



On the structural properties of the bounded control set of a linear control system

Víctor Ayala and Adriano Da Silva 

Abstract. The present paper shows that the closure of the bounded control set of a linear control system contains all the bounded orbits of the system. As a consequence, we prove that the closure of this control set is the continuous image of the cartesian product of the set of control functions by the central subgroup associated with the drift of the system.

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1. Introduction

Let $\mathcal{X}, Y^1, \dots, Y^m$ be smooth vector fields on a connected finite dimensional differentiable manifold M . A control system Σ_M on M is determined by a family of controlled differential equations

$$\dot{x}(t) = \mathcal{X}(x(t)) + \sum_{j=1}^m u_j(t) Y^j(x(t)), \quad (\Sigma_M)$$

which allows changing the behavior of \mathcal{X} according to the control vectors Y^1, Y^j, \dots, Y^m , and the set \mathcal{U} of admissible control functions

$$\mathcal{U} = \{u : \mathbb{R} \rightarrow \mathbb{R}^m : u \text{ is measurable with } u(t) \in \Omega \text{ a.e.}\},$$

where $\Omega \subset \mathbb{R}^m$ is a closed and convex set with $0 \in \text{int } \Omega$.

The controllability notion of Σ_M is one of the most relevant properties of the system. It allows connecting any two points of M through a concatenation of integral curves of Σ_M , in positive time. For instance, a necessary condition to solve any optimization problem between two states, like a time optimal or minimum energy trajectory, is the existence of a control $u \in \mathcal{U}$ such that the

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corresponding solution connects these two states. Despite the existence of nice examples of controllable systems, this global property is hard to be satisfied in general. A well known example is the class of linear control system on Euclidean spaces $\Sigma_{\mathbb{R}^n}$, where $\mathcal{X} = A$ is a matrix of order n , and $Y^j = b^j \in \mathbb{R}^n$ are constant vector fields. In this context, the Kalman rank condition characterizes controllability. However, to obtain that, you need to consider $\Omega = \mathbb{R}^m$, which is far from real life. A more realistic approach considers the case when $\Omega \subset \mathbb{R}^m$ is a compact subset, and the notion of control sets for Σ_M , which are roughly speaking maximal subsets of M where controllability holds. Recently, several papers are focused on studying the controllability and the existence, uniqueness and topological properties of the control sets of linear control systems on connected Lie groups. For this case, the drift \mathcal{X} is a linear vector field in the sense that its flow $\{\varphi_t\}_{t \in \mathbb{R}}$ is a 1-parameter group of automorphisms of G , and the control vectors are elements of the Lie algebra. It turns out that many properties of the system depend strongly of the dynamical behavior of \mathcal{X} . In fact, the flow of \mathcal{X} induce connected subgroups, called unstable, central and stable, which have a nice relationship with the set of the reachable points of the system and hence with its controllability and control sets (see [1, 2, 5, 7, 9, 10]).

Our aim here is to study the structural properties of the control set of a linear control system that contains the identity of the group. If such control set is bounded and contains the identity element in its interior, our main result shows that all bounded orbits of Σ_G are contained in its closure. As a consequence, such closure is the continuous image of the cartesian product of \mathcal{U} by the central subgroup.

The article is organized as follows: In Sect. 2 we introduce the decompositions induced by automorphisms and the notion of linear control systems and state the main results relating both notions. We finish the section with some new results needed in the prove of our main results. Section 3 is used to introduce and prove our main result. We prove here several properties concerning bounded orbits and the central subgroup of \mathcal{X} which implies the main result. Moreover, such properties also allows us to create a continuous function from the cartesian of \mathcal{U} by the central subgroup in the closure of the control set, showing that under our hypothesis the control set is the continuous image of this cartesian.

Notations

In the present paper, all the Lie groups and algebras considered are real and finite dimensional. Let G be a connected Lie group. A subgroup $H \subset G$ is said to be trivial if $H = \{e\}$, where $e \in G$ stands for the identity element of G . By L_g and R_g we denote, respectively, the left and right-translations by g . The conjugation of g is the map $C_g := L_g \circ R_{g^{-1}}$. The center $Z(G)$ of G is the set of elements in G that satisfy $C_g = \text{id}_G$. If $f : G \rightarrow H$ is a differentiable map between Lie groups, the differential of f at x is denoted by $(df)_x$.

2. Preliminaries

This section is devoted to present the background needed in order to establish our main results. We introduce here the decompositions induced on Lie groups and algebras by their automorphisms and also the notion of linear control systems and their control sets. The main results needed are also stated here. In the end of the section we proof some new results that will be necessary at Sect. 3.

2.1. Decomposition by automorphisms

Let \mathfrak{g} be a finite dimensional Lie algebra and $\rho \in \text{Aut}(\mathfrak{g})$ an automorphism. For any eigenvalue $\alpha \in \mathbb{C}$ of ρ , the real generalized eigenspaces of ρ associated with α are given by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : (\rho - \alpha I)^n X = 0 \text{ for some } n \geq 1\}, \quad \text{if } \alpha \in \mathbb{R} \text{ and}$$

$$\mathfrak{g}_\alpha = \text{span}\{\text{Re}(v), \text{Im}(v); v \in \bar{\mathfrak{g}}_\alpha\}, \quad \text{if } \alpha \in \mathbb{C}$$

where $\bar{\mathfrak{g}} = \mathfrak{g} + i\mathfrak{g}$ is the complexification of \mathfrak{g} and $\bar{\mathfrak{g}}_\alpha$ the generalized eigenspace of $\bar{\rho} = \rho + i\rho$, the extension of ρ to $\bar{\mathfrak{g}}$.

We define the **unstable**, **central** and **stable** ρ -invariant subspaces of \mathfrak{g} , respectively, by

$$\mathfrak{g}^+ = \bigoplus_{\alpha: |\alpha| > 1} \mathfrak{g}_\alpha, \quad \mathfrak{g}^0 = \bigoplus_{\alpha: |\alpha| = 1} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{g}^- = \bigoplus_{\alpha: |\alpha| < 1} \mathfrak{g}_\alpha.$$

Following [4] the fact that $[\bar{\mathfrak{g}}_\alpha, \bar{\mathfrak{g}}_\beta] \subset \bar{\mathfrak{g}}_{\alpha\beta}$ when $\alpha\beta$ is an eigenvalue of ρ and zero otherwise implies that $\mathfrak{g}^+, \mathfrak{g}^0, \mathfrak{g}^-$ are in fact ρ -invariant Lie subalgebras with $\mathfrak{g}^+, \mathfrak{g}^-$ nilpotent ones. Moreover, \mathfrak{g} is decomposed as $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$.

At the group level, let G be a connected Lie group with Lie algebra \mathfrak{g} . For any automorphism $\psi \in \text{Aut}(G)$ the **dynamical subgroups** of G induced by ψ are the Lie subgroups defined as follows: Since $(d\psi)_e$ is an automorphism of \mathfrak{g} it induces subalgebras $\mathfrak{g}^+, \mathfrak{g}^-$ and \mathfrak{g}^0 of \mathfrak{g} defined as previously. We can consider then connected subgroup G^+, G^- and G^0 with the Lie algebras $\mathfrak{g}^+, \mathfrak{g}^-$ and \mathfrak{g}^0 , respectively. As previously, the subgroups G^+, G^0 and G^- are called the **unstable**, **central** and **stable** subgroups of G , respectively.

Also, we denote by $G^{+,0}$ and $G^{-,0}$ the connected subgroups of G with Lie algebras given by $\mathfrak{g}^{+,0} := \mathfrak{g}^+ \oplus \mathfrak{g}^0$ and $\mathfrak{g}^{-,0} := \mathfrak{g}^- \oplus \mathfrak{g}^0$, respectively, and by $G^{+,-}$ the product $G^{+,-} = G^+ G^-$.

We say that G is **decomposable** by φ if

$$G = G^{+,0} G^- = G^{-,0} G^+ = G^{+,-} G^0.$$

If $G^0 = \{e\}$ we say that ψ is **hyperbolic**. In particular, if ψ is hyperbolic, $G = G^{+,-} := G^+ G^-$ is decomposable.

The next proposition summarizes the main properties of the previous subgroups. Its proof can be found in [4, Proposition 3.4].

Proposition 2.1. *For the dynamical subgroups of an automorphism ψ of G , it holds:*

1. *If G is solvable or if G^0 is a compact subgroup, then G is decomposable;*
2. *If ψ is hyperbolic, then G is a nilpotent Lie group.*

The previous decompositions for single automorphisms of G can be extended to flows of automorphisms as follows: Let $\{\varphi_t\}_{t \in \mathbb{R}}$ be a 1-parameter flow of automorphisms on G . By derivation, $\{(d\varphi_t)_e, t \in \mathbb{R}\}$ is a 1-parameter subgroup of $\text{Aut}(\mathfrak{g})$ and hence, there exists a derivation \mathcal{D} of \mathfrak{g} such that $(d\varphi_t)_e = e^{t\mathcal{D}}$ for any $t \in \mathbb{R}$. In particular, the subalgebras induced by $(d\varphi_t)_e$ coincide, for any $t \in \mathbb{R}$ with the sum of the generalized real eigenspaces of \mathcal{D} associated to eigenvalues with positive, zero and negative real parts. Therefore, we can define the dynamical subgroups of $\{\varphi_t\}_{t \in \mathbb{R}}$ to be the dynamical subgroups of φ_τ for some (and hence any) $\tau \in \mathbb{R}$. The properties of these subgroups were studied in previous works ([2, 3, 5, 7]) and their topological properties are much nicer than the ones associated with single automorphisms as the next result shows.

Proposition 2.2. *Let G^+, G^0 and G^- be the dynamical subgroups of the 1-parameter flow of automorphisms $\{\varphi_t\}_{t \in \mathbb{R}}$. It holds:*

1. G^+, G^0 and G^- are closed and have trivial intersections;
2. G^+ and G^- are simply connected nilpotent Lie subgroups;
3. Let N be the nilradical of G and assume that G^0 is a compact subgroup.

Then

- 2.1. $G^{+,-} \subset N$;
- 2.2. $N^0 = N \cap G^0$ is a compact, connected normal subgroup of G .

Remark 2.3. Following [3, Proposition 2.2] if G^+, G^0 and G^- are dynamical subgroups of a 1-parameter subgroups of automorphisms and G is decomposable, any element $g \in G$ can be written uniquely as

$$g = hk, \quad h \in G^{+,-}, k \in G^0.$$

In particular, such fact implies that $G^{+,-}$ is a closed submanifold of G .

The next example shows that the dynamical subgroups associated with single automorphisms does not need to be closed.

Example 2.4. Let us consider $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ the 2-dimensional torus. As a general result, the group of automorphisms $\text{Aut}(\mathbb{T}^2)$ is given by

$$\text{Aut}(\mathbb{T}^2) = \{A \in \text{Gl}(\mathbb{R}^2); AZ^2 = \mathbb{Z}^2\}.$$

In particular,

$$\psi = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \in \text{Aut}(\mathbb{T}^2), \quad \text{with} \quad \mathfrak{g}^+ = \mathbb{R}(1, \sqrt{2}) \quad \text{and} \quad \mathfrak{g}^- = \mathbb{R}(1, -\sqrt{2}).$$

Moreover, both G^+ and G^- are images of the well known irrational flows on \mathbb{T}^2 , and are henceforth not closed in \mathbb{T}^2 .

2.2. Linear control systems

Let G be a connected Lie group with Lie algebra \mathfrak{g} identified with the set of right-invariant vector fields of G . A **linear control system (LCS)** on G is given by a family of ordinary differential equations

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^m u_j(t)Y^j(g(t)), \quad (\Sigma_G)$$

where the **drift** \mathcal{X} is a linear vector field, that is, its associated flow $\{\varphi_t\}_{t \in \mathbb{R}}$ is a 1-parameter group of automorphisms, $Y^1, \dots, Y^m \in \mathfrak{g}$ and $u = (u_1, \dots, u_m) \in \mathcal{U}$. The set \mathcal{U} of admissible **control functions** is given by

$$\mathcal{U} = \{u : \mathbb{R} \rightarrow \mathbb{R}^m : u \text{ is measurable with } u(t) \in \Omega \text{ a.e.}\},$$

where Ω is a compact and convex subset of \mathbb{R}^m with $0 \in \text{int } \Omega$. Endowed with the weak*-topology of $L^\infty(\mathbb{R}, \mathbb{R}^m) = L^1(\mathbb{R}, \mathbb{R}^m)^*$ the set \mathcal{U} is a compact metrizable space and the **shift flow**

$$\theta : \mathbb{R} \times \mathcal{U} \rightarrow \mathcal{U}, \quad (t, u) \mapsto \theta_t u = u(\cdot + t),$$

is a continuous dynamical system (see [6, Section 4.2]). By \mathcal{U}_{per} we denote the set of periodic points of θ in \mathcal{U} , that is, $u \in \mathcal{U}_{\text{per}}$ if there exists $\tau > 0$ such that $\theta_\tau u = u$. By [6, Lemma 4.2.2] it holds that \mathcal{U}_{per} is dense in \mathcal{U} .

For any $g \in G$ and $u \in \mathcal{U}$, the solution $t \mapsto \phi(t, g, u)$ of Σ_G is defined for the whole real line and satisfies the **cocycle property**

$$\phi(t + s, g, u) = \phi(t, \phi(s, g, u), \theta_s u)$$

for all $t, s \in \mathbb{R}$, $g \in G$, $u \in \mathcal{U}$. The **control flow** of the system is the skew-product flow

$$\Phi : \mathbb{R} \times \mathcal{U} \times G \rightarrow \mathcal{U} \times G, \quad (t, u, x) \mapsto \Phi_t(u, x) = (\theta_t u, \phi(t, x, u)).$$

We also write $\phi_{t,u} : G \rightarrow G$ for the map $g \mapsto \phi(t, g, u)$. In the particular case of LCS, the intrinsic relations between the vector fields involved and the group structure implies the following relation

$$\phi_{\tau,u} \circ R_g = R_{\varphi_\tau(g)} \circ \phi_{\tau,u}, \quad \text{for any } \tau \in \mathbb{R}, g \in G, \tag{1}$$

where $\{\varphi_t\}_{t \in \mathbb{R}}$ is the flow of \mathcal{X} . We define the set of points reachable from $g \in G$ at time $\tau > 0$ and the **reachable set of g** , respectively, by

$$\mathcal{A}_\tau(g) := \{\phi_{\tau,u}(g), \quad u \in \mathcal{U}\} \quad \text{and} \quad \mathcal{A}(g) := \bigcup_{\tau > 0} \mathcal{A}_\tau(g).$$

We denote by $\mathcal{A}_\tau^*(g)$ and $\mathcal{A}^*(g)$ the respective reachable sets in negative time.

Next we define control sets for a LCS.

Definition 2.5. A nonempty subset $\mathcal{C} \subset G$ is said to be a **control set** of Σ_G if it is maximal (w.r.t. set inclusion) satisfying

- (i) For each $g \in \mathcal{C}$ there exists $u \in \mathcal{U}$ such that $\phi(\mathbb{R}^+, g, u) \subset \mathcal{C}$;
- (ii) For any $g \in \mathcal{C}$ it holds that $\mathcal{C} \subset \overline{\mathcal{A}(g)}$.

Since the identity $e \in G$ is a singularity of \mathcal{X} and $0 \in \text{int } \Omega$, there exists a control set \mathcal{C} of Σ_G containing the identity. Moreover, $e \in \text{int } \mathcal{C}$ if and only if $\mathcal{A}(e)$ (or equivalently $\mathcal{A}^*(e)$) is open. In this case, it holds that

$$\mathcal{C} = \overline{\mathcal{A}(e)} \cap \mathcal{A}^*(e).$$

The next result relates the dynamics of a LCS with the dynamical subgroups associated with the flow of \mathcal{X} (see [7, Lemma 3.1] and [2, Theorem 3.8]).

Theorem 2.6. *For a LCS Σ_G , it holds:*

1. If $\varphi_t(g) \in \mathcal{A}(e)$ for all $t \in \mathbb{R}$ then $\mathcal{A}(e)g \subset \mathcal{A}(e)$;
2. If $\varphi_t(g) \in \mathcal{A}^*(e)$ for all $t \in \mathbb{R}$ then $\mathcal{A}^*(e)g \subset \mathcal{A}^*(e)$;
3. If $\mathcal{A}(e)$ is open, then
 - 3.1. $G^{+,0} \subset \mathcal{A}(e)$ and $G^{-,0} \subset \mathcal{A}^*(e)$;
 - 3.2. If G is decomposable, \mathcal{C} is the only control set with nonempty interior of Σ_G ;
 - 3.3. G^0 is a compact subgroup if and only if \mathcal{C} is bounded.

2.3. Some results

In this section, we obtain some results related to the dynamical subgroups of automorphisms that will be necessary in the proof of the main result.

Proposition 2.7. *If $\psi \in \text{Aut}(G)$ is hyperbolic, the map*

$$f_\psi : g \in G \mapsto g\psi(g^{-1}) \in G$$

is an onto local diffeomorphism. Furthermore, f_ψ is injective if and only if the only fixed point of ψ is the identity element of G .

Proof. The differentiability of f_ψ follows directly from its definition. Moreover, for any G it holds that

$$\begin{aligned} f_\psi(hg) &= hg\psi((hg)^{-1}) = h(g\psi(g^{-1}))\psi(h^{-1}) = hf_\psi(g)\psi(h^{-1}) \implies f_\psi \circ L_h \\ &= L_h \circ R_{\psi(h^{-1})} \circ f_\psi, \end{aligned} \tag{2}$$

and since left and right translations are diffeomorphisms we get in particular that f_ψ has constant rank. On the other hand,

$$\begin{aligned} (df_\psi)_e X &= \frac{d}{dt} \Big|_{t=0} e^{tX} \psi(e^{-tX}) \\ &= \left((dL_{e^{tX}})_{\psi(e^{-tX})} \frac{d}{dt} \psi(e^{-tX}) + (dR_{\psi(e^{-tX})})_{e^{tX}} \frac{d}{dt} e^{tX} \right) \Big|_{t=0} = X - (d\psi)_e X \end{aligned}$$

and since ψ is hyperbolic, we have that $(d\psi)_e$ does not have 1 as an eigenvalue. Therefore, $(df_\psi)_e(\mathfrak{g}) = \mathfrak{g}$ and hence f_ψ is a local diffeomorphism.

The surjectiveness of f_ψ is proved by induction on the dimension of G . If G is an abelian group, for any $g, h \in G$, we get from Eq. (2) that

$$f_\psi(gh) = f_\psi(g)f_\psi(h)$$

showing that f_ψ is an homomorphism. Being f_ψ a local diffeomorphism we must have that $f_\psi(G)$ is a subgroup with nonempty interior in G and hence $f_\psi(G) = G$. In particular, the result is true when $\dim G = 1$.

Assume now that the result is true for any Lie group G with $\dim G < n$ and consider G to be a nonabelian Lie group of dimension n and $\psi \in \text{Aut}(G)$ to be hyperbolic. Since G is a nilpotent Lie group, the connected component $Z(G)_0$ of its center is a closed, connected, nontrivial, normal subgroup of G and so $\hat{G} = G/Z(G)_0$ is a connected Lie group satisfying $\dim \hat{G} = \dim G - \dim Z(G)_0 < n$. Let then $\hat{\psi} \in \text{Aut}(\hat{G})$ such that

$$\hat{\psi} \circ \pi = \pi \circ \psi, \quad \text{where } \pi : G \rightarrow \hat{G} \text{ is the canonical projection.}$$

A simple calculation shows that the associated map

$$f_{\hat{\psi}} : \hat{g} \in \hat{G} \mapsto \hat{g}\hat{\psi}(\hat{g}^{-1}) \in \hat{G}$$

also verifies $f_{\hat{\psi}} \circ \pi = \pi \circ f_{\psi}$. Since $\pi(G^{+,-}) = \hat{G}^{+,-}$ and $\pi(G^0) = \hat{G}^0$ it holds that $\hat{\psi}$ is hyperbolic and by the inductive hypothesis

$$\pi(f_{\psi}(G)) = f_{\hat{\psi}}(\pi(G)) = f_{\hat{\psi}}(\hat{G}) = \hat{G} = \pi(G). \tag{3}$$

On the other hand, since $\dim Z(G)_0 < n$ and $\psi|_{Z(G)_0} \in \text{Aut}(Z(G)_0)$ is hyperbolic, by the abelian case we get

$$f_{\psi}|_{Z(G)_0} = f_{\psi}|_{Z(G)_0} \text{ is surjective.}$$

Therefore, for any $g \in G$ there are by (3) elements $x \in G$ and $h \in Z(G)_0$ such that $g = f_{\psi}(x)h$. Since $f_{\psi}|_{Z(G)_0}$ is surjective it holds that $h = f_{\psi}(y)$ for some $y \in Z(G)_0$ and so

$$g = f_{\psi}(x)h = f_{\psi}(x)f_{\psi}(y) = f_{\psi}(xy),$$

where for the last equality we used (2) and the fact that $y \in Z(G)_0$. Therefore, f_{ψ} is surjective as stated.

For the last assertion, note that

$$f_{\psi}(g) = f_{\psi}(h) \iff g\psi(g^{-1}) = h\psi(h^{-1}) \iff \psi(g^{-1}h) = g^{-1}h$$

and consequently f_{ψ} is injective if and only if $e \in G$ is only fixed point of ψ as stated. □

The next result shows that for the dynamical subgroups associated with a flow of automorphisms, there is a diffeomorphism between the sum of the stable and unstable subalgebras and their respective subgroups that conjugates the flow and its differential.

Proposition 2.8. *For any 1-parameter flow of automorphisms $\{\varphi_t\}_{t \in \mathbb{R}}$, the map*

$$f : \mathfrak{g}^+ \oplus \mathfrak{g}^- \rightarrow G^{+,-}, \quad X + Y \mapsto e^X e^Y,$$

is a diffeomorphism and

$$\forall t \in \mathbb{R}, \quad f \circ (d\varphi_t)_e = \varphi_t \circ f. \tag{4}$$

Proof. The differentiability of f comes direct from its definition. Moreover, by Proposition 2.2, the Lie subgroups G^+ and G^- associated with $\{\varphi_t\}_{t \in \mathbb{R}}$ are simply connected and hence the exponential map restricted to both \mathfrak{g}^+ and \mathfrak{g}^- are diffeomorphisms. Consequently, if we denote by $\log^{\pm} := \exp|_{\mathfrak{g}^{\pm}}^{-1}$, the map

$$g : G^{+,-} \rightarrow \mathfrak{g}^+ \oplus \mathfrak{g}^-, \quad g(xy) = \log^+(x) + \log^-(y),$$

is a diffeomorphism and satisfies

$$g(f(X + Y)) = g(e^X e^Y) = (\log^+(e^X) + \log^-(e^Y)) = X + Y,$$

which shows that f is a diffeomorphism.

For the last assertion, the fact that \mathfrak{g}^+ and \mathfrak{g}^- are $(d\varphi_t)$ -invariant implies

$$\begin{aligned} f((d\varphi_t)_e(X + Y)) &= f((d\varphi_t)_eX + (d\varphi_t)Y) = e^{(d\varphi_t)_eX} e^{(d\varphi_t)_eY} \\ &= \varphi_t(e^X)\varphi_t(e^Y) = \varphi_t(e^Xe^Y) = \varphi_t(f(X + Y)), \end{aligned}$$

ending the proof. □

The previous proposition implies the following lemma.

Lemma 2.9. *Let $\{\varphi_t\}_{t \in \mathbb{R}}$ be a 1-parameter group of automorphisms. For any compact subset $K \subset G$ and any $x \in G^{+,-}$ with $x \neq e$ there exists $t_0 > 0$ such that 0*

$$\varphi_t(x) \notin K \quad \text{for } t \geq t_0 \quad \text{or} \quad \varphi_t(x) \notin K \quad \text{for } t \leq -t_0.$$

Proof. By Proposition 2.8 any $x \in G^{+,-}$ can be written as $x = f(X + Y)$, where $X \in \mathfrak{g}^+$ and $Y \in \mathfrak{g}^-$. Moreover,

$$\varphi_t(x) = \varphi_t(f(X + Y)) = f((d\varphi_t)_e)(X + Y),$$

and since f is a diffeomorphism, it is enough to show the equivalent assertion for $(d\varphi_t)_e$ restricted to $\mathfrak{g}^{+,-}$. However, by [8], there exists $c, \mu > 0$ such that

$$|(d\varphi_t)_e v| \geq ce^{\mu t}|v|, \quad t > 0, v \in \mathfrak{g}^+ \quad \text{and} \quad |(d\varphi_t)_e v| \geq ce^{\mu t}|v|, \quad t < 0, v \in \mathfrak{g}^-,$$

which implies the assertion. □

3. The main result and its consequences

Throughout the whole section, we will assume that

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^m u_j(t)Y^j(g(t)), \quad (\Sigma_G)$$

is a LCS such that the subgroup G^0 associated with \mathcal{X} is a compact subgroup. Moreover, we will assume that the control set \mathcal{C} containing the identity element of G satisfies $e \in \text{int } \mathcal{C}$.

Our aim here is to study the structural properties of \mathcal{C} under the previous hypothesis.

3.1. The main result

From here on let us denote by ϱ a left-invariant metric of G compatible with the topology. The next result characterizes bounded orbits of LCS.

Proposition 3.1. *For any $u \in \mathcal{U}$ there exists at most one $x \in G^{+,-}$ such that*

$$\{\phi_{t,u}(x), \quad t \in \mathbb{R}\} \quad \text{is bounded.}$$

Proof. In fact, if $x_1, x_2 \in G^{+,-}$ are such that $\{\phi_{t,u}(x_i), t \in \mathbb{R}\}$ is bounded for $i = 1, 2$ then

$$\varrho(\phi_{t,u}(x_1), \phi_{t,u}(x_2)) = \varrho(\varphi_t(x_1), \varphi_t(x_2)) = \varrho(\varphi_t(x_2^{-1}x_1), e)$$

is bounded. By Proposition 2.1, compactness of G^0 implies that $G = G^{+,-}G^0$ is decomposable and we can write $x_2^{-1}x_1 = h k$ with $h \in G^{+,-}$ and $k \in G^0$.

Now, by the left-invariance of the metric, it holds that

$$\begin{aligned} \varrho(\varphi_t(h^{-1}), e) &\leq \varrho(\varphi_t(h^{-1}), \varphi_t(k)) + \varrho(\varphi_t(k), e) = \varrho(e, \varphi_t(hk)) + \varrho(\varphi_t(k), e) \\ &= \varrho(\varphi_t(x_2^{-1}x_1), e) + \varrho(\varphi_t(k), e). \end{aligned}$$

Since $\varphi_t(k) \in G^0$ for all $t \in \mathbb{R}$, compactness of G^0 implies that $\varrho(\varphi_t(k), e)$ is bounded and hence $\varrho(\varphi_t(h^{-1}), e)$ is also bounded. Since $h^{-1} \in G^{+,-}$, Lemma 2.9 implies $h = e$ which gives us $x_1 = x_2 k$. Since G^+, G^0 and G^- are dynamical subgroups of a 1-parameter group of automorphisms, decomposition of the elements of G in the components $G^{+,-}$ and G^0 are unique (see Remark 2.3) which implies necessarily that $k = e$ and hence $x_1 = x_2$, concluding the proof. \square

Next we show that any element of \mathcal{U}_{per} is associated with a bounded orbit.

Proposition 3.2. *For any $u \in \mathcal{U}_{\text{per}}$ there exists a unique $x(u) \in G^{+,-}$ such*

$$\phi_{n\tau, u}(x(u)) \in x(u)G^0, \quad \text{for all } n \in \mathbb{Z}.$$

In particular, the orbit $\{\phi_{t,u}(x(u)), t \in \mathbb{R}\}$ is bounded.

Proof. Assume first that $G^{+,-}$ is a subgroup of G .

Let $u \in \mathcal{U}_{\text{per}}$ be a τ -periodic control function and decompose $\phi_{\tau, u}(e) = xy$ with $x \in G^{+,-}$ and $y \in G^0$. Since G^0 normalizes $G^{+,-}$ we have that $\psi := C_y \circ \varphi_\tau|_{G^{+,-}}$ is an automorphism of $G^{+,-}$. Moreover, the compactness of G^0 implies that G^+ and G^- are the unstable and stable subgroups of ψ and hence ψ is hyperbolic. Also, a simple calculation shows that

$$\psi^{n+1}(x) = (y\varphi_\tau(y) \cdots \varphi_{n\tau}(y)) \varphi_{(n+1)\tau}(x) (y\varphi_\tau(y) \cdots \varphi_{n\tau}(y))^{-1}$$

implying that

$$\psi(x) = x \iff (\varphi_{n\tau}(x))_{n \in \mathbb{N}} \text{ is bounded.}$$

However, as a consequence of Lemma 2.9, the only point $x \in G^{+,-}$ such that $(\varphi_{n\tau}(x))_{n \in \mathbb{N}}$ is bounded is the identity $e \in G$.

By Proposition 2.7, the map

$$f_\psi : g \in G^{+,-} \mapsto g\psi(g^{-1}) \in G^{+,-} \quad \text{is a diffeomorphism.}$$

Let then $x(u) \in G^{+,-}$ be the unique element satisfying $x = f_\psi(x(u))$. It holds that

$$x = f_\psi(x(u)) = x(u)\psi(x(u)^{-1}) = x(u)C_y(\varphi_\tau(x(u)^{-1})) = x(u)y\varphi_\tau(x(u)^{-1})y^{-1}$$

and hence

$$\phi_{\tau, u}(x(u)) = \phi_{\tau, u}(e)\varphi_\tau(x(u)) = xy\varphi_\tau(x(u)) = x(u)y \in x(u)G^0.$$

Also, if $\phi_{n\tau,u}(x(u)) \in x(u)G^0$ then

$$\begin{aligned} \phi_{(n+1)\tau,u}(x(u)) &= \phi_{\tau,u}(\phi_{n\tau,u}(x(u)) \in \phi_{\tau,u}(x(u)G^0) \\ &= \phi_{\tau,u}(x(u)) \cdot \varphi_{\tau}(G^0) \in x(u)G^0, \end{aligned}$$

where for the last equality we used Eq. (1). Let $n \in \mathbb{Z}^+$ and $g \in G^0$ such that $\phi_{n\tau,u}(x(u)) = x(u)g$. Since $\phi_{\tau,u}^{-1} = \phi_{-\tau,\theta_{\tau}u} = \phi_{-\tau,u}$ we have that

$$\begin{aligned} x(u) &= \phi_{-n\tau,u}(\phi_{n\tau,u}(x(u))) = \phi_{-n\tau,u}(x(u)g) = \phi_{-n\tau,u}(x(u))\varphi_{-n\tau}(g) \\ &\implies \phi_{-n\tau,u}(x(u)) = x(u)\varphi_{-n\tau}(g^{-1}) \in x(u)G^0 \end{aligned}$$

and hence

$$\phi_{n\tau,u}(x(u)) \in x(u)G^0, \quad \text{for all } n \in \mathbb{Z}.$$

For the general case, let us consider the induced LCS $\Sigma_{\hat{G}}$ on the Lie group $\hat{G} = G/N^0$, where N^0 is the subgroup defined in Proposition 2.2 given by the intersection of G^0 with the nilradical N of G . Since G^0 is compact, we have by the same proposition that $G^{+,-} \subset N$ and hence $\hat{G}^{+,-} = \pi(G^{+,-}) = \pi(G^{+,-}N^0) = \pi(N)$ implying that $\hat{G}^{+,-}$ is a subgroup. By the previous case, for any $u \in \mathcal{U}_{\text{per}}$ there exists a unique $\hat{x}(u) \in \hat{G}^{+,-}$ such that

$$\hat{\phi}_{n\tau,u}(\hat{x}(u)) \in \hat{x}(u)\hat{G}^0, \quad n \in \mathbb{Z}.$$

Since G is decomposable, there exists a unique $x(u) \in G^{+,-}$ such that $\pi(x(u)) = \hat{x}(u)$. Moreover,

$$\begin{aligned} \pi(\phi_{n\tau,u}(x(u))) &= \hat{\phi}_{n\tau,u}(\hat{x}(u)) \in \hat{x}(u)\hat{G}^0 = \pi(x(u)G^0) \\ &\implies \phi_{n\tau,u}(x(u)) \in \pi^{-1}(\pi(x(u)G^0)) = x(u)G^0N^0 = x(u)G^0 \end{aligned}$$

and hence

$$\phi_{n\tau,u}(x(u)) \in x(u)G^0, \quad \text{for all } n \in \mathbb{Z}.$$

For the boundedness of the whole orbit $\{\phi_{t,u}(x(u)), t \in \mathbb{R}\}$, let us consider $t \in \mathbb{R}$ and write it as $t = n\tau + r$ with $n \in \mathbb{Z}$ and $|r| \in [0, \tau)$. Then,

$$\phi_{t,u}(x(u)) = \phi_{r,u}(\phi_{n\tau,u}(x(u)) \in \phi_{r,u}(x(u)G^0) = \phi_{r,u}(x(u)) \cdot G^0,$$

showing that

$$\{\phi_{t,u}(x(u)), t \in \mathbb{R}\} \subset \mathcal{A}_{\tau}(x(u))G^0 \cup \mathcal{A}_{\tau}^*(x(u))G^0,$$

and consequently that $\{\phi_{t,u}(x(u)), t \in \mathbb{R}\}$ is bounded. □

The next lemma shows that the orbits associated with \mathcal{U}_{per} are contained in $\bar{\mathcal{C}}$.

Lemma 3.3. *For any $u \in \mathcal{U}_{\text{per}}$ it holds that $\phi_{t,u}(x(u)G^0) \subset \bar{\mathcal{C}}$, for any $t \in \mathbb{R}$.*

Proof. Let us first prove that $x(u)G^0 \subset \bar{\mathcal{C}}$. By Proposition 3.2 we have that,

$$\phi_{n\tau,u}(x(u)) \in x(u)G^0, \quad \text{for all } n \in \mathbb{Z},$$

where $\tau > 0$ is the period of u . Since G^0 is a compact subgroup, let us assume w.l.o.g. that $\phi_{n\tau,u}(x(u)) \rightarrow x(u)g_1$ with $g_1 \in G^0$. Decomposing $x(u) = gh$ with $g \in G^+$ and $h \in G^-$ gives us that

$$\begin{aligned} \varrho(x(u)g_1, \phi_{n\tau,u}(g)) &\leq \varrho(\phi_{n\tau,u}(x(u)), \phi_{n\tau,u}(g)) + \varrho(\phi_{n\tau,u}(x(u)), x(u)g_1) \\ &= \varrho(\phi_{n\tau,u}(g)\varphi_{n\tau}(h), \phi_{n\tau,u}(g)) + \varrho(\phi_{n\tau,u}(x(u)), x(u)g_1) \\ &= \varrho(\varphi_{n\tau}(h), e) + \varrho(\phi_{n\tau,u}(x(u)), x(u)g_1) \rightarrow 0 \quad \text{as } n \rightarrow +\infty, \end{aligned}$$

where in the last equality we used the left-invariance of the metric. As a consequence, $\phi_{n\tau,u}(g) \rightarrow x(u)g_1$ for some $g_1 \in G^0$. However, the fact that $G^{+,0} \subset \mathcal{A}(e)$ implies by invariance that $\phi_{n\tau,u}(g) \in \mathcal{A}(e)$ and hence $x(u)g_1 \in \overline{\mathcal{A}(e)}$. Let then $g_2 \in G^0$ arbitrary. By Theorem 2.6 it holds that

$$\begin{aligned} x(u)g_2 &= x(u)g_1(g_1^{-1}g_2) \in \overline{\mathcal{A}(e)}g_1^{-1}g_2 = \overline{\mathcal{A}(e)g_1^{-1}g_2} \subset \overline{\mathcal{A}(e)} \\ &\implies x(u)G^0 \subset \overline{\mathcal{A}(e)}. \end{aligned}$$

Consider now the decomposition $x(u) = h'g'$ with $h' \in G^-$ and $g' \in G^+$. By the same arguments as previously, we have that $\phi_{-n\tau,u}(h') \rightarrow x(u)h_1$, for some $h_1 \in G^0$. Using that $G^{-,0} \subset \mathcal{A}^*(e)$ gives us that $x(u)h_1 \in \overline{\mathcal{A}^*(e)}$ and again by Theorem 2.6 we get, for any $h_2 \in G^0$, that

$$\begin{aligned} x(u)h_2 &= x(u)h_1(h_1^{-1}h_2) \in \overline{\mathcal{A}^*(e)}h_1^{-1}h_2 = \overline{\mathcal{A}^*(e)h_1^{-1}h_2} \subset \overline{\mathcal{A}^*(e)} \\ &\implies x(u)G^0 \subset \overline{\mathcal{A}^*(e)}. \end{aligned}$$

Therefore,

$$x(u)G^0 \subset \overline{\mathcal{A}(e)} \cap \overline{\mathcal{A}^*(e)} \subset \overline{\mathcal{C}}.$$

Consider now $t > 0$. By invariance in positive time we already have that $\phi_{t,u}(x(u)G^0) \subset \overline{\mathcal{A}(e)}$. On the other hand, for any $n \in \mathbb{Z}$ we have that

$$\begin{aligned} \phi_{t,u}(x(u)) &= \phi_{t,u}(\phi_{-n\tau,u}(\phi_{n\tau,u}(x(u)))) \\ &= \phi_{t-n\tau,u}(\phi_{n\tau,u}(x(u))) \in \phi_{t-n\tau,u}(x(u)G^0), \end{aligned}$$

where for the last equality we used the cocycle property and the fact that u is τ -periodic.

Since $x(u)G^0 \subset \overline{\mathcal{C}} \subset \overline{\mathcal{A}^*(e)}$, if we take $n \in \mathbb{Z}$ such that $t - n\tau \leq 0$, the invariance in negative time of $\mathcal{A}^*(e)$ implies that

$$\phi_{t,u}(x(u)) \in \phi_{t-n\tau,u}(\overline{\mathcal{A}^*(e)}) \subset \overline{\mathcal{A}^*(e)}$$

and hence, for any $g \in G^0$, we get that

$$\begin{aligned} \phi_{t,u}(x(u)g) &= \phi_{t,u}(x(u))\varphi_t(g) \in \overline{\mathcal{A}^*(e)}\varphi_t(g) \\ &= \overline{\mathcal{A}(e)}\varphi_t(g) \subset \overline{\mathcal{A}^*(e)} \implies \phi_{t,u}(x(u)G^0) \subset \overline{\mathcal{A}^*(e)}, \end{aligned}$$

and consequently

$$\phi_{t,u}(x(u)G^0) \subset \overline{\mathcal{C}}, \quad \text{for all } t \geq 0.$$

By arguing analogously, we get that $\phi_{t,u}(x(u)G^0) \subset \overline{\mathcal{C}}$ for all $t < 0$ and so

$$\phi_{t,u}(x(u)G^0) \subset \overline{\mathcal{C}}, \quad \text{for all } t \in \mathbb{R},$$

concluding the proof. □

We can now state and prove the main result of this paper.

Theorem 3.4. *If a LCS Σ_G admits a bounded control set \mathcal{C} with $e \in \text{int } \mathcal{C}$ then all the bounded orbits of Σ_G are contained in $\bar{\mathcal{C}}$.*

Proof. Let $h \in G$ and decompose it as $h = xg$ with $x \in G^{+,-}$ and $g \in G^0$. If for some $u \in \mathcal{U}$ we have that $\{\phi_{t,u}(h), t \in \mathbb{R}\}$ is bounded, then

$$\phi_{t,u}(x) = \phi_{t,u}(hg^{-1}) = \phi_{t,u}(h)\varphi_t(g^{-1}) \in \{\phi_{t,u}(h), t \in \mathbb{R}\}G^0,$$

showing that $\{\phi_{t,u}(x), t \in \mathbb{R}\}$ is also bounded. Since $x \in G^{+,-}$ we have by Proposition 3.1 that $x = x(u)$ which by Lemma 3.3 and the previous discussion imply that

$$\{\phi_{t,u}(h), t \in \mathbb{R}\} = \{\phi_{t,u}(x(u)g), t \in \mathbb{R}\} \subset \bar{\mathcal{C}},$$

as stated. □

3.2. Consequences of the main result

Let us define the *lift* of the control set \mathcal{C} as the set

$$L(\mathcal{C}) := \{(u, g) \in \mathcal{U} \times G; \phi(\mathbb{R}, g, u) \subset \bar{\mathcal{C}}\}.$$

In this section/we show that if a LCS admits a bounded control set \mathcal{C} , it lift $L(\mathcal{C})$ is the continuous image of $\mathcal{U} \times G^0$. Moreover, if $G^{+,-}$ is a subgroup, the restriction of the control flow Φ_t to $L(\mathcal{C})$ coincides with the control flow of an induced LCS on G^0 .

Let us consider the map

$$x : \mathcal{U} \rightarrow G^{+,-},$$

which associated $u \in \mathcal{U}$ to the unique point $x(u) \in G^{+,-}$ such that $\{\phi_{t,u}(x(u)), t \in \mathbb{R}\}$ is bounded. For any $u \in \mathcal{U}_{\text{per}}$ the previous map is already well defined by Proposition 3.2. Let us consider now an arbitrary $u \in \mathcal{U}$. Since \mathcal{U}_{per} is dense in \mathcal{U} , there exists a sequence $(u_k)_{k \in \mathbb{N}} \subset \mathcal{U}_{\text{per}}$ with $u_k \rightarrow u$. Denote by $x_k := x(u_k)$ the unique point in $G^{+,-}$ with bounded orbit. By Lemma 3.3, it holds that

$$\forall t \in \mathbb{R}, \quad \phi_{t,u_k}(x_k) \in \bar{\mathcal{C}}, \text{ for all } k \in \mathbb{N},$$

and, in particular, the sequence $(x_k)_{k \in \mathbb{N}} \subset G^{+,-}$ is bounded. As a consequence, $(x_k)_{k \in \mathbb{N}}$ admits accumulation points in $G^{+,-}$. If $x \in G^{+,-}$ is one such point, then $x_{k_n} \rightarrow x$ for some subsequence $(x_{k_n})_{n \in \mathbb{N}}$. By continuity, we get

$$\phi_{t,u}(x) = \lim_{n \rightarrow +\infty} \phi_{t,u_{k_n}}(x_{k_n}) \in \bar{\mathcal{C}}, \text{ for all } t \in \mathbb{R},$$

showing that $\{\phi_{t,u}(x), t \in \mathbb{R}\} \subset \bar{\mathcal{C}}$ is a bounded orbit. Moreover, by Proposition 3.1 the point $x \in G^{+,-}$ with such property is unique, implying that the sequence $(x_k)_{k \in \mathbb{N}}$ is convergent. The previous arguments allows us to conclude that the map

$$u \in \mathcal{U} \mapsto x(u) \in G^{+,-},$$

is well defined and continuous. As a consequence,

$$H : \mathcal{U} \times G^0 \rightarrow L(\mathcal{C}), \quad (u, g) \mapsto (u, x(u)g),$$

is a continuous map. Moreover,

$$\begin{aligned} (u, x) \in L(\mathcal{C}) &\iff \phi(\mathbb{R}, x, u) \subset \bar{\mathcal{C}} \iff \{\phi_{t,u}(x), t \in \mathbb{R}\} \text{ is bounded} \\ &\iff (u, x) \in H(\mathcal{U} \times G^0). \end{aligned}$$

Hence $H(\mathcal{U} \times G^0) = L(\mathcal{C})$ and, since $\mathcal{U} \times G^0$ is compact and H is continuous, H is a closed map and hence a homeomorphism between $\mathcal{U} \times G^0$ and $L(\mathcal{C})$.

Let us assume that $G^{+,-}$ is a subgroup of G . Since G is decomposable, it holds that $G^{+,-}$ is a normal subgroup of G and hence the canonical projection $\pi : G \rightarrow G/G^{+,-}$ induces a LCS on $G/G^{+,-}$. Moreover, the fact that $G^0 \cap G^{+,-} = \{e\}$ implies that $\pi|_{G^0}$ is a group isomorphism and hence we can assume w.l.o.g. that $G/G^{+,-} = G^0$. Let us denote by Σ_{G^0} the LCS induced by π on G^0 and by Φ^0 its control flow. The next result shows that under the extra assumption that $G^{+,-}$ is a subgroup the dynamics of $\Phi_t|_{L(\mathcal{C})}$ is basically the same as the one from the Φ_t^0 .

Theorem 3.5. *If $G^{+,-}$ is a subgroup, then H conjugates $\Phi_t|_{L(\mathcal{C})}$ and Φ_t^0 .*

Proof. By the cocycle property,

$$\phi_{t+s,u}(x(u)) = \phi_{t,\theta_s u}(\phi_{s,u}(x(u))).$$

Hence,

$$\{\phi_{t,\theta_s u}(\phi_{s,u}(x(u))), t \in \mathbb{R}\} \text{ is a bounded orbit.}$$

In particular, we have that $\phi_{s,u}(x(u)) = x(\theta_s u)h$ for an unique $h \in G^0$. Furthermore, under the previous identification, it holds that

$$\begin{aligned} h &= \pi(h) = \pi(h(h^{-1}x(\theta_s u)h)) = \pi(x(\theta_s u)h) = \pi(\phi_{s,u}(x(u))) = \pi(\phi_{s,u}(e)) \\ &= \phi_{s,u}^0(e), \end{aligned}$$

showing that

$$\forall s \in \mathbb{R}, \quad \phi_{s,u}(x(u)) = x(\theta_s u)\phi_{s,u}^0(e).$$

Therefore,

$$\begin{aligned} H(\Phi_s^0(u, g)) &= H(\theta_s u, \phi_{s,u}^0(g)) = (\theta_s u, x(\theta_s u)\phi_{t,u}^0(e)\varphi_s(g)) \\ &= (\theta_s u, \phi_{s,u}(x(u))\varphi_s(g)) = (\theta_s u, \phi_{s,u}(x(u)g)) = \Phi_s(H(u, g)), \end{aligned}$$

showing that H conjugates $\Phi_t|_{L(\mathcal{C})}$ and Φ_t^0 as stated. \square

Remark 3.6. In [11, Theorem 3.4] the author shows that, under a hyperbolicity assumption, $L(\mathcal{C})$ is the graph of a continuous function whose domain is the set of control functions \mathcal{U} . In the context of LCS's, such an assumption is equivalent to $G^0 = \{e\}$. Therefore our previous result shows that when the system is not necessarily hyperbolic but has compact central manifold (that is G^0 is a compact subgroup) we still have a characterization of $L(\mathcal{C})$ in terms of the central manifold and the set \mathcal{U} .

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References

- [1] Ayala, V., Da Silva, A.: Central periodic points of a linear system. *J. Differ. Equ.* **272**(25), 310–329 (2021)
- [2] Ayala, V., Da Silva, A.: Controllability of linear control systems on lie groups with semisimple finite center. *SIAM J. Control. Optim.* **55**(2), 1332–1343 (2017)
- [3] Ayala, V., Da Silva, A., Jouan, P.: Jordan decompositions and the recurrent set of flows of automorphisms. *Discrete Contin. Dyn. Syst.* **41**(4), 1543–1559 (2021)
- [4] Ayala, V., Da Silva, A., Roman-Flores, H.: The dynamics of a Lie group endomorphism. *Open Math.* **15**, 1477–1486 (2017)
- [5] Ayala, V., Da Silva, A., Zsigmond, G.: Control sets of linear systems on Lie groups. *Nonlinear Differ. Equ. Appl. NoDEA* **24**(8), 1–15 (2017)
- [6] Colonius, F., Kliemann, W.: *The Dynamics of Control Systems & Control: Foundations & Applications*. Birkäuser Boston Inc., Boston (2000)
- [7] Da Silva, A.: Controllability of linear systems on solvable Lie groups. *SIAM J. Control. Optim.* **54**(1), 372–390 (2016)
- [8] Da Silva, A., Santana, A.J., Stelmastchuk, S.N.: Topological conjugacy of linear systems on Lie groups. *Discrete Contin. Dyn. Syst.* **37**(6), 3411–3421 (2017)
- [9] Jouan, P.: Equivalence of control systems with linear systems on Lie groups and homogeneous spaces. *ESAIM Control Optim. Calc. Var.* **16**, 956–973 (2010)
- [10] Jouan, P.: Controllability of linear systems on Lie groups. *J. Dyn. control syst.* **17**(4), 591–616 (2011)
- [11] Kawan, C.: On the structure of uniformly hyperbolic chain control sets. *Syst. Control Lett.* **90**, 71–75 (2016)

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