationships between three geometric structures on a connected Lie group G, generated by the same dynamics. Namely, Linear Control Systems, Almost Riemannian Structures, and Degenerate Dynamical Systems. These notions are generated by two ordinary differential equations on G: linear and invariant vector fields. A linear vector field on G is determined by its flow, a 1-parameter group of $Aut(G)$, the Lie group of G-automorphisms. An invariant vector field is just an element of the Lie algebra g of G. The Jouan Equivalence Theorem and the Pontryagin Maximum Principal are instrumental in this setup, allowing the extension of results from Lie groups to arbitrary manifolds for the same kind of structures which satisfy the Lie algebra finitude condition. For each structure, we present the first given examples; these examples generate the systems in the plane. Next, we introduce a general definition for these geometric structures on Euclidean spaces and G. We describe recent results of the theory. As an additional contribution, we conclude by formulating a list of open problems and challenges on these geometric structures. Since the involved dynamic comes from algebraic structures on Lie groups, symmetries are present throughout the paper.

Keywords: linear control systems; almost Riemannian structures; degenerate dynamical systems; singular locus

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1. Introduction

This review aims to give some information about three geometric structures on Lie groups that essentially depend on the same kind of dynamics. Namely, Linear Control Systems (LCS), Almost Riemannian Structures (ARS), and Degenerate Dynamical Systems (DDS). These notions are strongly related through two particular ordinary differential equations on G: linear and invariant vector fields. A linear vector field on G is determined by its flow, a 1-parameter group of $Aut(G)$, the Lie group of G-automorphisms. An invariant vector field is just an element of the Lie algebra $\mathfrak{g}$ of G.

Our contribution is two-fold. Firstly, we show relevant relationships between these geometric structures on Lie groups in more general setups. For instance, the Jouan Equivalence Theorem allows us to classify on an arbitrary differential manifold M: Nonlinear Control Systems, Almost Riemannian Structures, and Degenerate Dynamical Systems, which satisfy the finitude condition of the Lie algebra $\mathcal{L}$ generated by their dynamic. Therefore, from the corresponding linear structure on a determined Lie group G, it is possible to give information about the same type of structures on M when $\mathcal{L}$ is finite dimensional, determining G. This is one of the main reasons to develop linear structures on groups. Secondly, we formulate a list of open problems and challenges on these geometric structures.

From a linear control system point of view, the linear dynamics represents the drift to be controlled, and the invariant ones are the control vectors. Thus, the trajectories of the time-dependent vector fields induced by the family of the admissible control function $\mathcal{U}$ play the role of determining integral curves and strategies to move an initial condition to another desired one. So, the challenges here are first to characterize the controllability property of the system. Secondly, to analyze the existence, uniqueness and topological and algebraic properties of the control sets, i.e., regions of the manifold where controllability holds inside of its interior. Finally, if it is possible to reach a desired state, then: how to compute and optimal synthesis which reach the target at minimum time? or with minimum energy, etc. In this context, the Pontryagin Maximum Principle is instrumental [1].

An almost Riemannian structure can be seen, at least locally, as an orthonormal reference frame that degenerates in a singular set called the locus, where the frame lost dimension, see [2,3]. In our setup, there is a natural ARS called a simple almost Riemannian structure defined by linear and invariant vector fields such as a LCS. In [4], the authors establish the Hamiltonian equations of a simple ARS on G. This case is quite favorable because the co-tangent bundle T^*G of G is trivial. It allows determination and decomposition of the Hamiltonian equations generated by the system on the identity element of the manifold $T^*G \cong \mathfrak{g}^* \times G$. Here, $\mathfrak{g}^*$ is the dual space of the Lie algebra $\mathfrak{g}$. Thus, to construct the complete set of optimal arclength geodesics explicitly, the so-called optimal synthesis of a given ARS on a Lie group is a crucial issue to challenge.

A degenerate dynamical system is defined by a symplectic structure which becomes singular in a subset also called the locus. In this set up, the general problem is to understand the evolution of the dynamic near to this singular set. As a matter of fact, the classical Poincaré classification takes care of isolated singularities of the Hamiltonian. In this new context, the singularities comes from the degeneracy of the symplectic form, normally on submanifolds of co-dimension 1, which are the barriers, the walls. If some trajectories reach the locus, what kind of dynamic behavior could we expect inside the singular set, or when the trajectory leaves the locus? If the trajectory remains in the locus forever, it means that the “trajectory” freezes some coordinates. Therefore, the main challenge here is to extend the dynamic classification of Poincaré for regular systems to this particular degenerate situation. On the other hand, intend to face Arnold’s challenge on degenerate symplectic structures [5]. Following a suggestion of a reviewer, we also include the references [6,7].

A full classification of these three geometric structures in the low-dimensional case would answer this question and provide a series of examples of such structures, which is usually a starting point for more general results. Since the involved dynamics come from algebraic structures on Lie groups, many symmetries are present throughout the paper.

The structure of the review is as follows. In Section 2, we introduce the geometric structures on Euclidean spaces. We start with examples on the plane that motivated the study of these three geometric structures. Precisely, the time-optimal problem of a vehicle moving through a line appears in [1]. The Grushin plane [8] is the first known example of an almost Riemannian structure. Finally, we introduce a degenerate dynamic system example which appears in [9]. After that, we state these structures on $\mathbb{R}^n$. Section 3 explains the notion of linear vector fields on Lie groups and their associated derivation of the Lie algebra $\mathfrak{g}$. Some relationships involving the exponential maps are given. As a concrete example, we compute all the linear and invariant vector fields on solvable Lie groups of dimensions two and three. We do the same on the simple connected Heisenberg Lie group of dimension three. Section 4 contains the definitions of LCS, ARS, and DDS on the Lie group G. In Section 5, we establish the Pontryagin Maximum Principle on a differential manifold and the Hamiltonian function associated to an almost Riemannian structure appears in [4]. As a particular case, we obtain the Hamiltonian equations for a LCS on G. On the other hand, we also include the Jouan Equivalence Theorem [10], which is one of the reasons why it is relevant to develop LCS on Lie groups and homogeneous spaces, and ARS. In Section 6 we mention some recent results on any of the geometric structures. Finally, Section 7 includes challenges and open problems for LCS, ARS and DDS.

2. The Geometric Structures on Euclidean Spaces

In this section, we start by presenting three examples on the plane $\mathbb{R}^2$, which give rise to the geometrical structures we would like to show. From that, we define these structures on the n-dimensional Euclidean space $\mathbb{R}^n$.

2.1. A Pontryagin Example of a Linear Control System on the Plane

Let us consider a train moving through a line without friction. An optimization problem arises: stop the train in a given station in the minimum time [1].

We denote with $x(t)$ the distance from the train to the origin (the station). Therefore, $\dot{x}(t) = y(t)$ denotes the velocity, and the acceleration is $\dot{y}(t) = u(t)$. Here, u belongs to $\mathcal{U}$, the family of the local integrable measurable functions $u : [0, T_u] \to \Omega = [-1, 1] \subset \mathbb{R}$. In addition, the boundary of Ω represents the maximum and minimum normalized acceleration.

In a matrix form, this model reads as follows,

$$\Sigma_{\mathbb{R}^2} : \dot{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + u(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}, (x(t), y(t)) \in \mathbb{R}^2, t \in \mathbb{R}.$$

Any control $u \in \mathcal{U}$ generates an ordinary differential equation, and the optimization problem here is: given an initial condition $(x_0, y_0) \in \mathbb{R}^2$ find $u \in \mathcal{U}$ such that the integral curve of the system starting on (x_0, y_0) and with control u reaches the origin at minimum time.

The dynamic of the system comes from the linear vector field $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the invariant vector field $\frac{\partial}{\partial y}$ generated by $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\mathcal{U}$. By computing the Lie algebra of $\Sigma_{\mathbb{R}^2}$ through the Lie bracket $[A, b] = -Ab$, we obtain

$$Span_{\mathcal{LA}}\{A, b\}(x, y) = T_{(x,y)}\mathbb{R}^2 \cong T_{(0,0)}\mathbb{R}^2 = \mathbb{R}^2 = Span\{b, Ab\}. \tag{1}$$

2.2. Linear Control Systems on Euclidean Spaces

In a Euclidean general frame, a linear control system $\Sigma_{\mathbb{R}^n}$ on $\mathbb{R}^n$ is written as

$$\dot{x}(t) = Ax(t) + Bu(t) = Ax(t) + \sum_{j=1}^{m} u_j b_j, u \in \mathcal{U}.$$

The vector fields $b_j, j = 1, \ldots, m$ are the column vectors of the "cost matrix" B.

By definition [11], the system $\Sigma_{\mathbb{R}^n}$ depends on two classes of dynamics. The drift of the system, i.e., the linear differential equation $\dot{x}(t) = Ax(t)$, to be controlled. Moreover, the invariant control vectors b_j on $\mathbb{R}^n$.

The flow of the vector field induced by the matrix A applied to the starting point x reads as $x(t) = e^{tA}x_.$. And, the analytical solution $\phi(x, u, t)$ of $\Sigma_{\mathbb{R}^n}$ with initial condition $x \in \mathbb{R}^n$ and control $u \in \mathcal{U}$, is given by

$$\phi(x, u, t) = e^{tA}\left(x + \int_0^t e^{-\tau A}Bu(\tau)d\tau\right).$$

2.3. An almost Riemannian Structure Example. The Grushin Plane

Consider a 2-dimensional manifold M and X, Y two linear independent vector fields on M. The frame $\{X, Y\}$ induced a well defined Riemannian metric. On the other hand, assume

$$\dim\{X, Y\}(x) \leq 2, Span_{\mathcal{LA}}\{X, Y\}(x) = 2, x \in M. \tag{2}$$

According to the Chow–Rashevskii Theorem [12,13], still there exists a locally defined metric on M, but with singularities on the set called loci where X, Y are linear dependent.

This situation is shown by the following example, which appears in the study of *a certain class of hypoelliptic operators* [8].

The Grushin plane is the Abelian group $\mathbb{R}^2$ equipped with an almost Riemannian metric determined by the vector fields $X = \frac{\partial}{\partial x}$ and $Y = x\frac{\partial}{\partial y}$. Precisely, it is a Riemannian manifold except on the particular subset

$$\mathcal{Z} = \{(0, y) : y \in \mathbb{R}\} \tag{3}$$

called the (singular) locus, where $Y = 0$. It is worth mentioning that the Lie bracket $[X, Y] = \frac{\partial}{\partial y}$. Therefore, for any $(x, y) \in \mathbb{R}^2$

$$Span_{\mathcal{L}A}\{X, Y\}(x, y) \cong \mathbb{R}^2. \tag{4}$$

From [2], the associated metric and the curvature are given by

$$g = dx^2 + \frac{1}{x^2}dy^2, k = -\frac{2}{x^2}. \tag{5}$$

As for the LCS example above, we notice that $x\frac{\partial}{\partial y}$ is a linear vector field determined by the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $\frac{\partial}{\partial x} = (1, 0)$ which is invariant by translation on the plane. Thus, both systems depend on the same kind of dynamics.

2.4. Almost Riemannian Structure on Euclidean Spaces

On the Euclidean space of arbitrary dimension n, an almost Riemannian structure can be defined as follows.

Definition 1. *A simply Almost Riemannian Structure (ARS) on $\mathbb{R}^n$ is determined by the family*

$$\mathcal{F} = \{A, b_1, \ldots, b_m\}, \tag{6}$$

1. $A : \mathbb{R}^n \to \mathbb{R}^n$ *is a linear map*
2. $b_1, \ldots, b_m \in \mathbb{R}^n$
3. $\text{rank}(\mathcal{F}(x)) = n$, *for x in an open and dense subset of $\mathbb{R}^n$*
4. $T_x\mathbb{R}^n = \text{rank}(B, AB, \ldots, A^{n-1}B)$, *for any $x \in \mathbb{R}^n$.*

   Here, the column vectors of B are the invariant vector fields b_j, $j = 1, \ldots, m$.

2.5. A Degenerate Dynamic System Example

Similar dynamic behavior to the LCS in 2.0.2 and the ARS in 2.0.3 arises when considering the degenerate dynamical system model appearing in [9]. Let us consider the real matrix $Q(t) = \begin{pmatrix} 0 & x(t) \\ -x(t) & 0 \end{pmatrix}$ of order 2, and the associated singular differential equations

$$Q(t)\begin{pmatrix} \dot{x(t)} \\ y(t) \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, pq \neq 0. \tag{7}$$

If for any real time t, $x(t) \neq 0$ we obtain

$$\begin{pmatrix} \dot{x(t)} \\ y(t) \end{pmatrix} = \frac{1}{x(t)}\begin{pmatrix} -q \\ p \end{pmatrix} \tag{8}$$

The singular set $x = 0$ acts a barrier for the dynamics. However, geodesics can cross this barrier without any singularities.

2.6. Degenerate Dynamical Systems on Euclidean Spaces

On the Euclidean space $\mathbb{R}^n$, a degenerate structure can be defined through skew-symmetric vector fields $\mathcal{A}$ on the tangent bundle. Precisely, a family $\{\mathcal{A}(x) : x \in \mathbb{R}^n\}$ of skew-symmetric linear maps such that

$$\mathcal{A}(x) : T_x\mathbb{R}^n \to T_x\mathbb{R}^n, < \mathcal{A}(x)v, w >=< v, \mathcal{A}(x), w >, \tag{9}$$

and there exists an open and dense U in $\mathbb{R}^n$ with

$$\det \mathcal{A}(x) \neq 0, x \in U. \tag{10}$$

Let $b \in \mathbb{R}^n$. A Degenerate Dynamical System on Euclidean spaces reads

$$\mathcal{A}(x)(\dot{x}) = b, \tag{11}$$

which is determined by an algebraic-dynamical equation.

The examples in this section have several common issues. At first place, the dynamic of these structures is determined by linear and invariant vector fields. Secondly, by computing the corresponding Lie brackets, the Lie Algebra Rank Condition $(LARC)$ is satisfied, i.e.,

$$Span_{\mathcal{L}A}\{X, Y\}(x, y) \cong \mathbb{R}^2. \tag{12}$$

In any case, the existence of a metric is guaranteed by the Chow–Rashevskii Theorem [13]. To determine the geodesics, the Pontryagin Maximum Principle is instrumental. Because of that, in Section 5 we establish the Hamiltonian functions for an ARS on G, and the Pontryagin Maximum Principle for a time optimal problem in LCS [14]. In order to compute geodesics for an ARS on G, we note that reference [4] contains the Hamiltonian equation, including the normal and abnormal cases.

In the following, we define the structures on a connected Lie group G of arbitrary dimension. We establish some relevant results and list some challenges and open problems for research.

3. From Euclidean Spaces to Lie Groups

For the Lie theory we suggest to the readers the references [15–17]. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ considered as the set of left-invariant vector fields on G. In this section, we first explain how to extend the notion of a linear differential equation from $G = \mathbb{R}^n$ to arbitrary group G. After that, we list some of it properties, especially those related with the exponential map

$$\exp_G : \mathfrak{g} \to G, Y \in \mathfrak{g} \to Y_1(e). \tag{13}$$

where $\{Y_t : t \in \mathbb{R}\}$ is the flow associated to the left invariant vector field induced by the vector $Y \in T_eG$, and e denotes the identity element of G.

A particular case of this extension was first provided by [18]. In [19], the authors introduced a general definition involving the notion of the normalizer which is out of the scope of this review. Therefore, we prefer to give a direct generalization based on two different but dependent facts from the distinguished dynamics on Euclidean spaces. Let A be a real matrix of order n and $b \in \mathbb{R}^n$.

First, the flow induced by A satisfied

$$\left\{e^{tA} : t \in \mathbb{R}\right\} \subset Aut^+(n, \mathbb{R}). \tag{14}$$

Here, $Aut^+(n, \mathbb{R})$ denotes the connected component containing the identity element of $Aut(\mathbb{R}^n)$, the real invertible matrices of order n. On the other hand, the Lie bracket $[A, b] = -Ab \in \mathbb{R}^n$. It turns out that $[A,] : \mathfrak{r}^n \to \mathfrak{r}^n$ leaves the Lie algebra invariant $\mathfrak{r}^n \cong T_0\mathbb{R}^n$ of $\mathbb{R}^n$.

For the following definition and results, we follow references [15,19].

Definition 2. *A linear vector field $\mathcal{X}$ on G is determined by its flow $\{\mathcal{X}_t : t \in \mathbb{R}\}$ which is a 1-parameter subgroup of $Aut(G)$, the Lie group of G-automorphism.*

Even though in general $\mathcal{X}$ is a nonlinear vector field, we keep the linear name based on the picture coming from the following equivalence [20],

$$\mathcal{X}(gh) = (dL_g)_h \mathcal{X}(h) + (dR_h)_g \mathcal{X}(g), \quad \text{for all} \quad g, h \in G. \tag{15}$$

Recall that a derivation on a Lie algebra $(\mathfrak{g}, [,])$ is a linear map $\mathcal{D} : \mathfrak{g} \to \mathfrak{g}$ which satisfied the Leibnitz rule,

$$\mathcal{D}[X, Y] = [\mathcal{D}X, Y] + [X, \mathcal{D}Y], X, Y \in \mathfrak{g}. \tag{16}$$

We denote by $\partial\mathfrak{g}$ the Lie algebra of $\mathfrak{g}$-derivations.

From the Jacobi identity property of the Lie algebra, we can associate to each $\mathcal{X}$ an element $\mathcal{D} \in \partial\mathfrak{g}$ determined by the formula

$$\mathcal{D}Y = -[\mathcal{X}, Y](e), \text{ for all } Y \in \mathfrak{g}.$$

For a real time t the relationship between $\mathcal{X}_t$ and $\mathcal{D}$ is given through

$$(d\mathcal{X}_t)_e = \mathrm{e}^{t\mathcal{D}}, \text{ for any } t \in \mathbb{R}. \tag{17}$$

Furthermore, from the commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{(d\mathcal{X}_t)_e} & \mathfrak{g} \\ \exp\downarrow & & \downarrow\exp \\ G & \xrightarrow[\mathcal{X}_t]{} & G \end{array}$$

we obtain the formula

$$\mathcal{X}_t(\exp Y) = \exp(\mathrm{e}^{t\mathcal{D}} Y), \text{ for all } t \in \mathbb{R}, Y \in \mathfrak{g},$$

which allows computing of $\mathcal{X}_t(\exp Y)$. In fact, since the group is connected, any $g \in G$ can be expressed as a finite product of exponentials, and $\mathcal{X}_t$ respects the algebraic structure of G.

Reciprocally, if the group is simply connected, any derivation $\mathcal{D} \in \partial\mathfrak{g}$ has an associated linear vector field $\mathcal{X} = \mathcal{X}^D$ through the same formula above. For connected Lie groups, the same is true when $\mathcal{D} \in \mathfrak{aut}(G)$, the Lie algebra of $Aut(G)$ (see [19]). More precisely,

$$\mathfrak{aut}(G) \subsetneq \partial\mathfrak{g} \text{ and } \mathfrak{aut}(G) = \partial\mathfrak{g} \Leftrightarrow G \text{ is connected and simply connected.}$$

A particular linear dynamics comes from inner automorphisms. Consider an element $Z \in \mathfrak{g}$. Since Z is an invariant vector field, the solution starting on $g \in G$ is obtained by left translation of the solution through the identity element.

In other words,

$$Z_t(g) = \exp_G(tZ)g, g \in G$$

defines by conjugation a 1-parameter group of inner automorphisms on G by

$$\mathcal{X}_t(x) = Z_{-t}(e)\, g\, Z_t(e), g \in G.$$

Thus, $\mathcal{X}_t \in Aut(G)$ for any $t \in \mathbb{R}$. In this case, the associated derivation $\mathcal{D} : \mathfrak{g} \to \mathfrak{g}$ reads $\mathcal{D}(Y) = -[Z, Y]$, Y in $\mathfrak{g}$.

To better understand the algebraic objects introduced here, we finish this section with examples on low-dimensional groups. The Affin group on $\mathbb{R}^2$, a solvable 3-dimensional group, and the classical Heisenberg Lie group are teh examples given. In any case, we explicitly show a basis of the Lie algebra, the Lie algebra of derivations, the associated linear vector fields, and their corresponding flows.

Example 1. *The solvable* 2*-dimensional group.*

The 2-dimensional affine group is the semi-direct product $G = \mathbb{R} \times_\rho \mathbb{R}$, with Lie algebra the semi-direct product $\mathfrak{g} = \mathbb{R} \times_\theta \mathbb{R}$, see [21] for details. Here, the action on G and $\mathfrak{g}$ are given by $\rho_x = e^x$ and $\theta = Id_{\mathbb{R}}$, respectively.

The product in G reads

$$(x_1, y_1) * (x_2, y_2) = (x_1 + x_2, y_1 + e^{x_1} y_2).$$

Any $(\alpha, \beta) \in \mathbb{R}^2$ determines a left-invariant vector field as follows

$$Y(x, y) = (\alpha, e^x \beta).$$

Furthermore, the bracket between two elements of $\mathbb{R} \times_\theta \mathbb{R}$ is

$$[(\alpha_1, \beta_1), (\alpha_2, \beta_2)] = (0, \alpha_1\beta_2 - \alpha_2\beta_1).$$

Let Y^1 and Y^2 be the canonical basis of $\mathfrak{g}$. From the previous formula we obtain the rule $\left[Y^1, Y^2\right] = Y^2$. It follows that the Lie algebra is solvable [17].

The exponential map is explicitly given by,

$$\exp_G(a, b) = \begin{cases} (0, b), & \text{if } \ a = 0 \\ \left(a, \frac{1}{a}(e^a - 1)b\right) & \text{if } \ a \neq 0 \end{cases},$$

The Lie algebra of $\mathfrak{g}$-derivations is 2-dimensional. Precisely,

$$\partial\mathfrak{g} = \left\{ \mathcal{D} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} : a, b \in \mathbb{R}. \right\} \tag{18}$$

Since G is connected and simply connected, any pair $(a, b) \in \mathbb{R}^2$ induces a well defined linear vector field on G which reads as

$$\mathcal{X}(x, y) = (0, by + (e^x - 1)a). \tag{19}$$

The associated 1-parameter group of automorphisms defining its flow is given by the formula

$$\mathcal{X}_t(x, y) = \begin{cases} (x, y + t(e^x - 1)a) & \text{if } \ b = 0, \\ \left(x, e^{tb} y + \frac{1}{b}(e^{tb} - 1)(e^x - 1)a\right) & \text{if } \ b \neq 0. \end{cases}$$

For the next two examples, we consider connected and simple connected groups of three dimensions.

Example 2. *A non-nilpotent* 3*-dimensional solvable group.*

Let us consider the solvable Lie algebra $\mathfrak{g}$ as the semi-direct product $\mathbb{R} \otimes_\theta \mathbb{R}^2$ with the bracket rule

$$[(z_1, v_1), (z_2, v_2)] = (0, z_1\theta v_2 - z_2\theta v_1) \in \mathfrak{g}.$$

Here, $\theta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, see [22] for details.

By considering the canonical basis of $\mathfrak{g}$, we obtain

$$\mathfrak{g} = Span\Big\{Y^1 = (1,0,0), Y^2 = (0,1,0), Y^3 = (0,0,1)\Big\}.$$

Therefore, $\left[Y^1, Y^2\right] = 0$, and $\left[Y^1, Y^3\right] = Y^3$. The connected and simply connected Lie group G with Lie algebra $\mathfrak{g}$ is given by the semi-direct product $G = \mathbb{R} \otimes_\rho \mathbb{R}^2$ via $\rho(t) = e^{t\theta}$. Recall that

$$e^{t\theta} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\theta^n, \theta^0 = Id, t \in \mathbb{R}.$$

A left-invariant vector field is written as $Y = (a, w) \in \mathfrak{g}$ with flow

$$Y_t = (a, e^{t\theta} w), t \in \mathbb{R} \tag{20}$$

On the other hand, a general shape of a linear vector field on G reads as

$$\mathcal{X}(t, v) = (0, \mathcal{D}^* v + \Lambda_t \xi). \tag{21}$$

In this context, $\mathcal{D}^*$ is defined through the formula $\mathcal{D}(0, v) = (0, \mathcal{D}^* v)$, where $(0, \xi) = \mathcal{D}(1, 0)$, and $\Lambda_t = \begin{pmatrix} t & 0 \\ 0 & e^t - 1 \end{pmatrix}$.

Example 3. *Let us consider the* 3*-dimensional connected and simply connected Heisenberg Lie group G homemorphic to $\mathbb{R}^3$,*

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}; x, y, z \in \mathbb{R} \right\}, \tag{22}$$

with Lie algebra

$$\mathfrak{g} = Span\Big\{Y^1, Y^2, Y^3\Big\}, \tag{23}$$

where the only non null brackets are $\left[Y^1, Y^2\right] = Y^3$. The Lie algebra $\partial\mathfrak{g}$ has dimension six and is given by

$$\partial\mathfrak{g} = \left\{ \mathcal{D} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & a+d \end{pmatrix} : a, b, c, d, e, f \in \mathbb{R} \right\}.$$

Any invariant vector field is a linear combination of the basis of $\mathfrak{g}$. Furthermore, according to [23], the linear vector field associated with a derivation $\mathcal{D} \in \partial\mathfrak{g}$ reads as follows

$$\mathcal{X}(x, y, z) = (ax + dy)\frac{\partial}{\partial x} + (bx + ey)\frac{\partial}{\partial y} + (\frac{b}{2}x^2 + \frac{d}{2}y^2 + cx + fy + (a+e)z)\frac{\partial}{\partial z}. \tag{24}$$

4. The Corresponding Structures on Lie Groups

Here, we give a general definition of these three structures on a Lie group G.

4.1. Linear Control System

A linear control system Σ_G on G is determined by the family,

$$\Sigma_G : \dot{g(t)} = \mathcal{X}(g(t)) + \sum_{j=1}^{m} u_j(t) Y^j(g(t)), g(t) \in G, t \in \mathbb{R}, u \in \mathcal{U},$$

of ordinary differential equations parametrized by the class $\mathcal{U} = L^1_{loc}(\Omega)$, of admissible controls, i.e., by locally integrable functions $u : [0, T_u] \to \Omega \subset \mathbb{R}^m$. The set Ω is closed and $0 \in int(\Omega)$. The drift $\mathcal{X}$ is a linear vector field. Furthermore, for any j, the control vector $Y^j \in \mathfrak{g}$ is considered as a left-invariant vector field. If $\Omega = \mathbb{R}^m$, the system is called unrestricted. Otherwise, Σ_G is restricted [19].

We assume Σ_G satisfies the Lie algebra rank condition (*LARC*), which means that for any $g \in G$,

$$Span_{\mathcal{LA}}\left\{\mathcal{X}, Y^1, \dots, Y^m\right\}(g) = T_g G. \tag{25}$$

Denote by $\varphi(g, u, t)$ the solution of Σ_G associated to the control u with initial condition g at the time t. It turns out [24]

$$\varphi(g, u, t) = \mathcal{X}_t(g)\varphi(e, u, t). \tag{26}$$

Thus, to compute the system's solution through an initial condition g, we need to translate the solution through the identity element by the flow of the linear vector field acting on g. Just observe the symmetry with the solution of a classical linear system on Euclidean spaces

$$\phi(x, u, t) = e^{tA}\left(x + \int_0^t e^{-\tau A} Bu(\tau) d\tau\right). \tag{27}$$

4.2. Almost Riemannian Structures

Let $k \in \mathbb{N}$, an almost Riemannian structure ARS of order k on a Lie group G is determined by the data

$$\mathcal{F} = \{\mathcal{X}_1, \dots, \mathcal{X}_k, Y_1, \dots, Y_{n-k}\} : \tag{28}$$

1. $\mathcal{X}_1, \dots, \mathcal{X}_k$ are linear vector fields
2. $Y_1, \dots, Y_{n-k}$ are left invariant vector fields
3. $\text{rank}(\mathcal{F}(g)) = \dim(G)$ over an open and dense subset of G
4. $T_g G = \{X(g) : \ X \in \mathcal{F}_{\mathcal{LA}}\}$, where $\mathcal{F}_{\mathcal{LA}}$ is the $\mathcal{F}$-generated Lie algebra.

The ARS is called simply if $k = 1$ [4]. In this set up, the locus is given by

$$\mathcal{Z} = \{g \in G : \text{rank}(\mathcal{F}(g)) < n\}. \tag{29}$$

4.3. Degenerate Dynamical Systems

Let (M, g, ϖ) be a symplectic Riemannian manifold. An almost complex structure on M is a smooth field J of skew-symmetric structures on the tangent bundle as follows. The family of linear maps

$$x \in M \to J_x : T_x M \to T_x M, \tag{30}$$

satisfies $J_x^2 = -id_{T_x M}$.

We say that J is compatible with ϖ if the map

$$x \in M \to g_x^J : T_x M \times T_x M \to \mathbb{R}, g_x^J(p, q) = \varpi_x(p, J_x q) \tag{31}$$

is a Riemannian metric on M. It turns out that any symplectic manifold admits a compatible almost complex structure, but with a metric J_x different from the original one, XXX.

Let $f : M \to \mathbb{R}$ a smooth function $f \in \mathcal{C}^\infty(M)$ and denote with

$$\mathcal{Z} = \{x \in M : f(x) = 0\}, Crit(f) \tag{32}$$

the zeros and the critical points of f, respectively. A degenerate dynamical system is determined by an algebraic-dynamical equation

$$f(x) J_x(\dot{x}) = X_h(x), \tag{33}$$

where X_h is the Hamiltonian vector field associated with a smooth function $h \in \mathcal{C}^\infty(M)$ on M. In particular, this definition works when $M = G$ is a connected Lie group.

On the manifold $M - \mathcal{Z}$, the degenerate system is equivalent to the regular

$$\dot{x} = -\frac{1}{f(x)} grad_{g^J}(h(x)). \tag{34}$$

Assume f is a Morse–Bott type function on M, and denote with C a connected component of $\mathcal{Z}$. It turns out that

1. C is a M-submanifold of co-dimension one or;
2. C is a connected component of $Crit(f)$.

In the first case, C acts as a wall for the dynamic. In the second case, the dimension is arbitrary. Moreover, C can be seen as phase flow where the Hamiltonian gradient diverges as it approaches the locus. Recall that the gradient is computed concerning the Riemannian metric g^J.

5. Time Optimal Pontryagin Maximum Principle

In this setup, the space state M is a n-dimensional manifold and Σ_M is a control system on M, determined by the family of differential equations

$$\Sigma_M : x(\dot{t}) = f(x(t)) + \sum_{j=1}^{m} u_j(t) g^j(x(t)), x(t) \in M, t \in \mathbb{R}, u \in \mathcal{U},$$

where $f, g^1, \dots, g^m$ are arbitrary vector fields defined on M, and $\mathcal{U}$ as before.

The Hamiltonian function associated to the system reads as

$$\mathcal{H}(\lambda_x, x, u) = \langle \lambda_x, f(x) + \sum_{j=1}^{m} u_j Y^j(x) \rangle \quad \text{with} \ \ \lambda_x \in T_x^* M.$$

Here, $T_x^* M$ denotes the dual of the vector space $T_x M$, the tangent space of M at the state x. Moreover, $\lambda_x : T_x M \to \mathbb{R}$ is a linear transformation.

The symplectic structure of $T^* M$ comes from a canonical non-degenerate 2-differential form σ [15]. It turns out that for any admissible control $u \in \mathcal{U}$ the Hamiltonian function $\mathcal{H}_u : T^* M \to \mathbb{R}$ determines a well defined vector field $\vec{\mathcal{H}}_u$ on $T^* M$ through the identity

$$\sigma_\lambda(\cdot, \vec{\mathcal{H}}_u) = d_\lambda \mathcal{H}_u.$$

By considering a canonical locally coordinates (q_i, x_i) on $T_x^* M$, we obtain

$$d\mathcal{H}_u = \sum_{i=1}^{n} (\frac{\partial \mathcal{H}_u}{\partial q_i} dq_i + \frac{\partial \mathcal{H}_u}{\partial x_i} dx_i),$$

$$\dot{\lambda} = (\dot{q}_i, \dot{x}_i) = \vec{\mathcal{H}}_u = \sum_{i=1}^{n} (-\frac{\partial \mathcal{H}_u}{\partial x_i} \frac{\partial}{\partial q_i} + \frac{\partial \mathcal{H}_u}{\partial q_i} \frac{\partial}{\partial x_i}), \lambda \in T^* M \ .$$

Therefore, the Hamiltonian differential equations system induced by the vector field $\vec{\mathcal{H}}$ on $T^* M$ reads

$$\dot{q}_i = -\frac{\partial \mathcal{H}_u}{\partial x_i}, \dot{x}_i = \frac{\partial \mathcal{H}_u}{\partial q_i} = f(x) + \sum_{j=1}^{m} u_j Y^j(x), i = 1, \dots, n.$$

The Pontryagin Maximum Principle gives a non-null 1-parameter of covectors $\lambda = (\lambda_t)_{t \in \mathbb{R}}$, with several necessary conditions to find an optimal control. As a particular case, we consider the time-optimal problem steering the initial state x_0 to the desired condition x_1 at minimum time.

Theorem 1 (The time-optimal Pontryagin maximum principle)**.** *Let Σ_M be a control system on a manifold M as before. If the Σ_M-solution associated to the control $u^*(t)$, $t \in [0,T]$, minimizes the time, there exists a Lipschitzian curve $(\lambda(t), x(t))$ in the cotangent space T^*M with $\lambda(t) \neq 0$ for all $t \in [0,T]$, such that for almost all $t \in [0,T]$*

$$\mathcal{H}(\lambda(t), x(t), u^*(t)) = \max_{u \in U} \mathcal{H}(\lambda(t), x(t), u),$$

$$\mathcal{H}(\lambda(t), x(t), u^*(t)) \geq 0, \textit{and}$$
$$\dot{\lambda}_t = \vec{\mathcal{H}}_{u^*}(\lambda_t).$$

The curve λ_t is called an extremal and its projection on M is an optimal trajectory for the initial optimization problem of Σ_M.

Remark 1. *The Hamiltonian function for a simply almost Riemannian structure*

$$\mathcal{F} = \{\mathcal{X}, Y_1, \ldots, Y_{n-1}\} \tag{35}$$

on G reads,

$$\mathcal{H}_\xi(\lambda, g, v, u_1, \ldots, u_{n-1}) = \langle \lambda, v + \sum_1^{n-1} u_j Y_j) \rangle - \frac{1}{2}\xi \left(v^2 + \sum_1^{n-1} u_j^2 \right).$$

In our context, we used a particular case with $\xi = 0$ and $v = 1$. It means a time-optimal problem on LCS [14].

Since the co-tangent bundle of G is trivial, it turns out that $TG \cong \mathfrak{g}^* \times G$, where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$. This favorable situation allows translation $\mathcal{H}$ to the tangent space at the identity, and describe all the equations at e. This convenient situation is possible according to the formulas

$$Y_g^j = (dL_g)_e Y^j, F_g = (dL_{g^{-1}})_g \mathcal{X}_g \in T_e G, \lambda_g = \lambda \circ TL_{g^{-1}}. \tag{36}$$

Precisely, consider the optimal time Hamiltonian function

$$\mathcal{H}_v(\lambda, g, u_1, u_2, \ldots, u_m) = \langle \lambda, \mathcal{X} + \sum_{j=1}^{n-1} u_j Y^j \rangle$$

Following reference [14], the optimal time Hamiltonian equations of a LCS on in $\mathfrak{g}^* \times G$, read as

$$\begin{cases} \dot{g} & = \mathcal{X}_g + \sum_{j=1}^{n-1} u_j Y^j(g) \\ \dot{\lambda} & = (-\mathcal{D} + ad(vF(g) \sum_{j=1}^{n-1} u_j Y^j)^* \lambda \end{cases}$$

where $\mathcal{D}$ is the derivation of $\mathfrak{g}$ associated to $\mathcal{X}$.

When $v = 1$, we obtain the optimal time Hamiltonian equations of a linear control system on G. In this situation, the difference between the LCS and ARS is important. Contrary to the ARS case, the identity element $e \in G$ is not in general an interior point of the reachable set from e.

The Jouan Equivalence Theorem

This section shows why it is relevant to classify linear control systems and almost Riemannian structures on Lie groups. The equivalent theorem gives information to general

LCS and general ARS on manifolds the dynamics of which generate a finite dimensional Lie algebra. As in Section 5, let us consider an arbitrary affine control system

$$\Sigma_M : x\dot{(}t) = f(x(t)) + \sum_{j=1}^{m} u_j(t)Y^j(x(t)), x(t) \in M, t \in \mathbb{R}, u \in \mathcal{U},$$

on a differentiable connected manifold M.

We take use these available systems to introduce the notion of controllability and control sets. Roughly speaking, the controllability property of a system means that it is possible to connect any two points of the manifold through the solutions of the systems in non-negative time. A control set is a region of M where controllability holds in its interior.

Definition 3. *The system Σ_M is said to be controllable, if for any pair $x, y \in M$ there exists a control $u \in \mathcal{U}$ and a positive time t, such that the corresponding solution $\varphi(x, u, .)$ with control u as an initial condition x, reaches y at time t_y, i.e., $\varphi(x, u, t_y) = y$.*

Definition 4. *A subset $C \subset M$ is called a control set if for any $x \in M$,*

1. *There exists $u \in \mathcal{U}$ such that $\varphi(x, u, t) \in C$, for any $t > 0$;*
2. *$cl\mathcal{A}(x)$ contains C ;*
3. *C is maximal with respect (1), and (2).*

where the reachable set $\mathcal{A}(x)$ from x reads as

$$\mathcal{A}(x) = \{\varphi(x, u, t) : t \geq 0, u \in \mathcal{U}\}. \tag{37}$$

Next, we state the Jouan equivalence theorem [10],

Theorem 2. *Σ_M is equivalent by diffeomorphism to a linear control system on a Lie group or a homogeneous space if and only if the vector fields of the system are complete, and generate a finite dimensional Lie algebra, i.e.,*

$$Span_{\mathcal{L}A}\left\{f, g^1, \ldots, g^m\right\} < \infty. \tag{38}$$

Remark 2. *Equivalent systems share the same properties. For example, controllability, existence, uniqueness, and boundedness of control sets, and of course, optimal problems. In other words, it is possible to obtain properties of Σ_M through the knowledge of the equivalent system Σ_G.*

A similar result as Theorem 2 is valid for a general almost Riemannian structure on manifolds [4]. In particular, it is possible to obtain information for ARS on manifolds, and for degenerate dynamical systems by knowing topological–dynamical–algebraic properties of the locus, acting as a barrier for the degenerate structure.

6. Recent Results on the Geometric Structures

According to the Equivalence theorem, both for almost Riemannian structures and control systems, it is necessary to develop these structures on Lie groups and homogeneous spaces. Furthermore, it is worth understanding these structures on low-dimensional groups as much as possible for real applications. For more than 20 years, several researchers have been working on these topics. Here, we mention some references to characterize the controllability properties, the existence, uniqueness, and the boundedness of control systems. Moreover, time-optimal problems for different classes of groups: nilpotent, solvable and semi-simple [18,23–28].

For any result we obtain for the locus of an ARS, it is possible to adapt for the locus of the corresponding degenerate dynamical system reciprocally. So, we obtain relevant information to the singular set in both senses.

6.1. Recent Results on Linear Control Systems

We start by mentioning very recent results of a linear control system on homogeneous spaces of the solvable group G of two dimensions to appear in [29].

Let G be the solvable group of two dimensions as described in Example 1. According to the computation there, a general linear control system has the shape

$$\Sigma_G : \dot{g(t)} = \mathcal{X}(g(t)) + u(t)Y(g(t)), g(t) \in G, t \in \mathbb{R}, u \in \mathcal{U},$$

where $\mathcal{U} = L^1_{loc}(\Omega)$, $u : [0, T_u] \to \Omega \subset \mathbb{R}^m$, with a closed set Ω and $0 \in int(\Omega)$.

The drift $\mathcal{X}$ is a linear vector field depending on two parameters, a and b, coming from the associated derivation. Moreover, the left invariant control vector Y is defined by the pair $(\alpha, \beta) \in \mathfrak{g}$. In coordinates $g = (x, y)$ we obtain Σ_G

$$\dot{x(t)} = u(t)\alpha$$
$$\dot{y}(t) = by(t) + (e^{x(t)} - 1)a + u(t)e^{x(t)}\beta, u \in \mathcal{U},$$

We first consider the closed subgroup $L = \{0\} \times \mathbb{Z}$, and the homogeneous space L/G, which turns out to be a horizontal cylinder. The canonical projection $\pi : G \to L/G \cong \mathbb{R} \times S^1$ is well defined and given by $\pi(x, y) = (x, [y])$. In order to project the linear vector fields its flows must leave invariant the subgroup L. This condition implies that $\mathcal{X}(x, y) = (0, e^x - 1)a$.

On the other hand, any invariant vector field on G can be projected on every homogeneous space of G. As a result, we can project Σ_G in a homogeneous control system on the cylinder.

In the new coordinates $(z, w) \in \mathbb{R} \times S^1$, we obtain the system $\Sigma_{L/G}$

$$\dot{z(t)} = u(t)\alpha$$
$$\dot{w}(t) = (e^{z(t)} - 1)a + u(t)e^{z(t)}\beta, u \in \mathcal{U}.$$

Theorem 3. *$\Sigma_{L/G}$ is controllable $\Leftrightarrow$ LARC is satisfied $\Leftrightarrow$ $a\alpha \neq 0$ [29].*

From a geometric picture, controllability can be seen as follows. Take any state (z, w). By choosing $u = 0$, we obtain $z(t) = z$. Thus, the system turns around the circle $\{z\} \times \mathbb{R}$. However, if $u \neq 0$, the solution $(z(t), w(t), u)$ starting on (z, w) turns around the horizontal cylinder traveling to the right or to the left according to the signal and size of the selected control u. Therefore, if we consider an arbitrary point (z_1, w_1), we obtain: after the solution $(z(t), w(t), u)$ hit the circle $\{z_1\} \times \mathbb{R}$ with a control u_1, take the control $u = 0$, and rotate on the circle up to reach w_1.

Next, we consider the subgroup $L = \mathbb{Z} \times \{0\}$, and the homogeneous space L/G, which is a vertical cylinder. The canonical projection

$$\pi : G \to L/G \cong S^1 \times \mathbb{R},\ \pi(x, y) = ([x], y) \tag{39}$$

is well defined. As before, we compute the associated homogeneous system $\Sigma_{L/G}$. As a matter of fact, in this situation any linear vector fields are projectable. We obtain [29],

Theorem 4. *Assume $\Sigma_{L/G}$ satisfy LARC, $b < 0$ with $b \in int(\alpha\Omega)$. Then, there are two intervals $I, J \subset \mathbb{R}$ with $I \cap J = \phi$ which determines two control sets on the vertical cylinder as follows*

$$C_1 = S^1 \times I, C_2 = S^1 \times J. \tag{40}$$

Here, C_1 contains the identity element $(0, 0)$. In addition as we know, the homogeneous system $\Sigma_{L/G}$ is controllable at the interior of these control sets.

We present a controllability result on the nonnilpotent solvable Lie group of dimension three as explained in Example 3. However, we first need to introduce an especial decomposition induced by a derivation on any arbitrary Lie algebra $\mathfrak{g}$.

In the sequel, we follow [17,24]. Given a derivation $\mathcal{D} : \mathfrak{g} \to \mathfrak{g}$, and $\alpha \in Spec(\mathcal{D})$, a α-generalized eigenspace reads as

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : (\mathcal{D} - \alpha)^n X = 0 \text{ for some } n \geq 1\}.$$

As a matter of fact, if β is also an eigenvalue of $\mathcal{D}$,

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \text{ if } \alpha + \beta \text{ is an eigenvalue of } \mathcal{D}, [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0, otherwise.$$

The vector spaces $\mathfrak{g}^+$, $\mathfrak{g}^0$ and $\mathfrak{g}^-$ defined by

$$\mathfrak{g}^+ = \oplus_{\alpha\,:\,\mathrm{Re}(\alpha)>0}(\mathfrak{g}_\alpha), \qquad \mathfrak{g}^0 = \oplus_{\alpha\,:\,\mathrm{Re}(\alpha)=0}(\mathfrak{g}_\alpha) \qquad \text{and} \qquad \mathfrak{g}^- = \oplus_{\alpha\,:\,\mathrm{Re}(\alpha)<0}(\mathfrak{g}_\alpha),$$

are Lie algebras. Let G^+, G^0 and G^- be the connected Lie groups with Lie algebra $\mathfrak{g}^+$, $\mathfrak{g}^0$ and $\mathfrak{g}^-$, respectively. If $G = G^+G^0G^-$, the Lie group is said to be decomposable.

Example 4. *Consider the restricted linear system* Σ_G *on* $G = \mathbb{R} \otimes_\rho \mathbb{R}^2$ *with Lie algebra* $\mathfrak{g} = \mathbb{R} \otimes_\theta \mathbb{R}^2$ *determined by*

$$\Sigma_G : \dot{g(t)} = \mathcal{X}(g(t)) + u_1(t)Y^j(g(t)) + u_2(t)Y^2(g(t)), g(t) \in G, t \in \mathbb{R}, u \in \mathcal{U},$$

which satisfy LARC. Here, $\rho_t = e^{t\theta}$, $\Delta = Span\{Y^1, Y^2\}$ *has dimension* 2*, and* θ *is the real matrix of order* 2 *with all the coefficients* 0 *except* 1 *in the position* 22*. Therefore, we obtain [22].*

Theorem 5. *The system* Σ_G *is controllable if and only if* $dim(\mathfrak{g}^0) > 1$*, or* $dim(\mathfrak{g}^0) = 1$ *and* $[Y^1, Y^2] = Y^2$*.*

The following Theorem collects several results on Σ_G [24].

Theorem 6. *Let* Σ_G *be a restricted system on G such that* $\mathcal{A} = \mathcal{A}(e)$ *is open. If G is decomposable, there is precisely one control set* $\mathcal{C} = cl(\mathcal{A}) \cap \mathcal{A}^-$ *with a nonempty interior. Any solvable lie group is decomposable. Furthermore, if G is nilpotent and simply connected,* $\mathcal{C}$ *is bounded* $\Leftrightarrow cl(\mathcal{A} \cap G^-), cl(\mathcal{A}^- \cap G^+)$ *are compacts.*

As for the restricted classical linear system on Euclidean spaces, the next result characterizes the controllability property on nilpotent groups.

Theorem 7. *A restricted linear control system* Σ_G *on a nilpotent Lie group G is controllable if an only if* $\mathcal{A} = \mathcal{A}(e)$ *is open and* $G = G^0$*.*

Just observe that in Euclidean spaces the Kalman condition is equivalent to the openness of $\mathcal{A}$.

6.2. Some Results on Simply almost Riemannian Structures

In this section, we start by describing a satisfactory geometric result by Agrachev et al. for an ARS. In [2], the authors show a Gauss–Bonett-like formula for a 2-dimensional almost Riemannian manifold $\mathcal{F}$ as follows. In this context, an element $x \in \mathcal{Z}$ is said to be a tangency point if $\mathcal{F}(x)$ is tangent to $\mathcal{Z}$. $\mathcal{F}$ is triviallizable if a pair of vector fields globally generate it, and orientable if there exists a volume form. The curvature is denoted by K.

Theorem 8. *Let M be an oriented compact manifold of* 2 *dimensions. For a generic oriented* 2*-dimensional ARS on M without a tangency point it turns out*

$$\int K dA_s = 2\pi(\chi(M^+) - \chi(M^-)) \tag{41}$$

where χ *denotes the Euler Characteristic. In addition, if* $\mathcal{F}$ *is trivializable, then* $\int K dA_s = 0$*.*

In the same direction, we also mention the reference [3].

Next, we state a general result of a simply almost Riemannian structure on Lie groups appears [4]. Through a generic example, we show that the singular locus can be a very wild set.

Theorem 9. *Let $\mathcal{F} = \{\mathcal{X}, Y_1, \ldots, Y_{n-1}\}$ a simply almost Riemannian structure on $\mathfrak{g}$. Assume, $\Delta = Span\{Y_1, \ldots, Y_{n-1}\}$ is a subalgebra. Then, the singular locus $\mathcal{Z}$ is an analytic, embedded co-dimension 1 submanifold of G. Its tangent space at the identity element is given by $D^{-1}(\Delta)$. Furthermore, if G is solvable the locus is a subgroup with Lie algebra $D^{-1}(\Delta)$.*

If Δ is not a subalgebra the locus can be wild. On the Heisenberg group of Example 4 consider the distribution $\Delta = \{Y^1, Y^2\}$, the linear vector field $\mathcal{X}$ induced by a general derivation $\mathcal{D}$ and the almost Riemannian structure

$$\mathcal{F} = \left\{\mathcal{X}, Y^1, Y^2\right\}. \tag{42}$$

It turns out that the singular $\mathcal{Z}$ locus of $\mathcal{F}$ reads as [23],

$$\mathcal{Z} = \left\{ex + fy + (a+d)z - \frac{1}{2}cx^2 + \frac{1}{2}by^2 - dxy = 0\right\}. \tag{43}$$

Obviously, the locus generated by these quadratic forms need not be subgroups, not even manifolds. However, for $a = b = c = e = f = 0$, and $d = 1$, we obtain the hyperbolic paraboloid.

Finally, we give the reference *Isometries of almost-Riemannian structures on Lie groups* [20]. Here, the authors prove that two different ARS: $\mathcal{F}_1$ and $\mathcal{F}_2$ on a Lie group G are isometric if and only if there exists an isometry $\psi : G \to G$ that fixes the identity element. The isometry preserves the associated left-invariant distribution, i.e., $\psi(\Delta_1) = \Delta_2$, and the linear vector field. Furthermore, if the Lie group is nilpotent ψ is an automorphism.

This characterization makes it possible to obtain a complete classification of the ARS on the solvable 2-dimensional group and the Heisenberg group of dimension 3.

6.3. Some Results on Degenerate Dynamical Systems

As we say, Degenerate Dynamical Systems (DDS) is a new branch in Mathematics with relevant application in physics. Furthermore, this kind of structure needs to be developed. We hope that any result on almost Riemannian structures, especially on the corresponding singular locus, will help the analysis of degenerate behavior. The locus acts as a barrier for the corresponding degenerated dynamical system.

We finish this chapter inviting the readers to look at the very interesting article *Degeneracy Index and Poincaré-Hopf Theorem*, by Zanelli, J. and Ruan, H., which appears in [30]. It is about a relationship between a 2-dimensional degenerate dynamical system as explained in Section 2.5 and the Poincaré-Hopf Theorem.

7. Open Problems and Challenges

To finish this review, we write down a list of open problems and general challenges for each of the geometrical structures considered.

7.1. About Linear Control Systems on Lie Groups

The main challenge here is to classify several fundamental properties of a linear control system Σ_G on a connected Lie group G. Specifically, the controllability property, the existence, uniqueness, and other topological properties of control sets should be characterized on different classes of Lie groups: nilpotent, solvable, simple, semi-simple, semi-direct and direct product of groups. As we mention, many references are totally or partially answering these open problems. On the other hand, from the concrete applications point of view, the Jouan Equivalence Theorem gives an important reason why it is relevant to challenge this project [11,31].

Other relevant challenges are to extend all the Σ_G theory to $\Sigma_{G/H}$ on homogeneous spaces, and to affine control system as explained below.

In the following we list some specific open problems for Σ_G when G is an arbitrary connected Lie group.

1. To characterize controllability properties on every dimension;
2. To characterize the control sets and its topological properties;
3. To study topology equivalence and conjugacy of linear control systems;
4. To apply the Pontryagin Maximum Principle to solve.
   - The minimum time connecting two desired states;
   - Quadratic optimal problems, such a minimum energy, etc.
5. To study the previous problems on homogeneous spaces;
   - What is the relationship of controllability and control sets between $\Sigma_{G/H}$ *and* Σ_G?
6. Finally, to extend the Σ_G-theory to the affine control systems, i.e., by replacing in Σ_G the linear vector field $\mathcal{X}$ by $\mathcal{X} + Y$ with $Y \in \mathfrak{g}$ and for any $j = 1, \ldots, m$, by replacing $Y^j \in \mathfrak{g}$ by $\mathcal{X}^j + Y^j$, where $\mathcal{X}^j$ is linear.

7.2. About almost Riemannian Structures

We propose to work out the following issues involving the compute of geodesics associated to the Chow–Rashevskii metric.

1. To classify k-ARS on low dimensional Lie groups;
   - 1-ARS and and 2-ARS on solvable 3-dimensional Lie groups;
   - 1-ARS and 2-ARS in 3-dimensional semisimple Lie groups;
   - To study the relationships between $\mathcal{Z}$ and the manifolds G^+, G^-;
2. To classify k-ARS from an algebraic point of view;
   - Under which conditions $\mathcal{Z}$ is a Lie subgroup of G?
   - Under which condition the Lie algebra $\mathfrak{z}$ of $\mathcal{Z}$ is an ideal of $\mathfrak{g}$?
3. Are 2-ARS, k-ARS related with appropriate sub-Riemannian structures?
4. To find relationship between ARS and the Degenerate Dynamical System

7.3. About Degenerate Dynamical Systems

The general problem here is understanding the evolution of the dynamic near the locus set. Moreover, it is outside of the Poincaré classification, which takes care of the Hamiltonian isolated singularities. In this new context, the singularities comes from the degeneracy of the symplectic form, which normally are submanifold of co-dimension 1, including the barriers and the walls.

There are many open problems and challenges with this kind of structure. The theory should be developed in different situations. For instance, on Euclidean space, Compact symplectic manifolds, Hamiltonian actions on symplectic manifolds, and co-adjoint orbits of Lie groups.

Assume some trajectories reach the locus $\mathbb{Z}$. What kind of forwarding dynamic behavior could we expect inside the singular set or when the trajectory leaves the locus? If the trajectory remains in the locus forever, it means that the "trajectory" freezes some coordinates. Moreover, this study should be critical in real applications [9].

8. Conclusions

This review provides new information about three geometric structures on Lie groups that depend on the same kind of dynamics: Linear Control Systems, Almost Riemannian Structures, and Degenerate Dynamical Systems. We started with the examples that motivated the study of these linear structures on the plane. We computed all the linear and invariant vector fields on solvable Lie groups of dimensions two and three and the classical 3-dimensional Heisenberg Lie group. After providing the definitions of these structures on Euclidean spaces and Lie groups, we established the Pontryagin Maximum Principle and the Hamiltonian equations for an LCS on G. We also included the Jouan Equivalence Theorem, which gives one of the main reasons that it is worth developing linear structures on Lie groups and homogeneous spaces. We discussed some recent results on the three geometric structures studied here. The last section includes a list of open problems and challenges for LCS, ARS, and DDS. With the aim of possible applications, an effective dissemination of the theory for a broad audience is valuable. We would like to thank the Editor of this Special Issue for inviting us to publish this article in Symmetry.

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