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CONTROLLABILITY OF LINEAR SYSTEMS ON LIE GROUPS WITH FINITE SEMISIMPLE CENTER*

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Abstract. This paper studies controllability for a given linear system Σ on a connected Lie group G by taking into consideration the eigenvalues of an associated derivation $\mathcal{D}$. If we assume that the Lie group G has finite center and, for some $\tau > 0$, the identity element of G is an interior point of its reachable set at time τ, then the system is controllable if $\mathcal{D}$ has only eigenvalues with zero real part.

Key words. Lie group, derivation, linear system, controllability

AMS subject classifications. 16W25, 93B05, 93C05

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1. Introduction.

Throughout the paper G will stand for a connected Lie group with Lie algebra $\mathfrak{g}$, which we identify with the set of right invariant vector fields. In [2] the authors introduce the class of linear systems on G as the family of differential equations

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^{m} u_j(t) X^j(g(t)),$$

where drift $\mathcal{X}$ is a linear vector field, X^j are right invariant vector fields, and $u \in \mathcal{U} \subset L^\infty(\mathbb{R}, \Omega \subset \mathbb{R}^m)$ is the class of admissible controls. Here Ω is a convex subset of $\mathbb{R}^m$ such that $0 \in \operatorname{int}\Omega$. The system is called *restricted* if Ω is compact and *unrestricted* if $\Omega = \mathbb{R}^m$.

In [2] it is proved that the *ad-rank* condition

$$\operatorname{Span}\{\mathcal{D}^k(X^j) : j = 1, \ldots, m \text{ and } k \geq 0\} = \mathfrak{g}$$

implies local controllability of the linear system from the identity element $e \in G$, where $\mathcal{D}$ is the derivation induced by the linear vector field $\mathcal{X}$. In particular $e \in \operatorname{int}\mathcal{A}_\tau$ for any $\tau > 0$. Here, $\mathcal{A}_\tau$ stands for the set of reachable points from e at time τ, and $\operatorname{int}\mathcal{A}_\tau$ its interior. On the other hand, in [1] the authors give a first example of a linear system that is locally controllable around the identity, though its reachable set $\mathcal{A}$ is not a semigroup. Later, in [7], Jouan proves that $\mathcal{A}$ is a semigroup if and only if $\mathcal{A} = G$. This is a remarkable difference between the linear systems studied and the so-called invariant systems. Currently, it is known that an invariant system on G is controllable if and only if G is connected, and $e \in \operatorname{int}\mathcal{A}$ (see [12]). Therefore, to understand the class of linear systems from the controllability viewpoint is a really hard task. However, recently in [5] Da Silva showed that a linear system on a solvable Lie group

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is controllable if $\mathcal{A}$ is open and any eigenvalue of $\mathcal{D}$ has zero real part. Moreover, when G is a nilpotent Lie group and the linear system is restricted, controllability holds if and only if $\mathcal{A}$ is open and $\mathcal{D}$ has only eigenvalues with zero real part, showing that controllability of restricted linear systems on nilpotent Lie groups is a very exceptional issue.

The aim of this paper is to extend the results in [5] to more general classes of Lie groups. If the Lie group G has finite semisimple center (see Definition 3.1) such an extension is possible. Next, we state our main result.

THEOREM 1.1. *Let G be a connected Lie group with finite semisimple center. A linear system on G is controllable if $e \in \operatorname{int} \mathcal{A}_{\tau_0}$ for some $\tau_0 > 0$, and $\mathcal{D}$ has only eigenvalues with zero real part.*

Furthermore, we introduce a special class of linear systems which could be relevant for applications. It is a linear system on a direct product of Lie groups. More precisely, it is linear on the first component and homogeneous on the second one. Our main result shows that it is possible to connect any two configuration of a rolling sphere S over a revolving plane around the z-axis with constant angular velocity. The kinematic equations and the configuration space state $G = \mathbb{R}^2 \times SO(3)$ appears in the book of Jurdjevic [16].

The paper is organized as follows. Section 2 contains the definition of linear vector fields and linear systems on Lie groups. It also shows the decomposition of $\mathfrak{g}$ through the generalized eigenspaces of $\mathcal{D}$ and the subgroups of G associated with them. Finally, some basic properties of reachable sets of linear systems are given. In section 3 we prove the main results of the paper which strongly depend on the fact that the Lie subgroup $G^0 = \langle \exp(\mathfrak{g}^0) \rangle$ is contained in the reachable set $\mathcal{A}$ when group G has finite semisimple center. Here $\mathfrak{g}^0$ is the sum of all generalized eigenspaces of $\mathcal{D}$ for eigenvalues with zero real part. At the end of section 3 we introduce a special class of linear systems which could be relevant for concrete applications and give some examples.

2. Preliminaries.

This section contains definitions and basic results concerning linear vector fields and linear systems. For more on these subjects the reader may consult [1], [2], [5], [7], [8], and [9].

2.1. Linear vector fields and $\mathcal{D}$-decomposition.

Let G be a connected Lie group of dimension d with Lie algebra $\mathfrak{g}$ identified with the set of right invariant vector fields. The *normalizer* η of $\mathfrak{g}$ is defined by

$$\eta = \operatorname{norm}_{X^\infty(G)}(\mathfrak{g}) = \{F \in X^\infty(G);\ \text{for all } Y \in \mathfrak{g},\ \ [F, Y] \in \mathfrak{g}\},$$

where $X^\infty(G)$ denotes the Lie algebra of the $\mathcal{C}^\infty$ vector fields on G.

DEFINITION 2.1. *A vector field $\mathcal{X}$ on G is called linear if it belongs to η and $\mathcal{X}(e) = 0$.*

The following result gives equivalent conditions for a vector field on G to be linear (see [9, Theorem 1]).

THEOREM 2.2. *Let $\mathcal{X}$ be a vector field on a connected Lie group G. The following conditions are equivalent:*

1. *$\mathcal{X}$ is linear;*
2. *the flow of $\mathcal{X}$ is a one parameter group of automorphisms of G;*

3. $\mathcal{X}$ *satisfies*

$$\mathcal{X}(gh) = (dL_g)_h\mathcal{X}(h) + (dR_h)_g\mathcal{X}(g) \quad \text{for all} \quad g, h \in G,$$

where as usual L_g *and* R_h *denote the left and right translation by* g *and* h*, respectively.*

Let $\mathcal{X}$ be a linear vector field. Since the flow $(\varphi_t)_{t\in\mathbb{R}}$ of $\mathcal{X}$ is a one parameter group of automorphisms we can define a derivation $\mathcal{D}$ of $\mathfrak{g}$ by

$$\mathcal{D}Y = -[\mathcal{X}, Y](e) \text{ for all } Y \in \mathfrak{g}.$$

The relation between $(\varphi_t)_{t\in\mathbb{R}}$ and $\mathcal{D}$ is given by

$$(d\varphi_t)_e = \mathrm{e}^{t\mathcal{D}} \quad \text{for all} \quad t \in \mathbb{R} \tag{1}$$

and, therefore,

$$\varphi_t(\exp Y) = \exp(\mathrm{e}^{t\mathcal{D}}Y) \text{ for all } t \in \mathbb{R}, Y \in \mathfrak{g}.$$

Consider the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$ and the derivation $\mathcal{D}_\mathbb{C}$ on $\mathfrak{g}_\mathbb{C}$ induced by $\mathcal{D}$. For an eigenvalue α of $\mathcal{D}$ we define the α-generalized eigenspace of $\mathcal{D}_\mathbb{C}$ as

$$(\mathfrak{g}_\mathbb{C})_\alpha = \{X \in \mathfrak{g}_\mathbb{C} : (\mathcal{D}_\mathbb{C} - \alpha)^n X = 0 \text{ for some } n \geq 1\}.$$

Since the eigenvalues of $\mathcal{D}_\mathbb{C}$ coincide with the ones of $\mathcal{D}$ we have that

$$\mathfrak{g}_\mathbb{C} = \bigoplus_\alpha (\mathfrak{g}_\mathbb{C})_\alpha \quad \text{for every } \alpha \text{ eigenvalue of } \mathcal{D}.$$

Moreover, if β is also an eigenvalue of $\mathcal{D}$ [13, Proposition 3.1] this implies

$$[(\mathfrak{g}_\mathbb{C})_\alpha, (\mathfrak{g}_\mathbb{C})_\beta] \subset (\mathfrak{g}_\mathbb{C})_{\alpha+\beta}, \tag{2}$$

where $(\mathfrak{g}_\mathbb{C})_{\alpha+\beta} = \{0\}$ when $\alpha + \beta$ in not an eigenvalue of $\mathcal{D}$.

Let

$$\mathfrak{g}_\mathbb{C}^+ = \bigoplus_{\alpha\,:\,\mathrm{Re}(\alpha)>0} (\mathfrak{g}_\mathbb{C})_\alpha, \quad \mathfrak{g}_\mathbb{C}^0 = \bigoplus_{\alpha\,:\,\mathrm{Re}(\alpha)=0} (\mathfrak{g}_\mathbb{C})_\alpha, \quad \text{and} \quad \mathfrak{g}_\mathbb{C}^- = \bigoplus_{\alpha\,:\,\mathrm{Re}(\alpha)<0} (\mathfrak{g}_\mathbb{C})_\alpha.$$

Since $\mathfrak{g}_\mathbb{C}^+, \mathfrak{g}_\mathbb{C}^0$, and $\mathfrak{g}_\mathbb{C}^-$ are invariant by conjugation they coincide with the complexification of

$$\mathfrak{g}^+ := \mathfrak{g}_\mathbb{C}^+ \cap \mathfrak{g}, \qquad \mathfrak{g}^0 := \mathfrak{g}_\mathbb{C}^0 \cap \mathfrak{g}, \qquad \text{and} \qquad \mathfrak{g}^- := \mathfrak{g}_\mathbb{C}^- \cap \mathfrak{g}.$$

Moreover, $\mathfrak{g}^+, \mathfrak{g}^0$, and $\mathfrak{g}^-$ are $\mathcal{D}$-invariant Lie subalgebras of $\mathfrak{g}$ with $\mathfrak{g}^+$ and $\mathfrak{g}^-$ nilpotent ones, and such that $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$. Also, $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are ideals of the Lie subalgebras $\mathfrak{g}^{+,0} := \mathfrak{g}^+ \oplus \mathfrak{g}^0$ and $\mathfrak{g}^{-,0} := \mathfrak{g}^- \oplus \mathfrak{g}^0$, respectively.

2.2. Linear systems on Lie groups. Let Ω be a subset of $\mathbb{R}^m$ such that $0 \in \operatorname{int}\Omega$ and consider the class of admissible control functions $\mathcal{U} \subset L^\infty(\mathbb{R}, \Omega \subset \mathbb{R}^m)$.

Definition 2.3. *A linear system on a Lie group* G *is determined by the family of differential equations*

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^m u_j(t) X^j(g(t)), \tag{3}$$

where drift $\mathcal{X}$ is a linear vector field, X^j are right invariant vector fields, and $u = (u_1, \dots, u_m) \in \mathcal{U}$.

For $g \in G$, $u \in \mathcal{U}$, and $t \in \mathbb{R}$ the solution of (3) starting at g is given by

$$\phi_{t,u}(g) = L_{\phi_{t,u}}(\varphi_t(g)) = \phi_{t,u}\varphi_t(g),$$

where $\phi_{t,u} := \phi_{t,u}(e)$ denotes the solution of (3) starting at the identity element $e \in G$ (see, for instance, [6, Proposition 3.3]). The map

$$\phi : \mathbb{R} \times G \times \mathcal{U} \to G, \quad (t, g, u) \mapsto \phi_{t,u}(g),$$

satisfies the *cocycle property*

$$\phi_{t+s,u}(g) = \phi_{t,\Theta_s u}(\phi_{s,u}(g))$$

for all $t, s \in \mathbb{R}$, $g \in G$, $u \in \mathcal{U}$, where $\Theta_t : \mathcal{U} \to \mathcal{U}$ is the shift flow $u \in \mathcal{U} \mapsto \Theta_t u := u(\cdot + t)$. It follows directly from the cocycle property that the diffeomorphism $\phi_{t,u}$ has inverse $\phi_{-t,\Theta_t u}$ for any $t \in \mathbb{R}$ and $u \in \mathcal{U}$. Since for each $t > 0$, $\phi_{t,u}(g)$ depends only on $u|_{[0,t]}$ we obtain

$$\phi_{t,u_2}(\phi_{s,u_1}(g)) = \phi_{t+s,u}(g).$$

Here, the control $u \in \mathcal{U}$ is defined by concatenation between u_1 and u_2,

$$u(\tau) = \begin{cases} u_1(\tau) & \text{for } \tau \in [0, s], \\ u_2(\tau - s) & \text{for } \tau \in [s, t + s]. \end{cases}$$

For any $g \in G$ the *set of reachable points from* g *at time* τ and the *reachable set of* g are defined, respectively, by

$$\mathcal{A}_\tau(g) := \{\phi_{\tau,u}(g), u \in \mathcal{U}\} \quad \text{and} \quad \mathcal{A}(g) := \bigcup_{\tau > 0} \mathcal{A}_\tau(g). \tag{4}$$

When $g = e$, the sets $\mathcal{A}(e) = \mathcal{A}$ and $\mathcal{A}_\tau(e) = \mathcal{A}_\tau$ are called the *reachable set* and the *reachable set at time* τ, respectively.

Definition 2.4. *The linear system* (3) *on* G *is said to be controllable if, given* $g, h \in G$*, there are* $u \in \mathcal{U}$ *and* $\tau > 0$ *such that* $\phi_{\tau,u}(g) = h$.

Remark 2.5. It is straightforward to see that the linear system is controllable if and only if $G = \mathcal{A} \cap \mathcal{A}^*$. Here $\mathcal{A}^*$ stands for the reachable set of Σ in negative time.

The next proposition states the main properties of the reachable sets.

Proposition 2.6. *It holds that*

1. $0 \leq \tau_1 \leq \tau_2$ *implies* $\mathcal{A}_{\tau_1} \subset \mathcal{A}_{\tau_2}$;
2. *for all* $g \in G$, $\mathcal{A}_\tau(g) = \mathcal{A}_\tau \varphi_\tau(g)$;
3. *for all* $\tau, \tau' \geq 0$,
$$\mathcal{A}_{\tau+\tau'} = \mathcal{A}_\tau \varphi_\tau(\mathcal{A}_{\tau'}) = \mathcal{A}_{\tau'} \varphi_{\tau'}(\mathcal{A}_\tau),$$
and, inductively for any positive real numbers $\tau_1, \dots, \tau_n$,
$$\mathcal{A}_{\tau_1}\varphi_{\tau_1}(\mathcal{A}_{\tau_2})\varphi_{\tau_1+\tau_2}(\mathcal{A}_{\tau_3}) \cdots \varphi_{\sum_{i=1}^{n-1}\tau_i}(\mathcal{A}_{\tau_n}) = \mathcal{A}_{\sum_{i=1}^{n}\tau_i};$$
4. *for all* $u \in \mathcal{U}$, $g \in G$ *and* $t \geq 0$, *it follows that* $\phi_{t,u}(\mathcal{A}(g)) \subset \mathcal{A}(g)$;
5. $e \in \operatorname{int} \mathcal{A}$ *if and only if* $\mathcal{A}$ *is open.*

The proof of items 1 to 3 can be found in [7, Proposition 2]; items 4 and 5 in [5, Proposition 2.13].

Remark 2.7. We notice that item 4 together with the fact that $0 \in \operatorname{int}\Omega$ shows us that $\mathcal{A}$ is invariant by the flow φ_t in positive time, that is, $\varphi_t(\mathcal{A}) \subset \mathcal{A}$ for any $t \geq 0$.

We finish this section with a technicall lemma that will be needed ahead. For its proof, the reader may consult [14, Lemma 4.1].

LEMMA 2.8. *Let G be a connected semisimple Lie group with finite center and $\mathcal{S}$ a semigroup in G. Suppose that there exists $X \in \mathfrak{g}$ such that* $\operatorname{ad}(X)$ *is nilpotent and* $\exp X \in \operatorname{int}\mathcal{S}$. *Then $\mathcal{S} = G$.*

3. Controllability of linear systems. In this section we extend the results obtained in [5] about controllability of linear systems on solvable Lie groups to the following class.

DEFINITION 3.1. *Let G be a connected Lie group. We say that G has a finite semisimple center if all semisimple Lie subgroups of G have finite center.*

Remark 3.2. Several Lie groups meet Definition 3.1. For instance, any solvable Lie group, any semisimple Lie group with finite center, and any direct or semidirect product of these two classes have finite semisimple center. Moreover, by Malcev's theorem (see [11, Theorem 4.3]) and its corollaries, any connected Lie group G with one Levi subgroup L with finite center also has a semisimple finite center.

Let us denote by G_0, G^+, G^-, G^0, $G^{+,0}$, and $G^{-,0}$ the connected Lie subgroups of G with Lie algebras $\mathfrak{g}_0$, $\mathfrak{g}^+$, $\mathfrak{g}^-$, $\mathfrak{g}^0$, $\mathfrak{g}^{+,0}$, and $\mathfrak{g}^{-,0}$, respectively.

Here are some elementary properties of these subgroups. For the proofs we refer the reader to [5, Proposition 2.9].

PROPOSITION 3.3. *It holds that*

1. $G^{+,0} = G^+G^0 = G^0G^+$ *and* $G^{-,0} = G^-G^0 = G^0G^-$;
2. $G^+ \cap G^- = G^{+,0} \cap G^- = G^{-,0} \cap G^+ = \{e\}$;
3. $G^{+,0} \cap G^{-,0} = G^0$;
4. $G^+, G^0, G^-, G^{+,0}$, *and* $G^{-,0}$ *are closed in G;*
5. *if G is solvable then*
$$G = G^{+,0}G^- = G^{-,0}G^+.$$

Moreover, the fixed points of $\mathcal{X}$ are elements of G^0.

The next three lemmas from [5] relate φ-invariant subsets of G with the reachable set of the linear system.

LEMMA 3.4. *Let $g \in \mathcal{A}$ and assume that $\varphi_t(g) \in \mathcal{A}$ for any $t \in \mathbb{R}$. Then $\mathcal{A}\, g \subset \mathcal{A}$.*

PROPOSITION 3.5. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{n}$ an ideal of $\mathfrak{h}$ such that $\mathcal{D}(\mathfrak{h}) \subset \mathfrak{n}$. Denote by H and N the connected Lie subgroups of G with Lie subalgebras $\mathfrak{h}$ and $\mathfrak{n}$ respectively. Then*
$$N \subset \mathcal{A} \implies H \subset \mathcal{A}.$$

PROPOSITION 3.6. *If $K \subset G^0$ is a connected φ-invariant solvable Lie subgroup of G^0 and $\mathcal{A}$ is open then $G^0 \subset \mathcal{A}$.*

Next, we sharpen these results and prove an extension of Proposition 3.6 from solvable Lie groups to groups with semisimple finite center. In order to do that we just need to show that the semisimple component of G^0 is contained in $\mathcal{A}$, which will be done by constructing a special semigroup associated with the semisimple part.

Here and subsequently, we assume that $e \in \operatorname{int} \mathcal{A}_{\tau_0}$ for some $\tau_0 > 0$. Essentially, we generalize Proposition 3.6 above and [5, Theorem 3.7] for Lie groups with finite semisimple center.

Since $\mathfrak{g}^0$ is $\mathcal{D}$-invariant we can restrict $\mathcal{D}$ as a derivation of $\mathfrak{g}^0$. Let us denote by $\mathfrak{r}$ the solvable radical of the subalgebra $\mathfrak{g}^0$ and by R its associated connected solvable subgroup. Since $\mathfrak{r}$ is invariant under automorphisms we get $\mathrm{e}^{\tau\mathcal{D}}\mathfrak{r} = \mathfrak{r}$ for any $\tau \in \mathbb{R}$ which implies that $\mathfrak{r}$ is $\mathcal{D}$-invariant. Thus, we obtain a well-defined derivation $\mathcal{D}_*$ on the semisimple Lie algebra $\mathfrak{l} := \mathfrak{g}^0/\mathfrak{r}$ with a commuting property, i.e., $\mathcal{D}_* \circ \pi_* = \pi_* \circ \mathcal{D}$. Here $\pi_* : \mathfrak{g}^0 \to \mathfrak{l}$ is the canonical projection. Moreover, since any derivation in a semisimple Lie algebra is inner there is $Y \in \mathfrak{g}^0$ such that $\mathcal{D}_* = \mathrm{ad}_{\mathfrak{l}}(\pi_*(Y))$.

By fixing $Y \in \mathfrak{g}^0$ as before, it turns out that $\mathcal{D}^n(Z) - \mathrm{ad}(Y)^n Z \in \mathfrak{r}$ for every $n \in \mathbb{N}$ and $Z \in \mathfrak{g}^0$. Consequently, for any $\tau \in \mathbb{R}$ we have

$$\mathrm{e}^{\tau\mathcal{D}} Z = \mathrm{e}^{\tau\,\mathrm{ad}(Y)Z} + W \quad \text{for some} \quad W = W_{\tau,Y,Z} \in \mathfrak{r}. \tag{5}$$

Consider the one parameter group of inner automorphisms of G^0 induced by

$$\varphi_\tau^Y(h) := \exp(\tau Y) h \exp(-\tau Y).$$

Hence, for any $Z \in \mathfrak{g}^0$

$$\varphi_\tau(\exp Z) = \exp(\mathrm{e}^{\tau\mathcal{D}} Z) \quad \text{and} \quad \varphi_\tau^Y(\exp Z) = \exp(\mathrm{e}^{\tau\,\mathrm{ad}(Y)} Z).$$

From (5) and [15, Lemma 3.1] we get

$$\varphi_\tau(\exp Z) = \exp\left(\mathrm{e}^{\tau\,\mathrm{ad}(Y)} Z + W\right) = \exp\left(\mathrm{e}^{\tau\,\mathrm{ad}(Y)} Z\right) g = \varphi_\tau^Y(\exp Z) g,$$

where $g = g_{\tau,Y,Z} \in R$. Actually, since G^0 is connected, it holds that

$$\varphi_\tau(h) = \varphi_\tau^Y(h) g \quad \text{for any } h \in G_0, \tag{6}$$

where $g = g_{\tau,Y,h} \in R$. In particular, for any $\tau, \tau' \in \mathbb{R}$ it holds that

$$\varphi_\tau(\exp(\tau' Y)) = \exp(\tau' Y) g, \quad \text{where} \quad g = g_{\tau,\tau',Y} \in R.$$

Let us consider now the semisimple connected Lie group $L := G^0/R$, which has Lie algebra $\mathfrak{l}$. By the hypothesis, L has finite center. By the previous analysis

$$\mathcal{A}_\tau(\exp(\tau Y)) = \mathcal{A}_\tau \varphi_\tau(\exp(\tau Y)) = \mathcal{A}_\tau \exp(\tau Y) g, \quad g \in R,$$

and, consequently,

$$\pi(G^0 \cap \mathcal{A}_\tau(\exp(\tau Y))) = \pi((G^0 \cap \mathcal{A}_\tau) \exp(\tau Y)).$$

Here $\pi : G^0 \to L$ is the canonical projection. Consider for any $\tau > 0$ the subsets of L defined by

$$\mathcal{S}_\tau := \pi((G^0 \cap \mathcal{A}_\tau) \exp(\tau Y)) \quad \text{and} \quad \mathcal{S} := \bigcup_{\tau > 0} \mathcal{S}_\tau.$$

It holds that $\mathcal{S}$ is a semigroup of L. In fact, let $x_1, x_2 \in \mathcal{S}$. By definition, there are $\tau_i > 0$, $u_i \in \mathcal{U}$ such that

$$x_i = \pi(\phi_{\tau_i,u_i} \exp(\tau_i Y)) \quad \text{with} \quad \phi_{\tau_i,u_i} \in \mathcal{A}_{\tau_i} \cap G^0 \quad \text{for} \quad i = 1, 2.$$

It follows that

$$x_2x_1 = \pi(\phi_{\tau_2,u_2}\exp(\tau_2 Y)\phi_{\tau_1,u_1}\exp(\tau_1 Y)).$$

Furthermore,

$$\begin{aligned}\phi_{\tau_1,u_1}(\exp((\tau_2+\tau_1)Y)) &= \phi_{\tau_1,u_1}\varphi_{\tau_1}(\exp((\tau_2+\tau_1)Y))\\ &= \phi_{\tau_1,u_1}\exp((\tau_2+\tau_1)Y)g_1 = (\phi_{\tau_1,u_1}\exp(\tau_1 Y))\exp(\tau_2 Y)g_1,\end{aligned}$$

where $g_1 \in R$. By considering the concatenation $u \in \mathcal{U}$ between u_1 and u_2 we obtain

$$\begin{aligned}\phi_{\tau_2+\tau_1,u}(\exp((\tau_2+\tau_1)Y)) &= \phi_{\tau_2,\Theta_{\tau_1}u}(\phi_{\tau_1,u}(\exp((\tau_2+\tau_1)Y)))\\ &= \phi_{\tau_2,u_2}\varphi^Y_{\tau_2}(\phi_{\tau_1,u_1}(\exp((\tau_2+\tau_1)Y)))g_2\\ &= \phi_{\tau_2,u_2}\exp(\tau_2 Y)\phi_{\tau_1,u_1}\exp(\tau_1 Y)g,\end{aligned}$$

where $g = \varphi^Y_{\tau_2}(g_1)g_2 \in R$.

Therefore

$$\phi_{\tau_2,u_2}\exp(\tau_2 Y)\phi_{\tau_1,u_1}\exp(\tau_1 Y) \in G^0 \cap \mathcal{A}_{\tau_2+\tau_1}(\exp(\tau_2+\tau_1)Y)g^{-1}.$$

Since $(\exp(\tau_2+\tau_1)Y)g^{-1} \in G^0$, we obtain

$$G^0 \cap \mathcal{A}_{\tau_2+\tau_1}(\exp(\tau_2+\tau_1)Y)g^{-1} = (G^0 \cap \mathcal{A}_{\tau_2+\tau_1})(\exp(\tau_2+\tau_1)Y)g^{-1}.$$

By taking the projection

$$x_2x_1 \in \pi((G^0 \cap \mathcal{A}_{\tau_2+\tau_1})\exp((\tau_2+\tau_1)Y)) = \mathcal{S}_{\tau_2+\tau_1} \subset \mathcal{S},$$

showing that $\mathcal{S}$ is a semigroup of L.

Moreover, if we consider $X = \tau_0\pi_*(Y) \in \mathfrak{l}$, the fact that we are assuming that $e \in \operatorname{int}\mathcal{A}_{\tau_0}$ for some $\tau_0 > 0$ led us to conclude that

$$\exp_L(X) = \pi(\exp(\tau_0 Y)) \in \pi((\operatorname{int}\mathcal{A}_{\tau_0} \cap G^0)\exp(\tau_0 Y)) \subset \operatorname{int}\mathcal{S}$$

which implies that $\mathcal{S}$ has a nonempty interior in L.

Now we are now able to extend Proposition 3.6 as follows.

PROPOSITION 3.7. *Let G be a connected Lie group with finite semisimple center. If $e \in \operatorname{int}\mathcal{A}_{\tau_0}$, for some $\tau_0 > 0$, then $G^0 \subset \mathcal{A}$.*

Proof. Let us consider $\mathcal{S}$ and L defined as above. We only have two topological possibilities for the semisimple Lie group L:

1. L is compact: It is a well-known fact that any semigroup with nonempty interior of a compact group contains the identity component of the group. Being that L is also connected we must have, by our assumptions, that $\mathcal{S} = L$.
2. L is noncompact: By [13, Corollary 3.15] we have the decomposition $X = X_S + X_N$, where $\operatorname{ad}_{\mathfrak{l}}(X_S)$ is semisimple[1] and $\operatorname{ad}_{\mathfrak{l}}(X_N)$ is nilpotent. Since $\mathcal{D}$ restricted to $\mathfrak{g}^0$ has only eigenvalues with zero real part, the same holds for $\operatorname{ad}_{\mathfrak{l}}(X_S)$ and, therefore, $\operatorname{Ad}_L(\exp_L X) = \mathrm{e}^{\operatorname{ad}_{\mathfrak{l}}(X_S)}$ is an elliptic element, that is, all its eigenvalues have modulus 1. By [10, Theorem 7.2] and the fact that L has finite center we have that

$$K := \operatorname{cl}\{\exp(tX_S), t \in \mathbb{R}\}$$

[1]That is, its extension to the complexification of $\mathfrak{l}$ is diagonalizable.

is a compact subgroup of L. Let us consider the subset

$$K_{\mathcal{S}} := \{x \in K; \exists \tau \in \mathbb{R} \text{ such that } x \exp(\tau X_N) \in \operatorname{int} \mathcal{S}\}.$$

By the continuity of the product and the fact that $\exp X \in \operatorname{int} \mathcal{S}$ we have that $K_{\mathcal{S}}$ is a subset with nonempty interior in K. Moreover, for any $x_1, x_2 \in K_{\mathcal{S}}$ there exists $\tau_1, \tau_2 \in \mathbb{R}$ such that $x_i \exp(\tau_i X_N) \in \operatorname{int} \mathcal{S}$, $i = 1, 2$. Since any element of K commutes with $\exp(tX_N)$ for all $t \in \mathbb{R}$, we get that

$$x_1 x_2 \exp((\tau_1 + \tau_2) X_N) = (x_1 \exp(\tau_1 X_N))(x_2 \exp(\tau_2 X_N)) \in \operatorname{int} \mathcal{S}$$

which shows that $K_{\mathcal{S}}$ is a semigroup of K. By the same argument as above, we have that K_S contains the identity component of K and so there exists $\tau \in \mathbb{R}$ such that $\exp(\tau X_N) \in \operatorname{int} \mathcal{S}$. The assumption that G has a finite semisimple center implies that L has a finite center and, since $\operatorname{ad}_{\mathfrak{l}}(\tau X_N)$ is nilpotent, we have by Lemma 2.8 that $\mathcal{S} = L$.

Therefore, in either case, we have that $\mathcal{S} = L$. Let us consider

$$\mathfrak{h} := \{Z \in \mathfrak{g}^0; \ \ \mathcal{D}(Z) \in \mathfrak{r}\}.$$

Since $\mathcal{D}$ is a derivation of $\mathfrak{g}^0$ it holds that $\mathfrak{h}$ is a $\mathcal{D}$-invariant Lie subalgebra such that $\mathcal{D}(\mathfrak{h}) \subset \mathfrak{r}$. Moreover, the fact that $\mathcal{D}(Z) - \operatorname{ad}(Y)Z \in \mathfrak{r}$ for any $Z \in \mathfrak{g}^0$ implies that $Y \in \mathfrak{h}$.

Since $R \subset G^0$ is a φ-invariant connected subgroup, Proposition 3.6 implies that $R \subset \mathcal{A}$. Therefore, by Proposition 3.5 it holds that $H \subset \mathcal{A}$, where $H \subset G^0$ is the connected subgroup with Lie algebra $\mathfrak{h}$. By Lemma 3.4 we get that

$$\mathcal{A}_\tau \exp(\tau Y) \subset \mathcal{A} \exp(\tau Y) \subset \mathcal{A}, \quad \tau > 0.$$

In particular, $\mathcal{S} \subset \pi(\mathcal{A} \cap G^0)$ which implies that $G^0 \subset (\mathcal{A} \cap G^0)R$. Applying Lemma 3.4 again, we obtain

$$G_0 = (\mathcal{A} \cap G^0)R \subset \mathcal{A}R \subset \mathcal{A}$$

ending the proof. ∎

We are now in a position to show a generalization of [5, Theorem 3.7].

THEOREM 3.8. *Let G be a connected Lie group with finite semisimple center. If $e \in \operatorname{int} \mathcal{A}_{\tau_0}$ for some $\tau_0 > 0$, then $G^{+,0} \subset \mathcal{A}$.*

Proof. Let $g \in G^+$. Since $\mathrm{e}^{t\mathcal{D}}|_{\mathfrak{g}^+}$ has only eigenvalues with positive real part and $\mathcal{A}$ is open, there is $t > 0$ quite large such that $\varphi_{-t}(g) \in \mathcal{A}$ and, consequently, $g \in \varphi_t(\mathcal{A}) \subset \mathcal{A}$. Since $g \in G^+$ is arbitrary, we conclude that $G^+ \subset \mathcal{A}$. Also, by Proposition 3.7 we have that $G^0 \subset \mathcal{A}$ and since $G^{+,0} = G^+ G^0$, Lemma 3.4 implies that $G^{+,0} \subset \mathcal{A}$, as desired. ∎

Remark 3.9. The proof of our main result uses the notion of a reverse time system. Hence, we will show some relationship between both systems.

By considering the linear vector field $\mathcal{X}^*$ on G whose flow is given by $\varphi^*_\tau := \varphi_{-\tau}$ it is straightforward to see that the derivation $\mathcal{D}^*$ associated with $\mathcal{X}^*$ satisfies $\mathcal{D}^* = -\mathcal{D}$. The Lie subalgebras and Lie subgroups induced by the derivation $\mathcal{D}^*$ and $\mathcal{D}$ are related as follows:

$$\mathfrak{g}^+_* = \mathfrak{g}^-, \quad \mathfrak{g}^-_* = \mathfrak{g}^+, \quad \text{and} \quad \mathfrak{g}^0_* = \mathfrak{g}^0,$$

and

$$G_*^+ = G^-, \quad G_*^- = G^+, \quad \text{and} \quad G_*^0 = G^0.$$

Moreover, if we consider the linear system (3) with drifts $\mathcal{X}^*$, $\mathcal{X}$, and the same right invariant vector fields, their respective solutions are related by $\phi^*_{t,u}(g) = \phi_{-t,u}(g)$ which implies that $\mathcal{A}^*_\tau = \varphi_{-\tau}((\mathcal{A}_\tau)^{-1})$.

With the previous analysis we are now able to prove the main result of the paper.

THEOREM 3.10. *Let G be a Lie group with finite semisimple center. Then, the linear system* (3) *on G is controllable if $e \in \operatorname{int} \mathcal{A}_{\tau_0}$ for some $\tau_0 > 0$ and $\mathcal{D}$ has only eigenvalues with zero real part.*

Proof. Since $\mathcal{A}^*_\tau = \varphi_{-\tau}((\mathcal{A}_\tau)^{-1})$ for any $\tau > 0$ we have that $e \in \operatorname{int} \mathcal{A}_{\tau_0}$ for some $\tau_0 > 0$ if and only if $e \in \operatorname{int} \mathcal{A}^*_{\tau_0}$. By Theorem 3.8 it follows that $G^{+,0} \subset \mathcal{A}$ and $G^{-,0} = G_*^{+,0} \subset \mathcal{A}^*$. Since $\mathcal{D}$ has only eigenvalues with zero real part $G^{+,0} = G^0 = G^{-,0}$ and, hence, $G = G^0 = \mathcal{A} \cap \mathcal{A}^*$ which implies that the linear system is controllable. ∎

Remark 3.11. Although the condition $e \in \operatorname{int} \mathcal{A}_{\tau_0}$ for some $\tau_0 > 0$ is usually stronger than the openness of $\mathcal{A}$ (as required for the solvable case in [5]), for restricted systems [4, Lemma 4.5.2] it implies that both conditions are actually equivalent.

Example 3.12. We use the classical linear system, consider $G = \mathbb{R}^d$ and the dynamic on G determined by

$$\dot{x}(t) = Ax(t) + Bu(t); \quad A \in \mathbb{R}^{d\times d}, B \in \mathbb{R}^{d\times m}, u \in \mathcal{U}.$$

Since the right (left) invariant vector fields on the abelian Lie group $\mathbb{R}^d$ are given by constant vectors we can write the system as

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t) b_i, \qquad B = (b_1|b_2|\cdots|b_m),$$

showing that it is a linear system in the sense of (3). Of course the set of derivations on $\mathbb{R}^d$ is the Lie algebra of the real matrices of order d.

In the book *The Dynamics of Control* [4] the authors proved that for a restricted linear system on the Euclidean space $\mathbb{R}^d$ satisfying the Kalman condition, there exists one and only one control set C with nonempty interior. Recall that a control set C is a maximal controlled invariant set such that $C \subset \operatorname{cl}(\mathcal{A}(g))$ for every $g \in C$. The mentioned control set is given explicitly by

$$C = \operatorname{cl}(\mathcal{A}) \cap \mathcal{A}^*.$$

If we assume that A has only eigenvalues with zero real part, then $\mathbb{R}^d = (\mathbb{R}^d)^0 = \mathcal{A} \cap \mathcal{A}^*$. It turns out that $C = \mathbb{R}^d$ and the system is controllable. Therefore, our main result is also a generalization for restricted linear systems from Euclidean spaces to Lie groups with finite semisimple center.

Example 3.13. We use a special class of linear control systems. Consider two Lie algebras $\mathfrak{e}$ and $\mathfrak{h}$ with respective connected Lie groups E and H, and the direct product $\mathfrak{g} = \mathfrak{e} \times \mathfrak{h}$ with the canonical product Lie algebra structure. Let $X^{\mathfrak{e},1}, \dots, X^{\mathfrak{e},m}$ be right invariant vector fields of $\mathfrak{e}$ and $X^{\mathfrak{h},1}, \dots, X^{\mathfrak{h},n}$ be a basis of right invariant vector fields of $\mathfrak{h}$. Take any derivation $\mathcal{D}^{\mathfrak{e}}$ of $\mathfrak{e}$ with associated linear vector field $\mathcal{X}^{\mathfrak{e}}$.

Therefore,

$$\dot{g}(t) = X^{\mathfrak{e}}(e(t)) + \sum_{j=1}^{m} u_j(t) X^{\mathfrak{e},j}(e(t)) + \sum_{j=1}^{n} v_j(t) X^{\mathfrak{h},j}(h(t))$$

is a linear control system that we call an $\mathfrak{h}$-homogeneous linear system on $G = E \times H$. Here $g(t) = (e(t), h(t))$ for any $t \in \mathbb{R}$ and $w = (u, v)$ belongs to $\mathcal{U} \subset L^\infty(\mathbb{R}, \Omega \subset \mathbb{R}^m \times \mathbb{R}^n)$. Just observe that $\mathcal{D} = (\mathcal{D}^{\mathfrak{e}}, 0)$ is a derivation of $\mathfrak{g}$ and $\mathfrak{g}^0 = \mathfrak{g}$.

Remark 3.14. Obviously, an $\mathfrak{h}$-homogeneous linear system on G is controllable if and only if the linear system is controllable when $v = 0$.

In Jurdjevic's book [16] the kinematic equations of a rolling two dimensional sphere S over a revolving plane around the z-axis with constant angular velocity ω, are established as follows: The assumption that the sphere rolls without slipping implies that the movement is described by the center of S as a curve $e(t)$ on $\mathbb{R}^2$ and by the family of orthogonal matrices $g(t)$, i.e., elements of the orthogonal group $SO(3)$, which transform the attached frame on $S(t)$ to the canonical coordinates on R^3 at the origin. So, the configuration space is the Lie group $G = \mathbb{R}^2 \times \mathrm{SO}(3)$. Then, $\mathfrak{e} = \mathbb{R}^2$, $\mathfrak{h} = \mathfrak{so}(3)$, and $\mathfrak{g} = \mathbb{R}^2 \times \mathfrak{so}(3)$, where the second component denotes the Lie algebra of a skew-symmetric matrix of order three. After this analysis, the following linear system is obtained:

$$\dot{g}(t) = X^{\mathfrak{e}}(e(t)) + \sum_{j=1}^{3} v_j(t) X^{\mathfrak{h},j}(h(t)),$$

an $\mathfrak{so}(3)$-homogeneous linear system in our context. Precisely

$$X^{\mathfrak{h},1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; X^{\mathfrak{h},2} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; X^{\mathfrak{h},3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\mathcal{D}^{\mathfrak{e}} = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Of course $\mathcal{D} = (\mathcal{D}^{\mathfrak{e}}, 0)$ does not act on $\mathfrak{h}$. So, in order to study the controllability property of the rolling sphere it is necessary to consider an extension of the system, an $\mathfrak{so}(3)$-homogeneous linear system on G as follows:

$$\dot{g}(t) = \mathcal{X}^{\mathfrak{e}}(e(t)) + ub + \sum_{j=1}^{3} v_j(t) X^{\mathfrak{h},j}(h(t)) \text{ with } \Omega = \mathbb{R} \times \mathbb{R}^3, \tag{7}$$

where $b \in \mathbb{R}^2$ is a vector such that the system satisfies the Kalman rank condition for $v = 0$. For example if $b = e_1$ we obtain

$$\mathbb{R}^2 = \mathrm{Span}\{b, \mathcal{D}^{\mathfrak{e}}(b)\}.$$

The Lie group G has finite semisimple center, the system satisfies the ad-rank condition, and $\mathcal{D}$ is a $\mathfrak{g}$-derivation with $\mathrm{Spec}(\mathcal{D}) = \{0, \pm\omega i\}$. Thus, our main theorem applies to the $\mathfrak{so}(3)$-homogeneous linear system showing that (7) is controllable.

Example 3.15. *Let $G = SL(2,\mathbb{R})$ be the Lie group of determinant* 1 *matrices with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, the matrices of* 0 *trace, and consider the basis $\{X, Y, Z\}$ of $\mathfrak{sl}(2,\mathbb{R})$ given by*

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and } \; Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

that satisfies

$$[X, Y] = 2Y, \quad [X, Z] = -2Z, \quad \text{and} \quad [Y, Z] = X.$$

(i) *In* [1] *the authors show the existence of a local controllable linear control system from the identity on G such that the accessibility set $\mathcal{A}$ is not a semigroup and thus is not controllable (see* [7, Proposition 7]*). Precisely, let*

$$\mathcal{D} = \operatorname{ad}(X), \; H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and consider the linear system

$$\dot{g}(t) = \mathcal{X}(g(t)) + uH(g(t)), \quad u \in \mathcal{U},$$

where $\mathcal{X} = \mathcal{X}^{\mathcal{D}}$ is the linear vector field associated with $\mathcal{D}$. We have

$$\operatorname{Span}\left\{H = X + Y + Z, \mathcal{D}(H) = 2(Y - Z), \mathcal{D}^2(H) = 4(Y + Z)\right\} = \mathfrak{sl}(2,\mathbb{R})$$

and the rank condition follows. On the other hand, the center of $\mathrm{SL}(2,\mathbb{R}) = \mathbb{Z}_2$ *and we observe that $\mathcal{D}$ has eigenvalues* $-2, 0,$ *and* 2.

(ii) *Consider the derivation*

$$\mathcal{D} = \operatorname{ad}(Y) = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It induces the linear vector field $\mathcal{X} = \mathcal{X}^{\mathcal{D}}$ on G. Consider the linear system

$$\dot{g}(t) = \mathcal{X}(g(t)) + uZ(g), \quad u \in \mathcal{U}.$$

Since

$$\operatorname{Span}\{Z, \mathcal{D}(Z) = -X, \mathcal{D}^2(Z) = -2Y\} = \mathfrak{sl}(2,\mathbb{R})$$

and the only eigenvalue of $\mathcal{D}$ is zero, we have by Theorem 3.10 *that the linear system is controllable.*

Conclusions and further work. The present paper shows that if for some $\tau_0 > 0$ we have that the identity element $e \in G$ belongs to the interior of the set $\mathcal{A}_\tau$, then the controllability of the system is obtained if the derivation $\mathcal{D}$ associated with the drift of the system has only eigenvalues with zero real part.

For restricted systems on nilpotent Lie groups it is proved in [6] that the above condition is also necessary, that is, if the system is controllable then $\mathcal{D}$ has only eigenvalues with zero real part. The next natural step would be to analyze if the above is also true for general Lie groups. Another natural direction would be to analyze the control sets of linear systems and see the drift's influence on such sets. In [3] the authors give a full description of such sets for solvable Lie groups. The intention is to understand such sets when G is a semisimple Lie group and use it to analyze the general case.

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