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CONTROL SYSTEMS ON FLAG MANIFOLDS AND THEIR CHAIN CONTROL SETS

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Abstract. A right-invariant control system Σ on a connected Lie group G induce affine control systems Σ_Θ on every flag manifold $\mathbb{F}_\Theta = G/P_\Theta$. In this paper we show that the chain control sets of the induced systems coincides with their analogous one defined via semigroup actions. Consequently, any chain control set of the system contains a control set with nonempty interior and, if the number of the control sets with nonempty interior coincides with the number of the chain control sets, then the closure of any control set with nonempty interior is a chain control set. Some relevant examples are included.

1. Introduction.

Let G be a noncompact connected semisimple Lie group. A right-invariant control system on a connected Lie group G is determined by a family of differential equations given by

$$\dot{g}(t) = X_0(g(t)) + \sum_{j=1}^{m} u_i(t) X_j(g(t)), \quad u \in \mathcal{U} \tag{Σ}$$

where $X_0, X_1, \ldots, X_m$ are right invariant vector fields and $\mathcal{U} \subset L^\infty(\mathbb{R}, U \subset \mathbb{R}^m)$. Here $U \subset \mathbb{R}^m$ is a compact and convex set with $0 \in \operatorname{int} U$.

This class of system induce an affine control system on any flag manifold $\mathbb{F}_\Theta = G/P_\Theta$ of G defined by

$$\dot{x}(t) = f_0^\Theta(x(t)) + \sum_{j=1}^{m} u_j(t) f_j^\Theta(x(t)), \quad u \in \mathcal{U} \tag{Σ_Θ}$$

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where $f_i^\Theta = (d\pi_\Theta)X_i$ for $i = 0, \dots, m$ and $\pi_\Theta : G \to \mathbb{F}_\Theta$ is the canonical projection.

It is worth pointing out that to study systems on flag manifolds is desirable for at least two reasons: First, flag manifolds include, spheres, projective spaces and Grassmannian spaces. Secondly, right-invariant systems appears in many important applications in engineering and physics such as the orientation of rigid bodies and optimal problems (see [4], [8] and [9]).

In order to understand the dynamical behavior of Σ_Θ one have to analyze control and chain control sets of the system. There is a way to analyze the chain control sets of Σ_Θ through the induced control flow ϕ^Θ on the fiber bundle

$$\phi^\Theta : \mathbb{R} \times \mathcal{U} \times \mathbb{F}_\Theta \to \mathcal{U} \times \mathbb{F}_\Theta.$$

In fact, since $\mathbb{F}_\Theta$ is a compact manifold, Theorem 4.3.11 of [5] implies that there exists a bijection between the Morse components of the control flow ϕ^Θ and the chain control sets of the system Σ_Θ. Also, by [3], all the Morse sets of the flow ϕ^Θ are given fiberwise as fixed points in $\mathbb{F}_\Theta$ of split elements of G.

On the other hand, since the positive orbit $\mathcal{S} := \mathcal{O}^+(1)$ is a semigroup we can use the notion of control and chain control sets associated with $\mathcal{S}$ (see for instance [2], [12] and [14]). In particular, it is know that any effective chain control set of $\mathcal{S}$ contains a control set with nonempty interior.

Our viewpoint sheds some new light on invariant control systems and its relation with the Lie semigroup theory. In fact, in this paper we show that both notion actually agree, that is, the chain control sets of the induced systems Σ_Θ and the effective chain control sets associated with $\mathcal{S}$ are actually the same. That implies, in particular, that any given chain control set of Σ_Θ contains a control set with nonempty interior. Moreover, if the number of chain control sets coincides with the number of the control sets with nonempty interior then any chain control set is the closure of the control set that it contains.

The paper is structured as follows: In Section 2 we give some preliminaries on control and Lie theory and enunciate some results that will be needed. In Section 3 we state and prove our main result, showing that both notions of chain control sets coincides. As a consequence, we prove that if the number of chain control sets and of control sets with nonempty interior coincide, then any chain control set is the closure of the only control set with nonempty interior that it contains. Finally, we include a couple of relevant examples.

2. **Preliminaries.**

In this section we introduce all the concepts of control and Lie theory needed in order to present and prove our main result.

2.1. **Control theory.**

Let M be a Riemannian differentiable manifold and consider an admissible class of control $\mathcal{U} \subset L^\infty(\mathbb{R}, U \subset \mathbb{R}^m)$. A control affine system on M is defined by the family of differential equations

$$\dot{x}(t) = f_0(x(t)) + \sum_{j=1}^m u_j(t) f_j(x(t)), \quad x(t) \in M. \tag{1}$$

where $f_0, f_1, \dots, f_m$ are smooth vector fields on M. The set $\mathcal{U}$ is a compact metrizable space and the shift flow

$$\theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \quad (t, u) \mapsto \theta_t u := u(\cdot + t)$$

is a continuous chain transitive dynamical system (see [5]).

For $u \in \mathcal{U}$ fixed, we denote by $\varphi(\cdot, x, u)$ the unique solution of (1) with $\varphi(0, x, u) = x$. If the vector fields $f_0, \dots, f_m$ are elements of $\mathcal{C}^\infty$, then φ has also the same class of differentiability with respect to M and the corresponding partial derivatives depend continuously on $(t, x, u) \in \mathbb{R} \times M \times \mathcal{U}$ (see Thm. 1.1 of [10]).

If we assume that all the solutions determined by $\mathcal{U}$ are defined in the whole time axis[1], we obtain the well defined map

$$\varphi : \mathbb{R} \times M \times \mathcal{U} \to M, \quad (t, x, u) \mapsto \varphi(t, x, u),$$

called the transition map of the system. It turns out that this map together with the shift flow determines a skew-product flow as follows

$$\phi : \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \;\; (t, x, u) \mapsto \phi_t(u, x) = (\theta_t u, \varphi(t, x, u)),$$

which is called the control flow of the system (see Sec. 4.3 of [5]).

We denote by

$$\mathcal{O}^+_\tau(x) := \{\varphi(\tau, x, u) \;:\; u \in \mathcal{U}\},$$
$$\mathcal{O}^+_{\leq\,\tau}(x) := \bigcup_{t\,\in\,[0,\tau]} \mathcal{O}^+_t(x) \quad \text{and} \quad \mathcal{O}^+(x) := \bigcup_{\tau\,>\,0} \mathcal{O}^+_\tau(x).$$

the set of reachable points from $x \in M$ at time $\tau > 0$, the set of reachable points from x up to time τ, and the **positive orbit of** x, respectively. Analogously we denote by $\mathcal{O}^-_\tau(x), \mathcal{O}^-_{\leq\,\tau}(x)$ and $\mathcal{O}^-(x)$ the set of the controllable points to x in time $\tau > 0$, the set of controllable points to x up to time τ and the **negative orbit of** x, respectively.

The affine control system (1) is called locally accessible at x if for all $\tau > 0$ the sets $\mathcal{O}^+_{\leq\,\tau}(x)$ and $\mathcal{O}^-_{\leq\tau}(x)$ have nonempty interiors. It is called **locally accessible** if it is locally accessible at every point $x \in M$. Under the assumption of locally accessibility, it holds that $\operatorname{int}\mathcal{O}^+(x)$ is a dense subset of $\mathcal{O}^+(x)$, for any $x \in M$ (see Lemma 1.2 of [10]).

The next definition yields information about completly controllable regions of the manifold M.

Definition 2.1. A **control set** of the system (1) is a subset $D \subset M$ satisfying:

(i) for each $x \in D$ there is $u \in \mathcal{U}$ with $\varphi(\mathbb{R}_+, x, u) \subset D$
(ii) $D \subset \operatorname{cl}\mathcal{O}^+(x)$ for all $x \in D$
(iii) D is maximal with respect to the set inclusion and properties (i) and (ii).

In [5], Proposition 3.2.4, it is shown that a subset D with nonempty interior, that is maximal with the property (ii) above is also a control set.

Let us now fix a metric ϱ on M. For $x, y \in M$ and $\varepsilon, \tau > 0$, a **controlled** (ε, τ)**-chain** from x to y is given by an integer $n \in \mathbb{N}$ and sequences

$$x_0, \ldots, x_n \in M, \; u_0, \ldots, u_{n-1} \in \mathcal{U}, \text{ and } t_0, \ldots, t_{n-1} \geq \tau$$

such that

$$x = x_0, \; y = x_n \text{ and } \varrho(\varphi(t_i, x_i, u_i), x_{i+1}) < \varepsilon, \quad \text{for } i = 0, \ldots, n-1.$$

Definition 2.2. A set $E \subset M$ is called a **chain control set** of (1) if it satisfies the following properties:

$i)$ for each $x \in E$ there is $u \in \mathcal{U}$ with $\varphi(\mathbb{R}, x, u) \subset E$
$ii)$ For all $x, y \in E$ and $\varepsilon, \tau > 0$ there exists a controlled (ε, τ)-chain from x to y in M

[1]This condition is in general restrictive, but for the class of affine control systems under study it holds.

iii) E is maximal with respect to the set inclusion and the properties (i) and (ii) above.

A set E satisfying condition (ii) in the above definition is called **controlled chain transitive**.

Remark 1. From the general theory, every chain control set of the control system (1) is closed, but this is not necessarely true for control sets. Moreover, every control set with nonempty interior is contained in a chain control set if local accessibility holds (see Sec. 4.3 of [5]).

2.2. **Lie theory and flag manifolds.** In order to state and prove our main result, in this section we give some facts about semisimple Lie groups and their induced flag manifolds.

2.2.1. *Semisimple Lie groups.* Let G be a connected non-compact semisimple Lie group G with finite center and Lie algebra $\mathfrak{g}$. Fix a Cartan involution $\zeta : \mathfrak{g} \to \mathfrak{g}$ with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and let $\mathfrak{a} \subset \mathfrak{s}$ be a maximal Abelian subalgebra and $\mathfrak{a}^+ \subset \mathfrak{a}$ a Weyl chamber. Let us denote by Π, Π^+ and $\Pi^- := -\Pi^+$ the **set of roots**, the **set of positive roots** and **set of negative roots**, respectively, associated with $\mathfrak{a}^+$. The **Iwasawa decomposition** of the Lie algebra $\mathfrak{g}$ reads as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^{\pm} \text{ where } \mathfrak{n}^{\pm} := \sum_{\alpha \,\in\, \Pi^{\pm}} \mathfrak{g}_{\alpha},$$

and $\mathfrak{g}_{\alpha} := \{X \in \mathfrak{g}, \ \ [H, X] = \alpha(H)X, \quad \text{for all} \ \ H \in \mathfrak{a}\}$.

Let K, A and $N^{\pm}$ be the connected Lie subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}^{\pm}$, respectively. The Iwasawa decomposition of the Lie group G is given by $G = KAN^{\pm}$. The Weyl group of $\mathfrak{g}$ is the finite group generated by the reflections on the hyperplanes $\ker \alpha$ for $\alpha \in \Pi$. Alternatively, the Weyl group can be obtained by the quotient M^*/M, where M^* and M are the normalizer and the centralizer of $\mathfrak{a}$ in K, respectively.

We denote by $\Lambda \subset \Pi^+$ the set of the positive roots which cannot be written as linear combinations of other positive roots. The Weyl group coincides with the group generated by the reflections associated to $\alpha \in \Lambda$. There is only one involutive element $w_0 \in \mathcal{W}$ such that $w_0 \Pi^+ = \Pi^-$.

For $\Theta \subset \Lambda$ the parabolic subalgebra of type Θ is $\mathfrak{p}_{\Theta} := \mathfrak{n}^-(\Theta) \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$. Here $\mathfrak{m}$ is the Lie algebra of M, $\mathfrak{n}^-(\Theta)$ is the sum of the eigenspaces $\mathfrak{g}_{\alpha}$ when $\alpha \in \langle\Theta\rangle \cap \Pi^-$ and $\langle\Theta\rangle \subset \Pi$ is the set of roots given as linear combination of the simple roots in Θ. The corresponding parabolic subgroup P_{Θ} is the normalizer of $\mathfrak{p}_{\Theta}$ in G. The flag manifold $\mathbb{F}_{\Theta}$ is the orbit $\mathbb{F}_{\Theta} := \mathrm{Ad}(G)\mathfrak{p}_{\Theta}$ on a Grasmannian manifold. It is also identified with the homogeneous space G/P_{Θ}. When $\Theta = \emptyset$, the sets $\mathfrak{p} := \mathfrak{p}_{\emptyset}$ and $\mathbb{F} := \mathbb{F}_{\emptyset}$ are called the minimal parabolic subalgebra and the maximal flag manifold, respectively.

An element of $\mathfrak{g}$ of the form $Y = \mathrm{Ad}(g)H$ with $g \in G$ and $H \in \mathrm{cl}\,\mathfrak{a}^+$ is called a **split element**. When $H \in \mathfrak{a}^+$ we say that Y is a **split regular** element. The flow induced by a split element $H \in \mathrm{cl}\,\mathfrak{a}^+$ on $\mathbb{F}_{\Theta}$, is given by

$$(t, \mathrm{Ad}(g)\mathfrak{p}) \mapsto \mathrm{Ad}(\mathrm{e}^{tH} g)\mathfrak{p}_{\Theta}.$$

It turns out that the associated vector field is a gradient vector field with respect to an appropriate Riemannian metric on $\mathbb{F}_{\Theta}$.

The connected components of the fixed point set of this flow are given by

$$\mathrm{fix}_\Theta(H,w) = Z_H \cdot wb_\Theta = K_H \cdot wb_\Theta, \quad w \in \mathcal{W}.$$

Here b_Θ is the origin of $\mathbb{F}_\Theta$, Z_H is the centralizer of H in G and $K_H = Z_H \cap K$. The sets $\mathrm{fix}_\Theta(H,w)$ are in bijection with the double coset space $\mathcal{W}_H\backslash\mathcal{W}/\mathcal{W}_\Theta$ where $\mathcal{W}_H$ and $\mathcal{W}_\Theta$ are the subgroups of the Weyl group generated by the simple roots in $\Theta(H) := \{\alpha \in \Lambda,\ \alpha(H) = 0\}$ and in Θ, respectively.

Each component $\mathrm{fix}_\Theta(H,w)$ is a compact connected submanifold of $\mathbb{F}_\Theta$. Moreover, for $Y = \mathrm{Ad}(g)H$ with $H \in \mathrm{cl}\,\mathfrak{a}^+$ we get that

$$\mathrm{fix}_\Theta(Y,w) = \mathrm{fix}_\Theta(\mathrm{Ad}(g)H,w) = g \cdot \mathrm{fix}_\Theta(H,w).$$

When H is a split regular element the set $\mathrm{fix}_\Theta(H,w)$ reduces to the point wb_Θ and consequently $\mathrm{fix}_\Theta(Y,w) = gwb_\Theta$.

2.2.2. *Semigroups.* In what follows we introduce the notion of control and chain control sets via semigroup action. Let $\mathcal{S}$ be a semigroup of a Lie group G and M a space provided with a G-transitive action. A $\mathcal{S}$-control set on M is a subset $D \subset M$ satisfying

(i) $\mathrm{int}\, D \neq \emptyset$
(ii) $D \subset \mathrm{cl}(\mathcal{S}x)$ for all $x \in D$
(iii) D is maximal (w.r.t. set inclusion) with the properties (i) and (ii).

A $\mathcal{S}$-control set D is said to be **effective** if the set $D_0 = \{x \in D,\ \ x \in (\mathrm{int}\, \mathcal{S})x\}$ is nonempty. The subset D_0 is said to be the **core** of D.

For semigroups with nonempty interior of a semisimple Lie group G we have the following result from [14].

Theorem 2.3. *For any $w \in \mathcal{W}$ there is an effective $\mathcal{S}$-control set $D_\Theta(w) \subset \mathbb{F}_\Theta$ whose core is given by*

$$D_\Theta(w)_0 = \left\{\mathrm{fix}_\Theta(Y,w); \quad Y \textit{ is split regular and } \mathrm{e}^Y \in \mathrm{int}\, \mathcal{S}\right\}.$$

Moreover,

(i) $D_\Theta(1)$ is the only invariant $\mathcal{S}$-control set;
(ii) $D_\Theta(w_0)$ the only invariant $\mathcal{S}^{-1}$-control set;
(iii) any effective $\mathcal{S}$-control set in $\mathbb{F}_\Theta$ is of the form $D_\Theta(w)$ for some $w \in \mathcal{W}$.

It turns out that there exists a unique $\Theta(\mathcal{S}) \subset \Lambda$ such that the set $\{w \in \mathcal{W} : D(w) = D(1)\} \subset \mathcal{W}$ coincides with the subgroup $\mathcal{W}_{\Theta(\mathcal{S})}$ of $\mathcal{W}$. Moreover,

$$\mathcal{W}_{\Theta(\mathcal{S})} w_1 \mathcal{W}_\Theta = \mathcal{W}_{\Theta(\mathcal{S})} w_2 \mathcal{W}_\Theta \Leftrightarrow D_\Theta(w_1) = D_\Theta(w_2)$$

and the number of effective control sets in $\mathbb{F}_\Theta$ is equal to the number of elements in the coset space $\mathcal{W}_{\Theta(S)} \backslash \mathcal{W}/\mathcal{W}_\Theta$. The subset $\Theta(S)$ is called the **flag type of the semigroup $\mathcal{S}$**.

There exists also a concept of chain control sets associated with a semigroup as follows: Let $\mathcal{F}$ be a family of subsets of $\mathcal{S}$. For $x, y \in M$, $\varepsilon > 0$ and $A \in \mathcal{F}$ a $(\mathcal{S}, \varepsilon, A)$**-chain** from x to y is given by an integer $n \in \mathbb{N}$ and sequences $x_0, \ldots, x_n \in M$, $g_0, \ldots, g_{n-1} \in A$ such that $x_0 = x$, $x_n = y$ and

$$\varrho(g_i x_i, x_{i+1}) < \varepsilon, \quad \text{for} \quad i = 0, \ldots, n-1.$$

A subset $E_\mathcal{F} \subset M$ is called a $\mathcal{F}$**-chain control set** of $\mathcal{S}$ if it satisfies

(i) $\mathrm{int}\, E_\mathcal{F} \neq \emptyset$.
(ii) For all $x, y \in E_\mathcal{F}$, $\varepsilon > 0$ and $A \in \mathcal{F}$ there exists a $(\mathcal{S}, \varepsilon, A)$-chain from x to y.

(iii) $E_{\mathcal{F}}$ is maximal (w.r.t. set inclusion) with the properties *(i)* and *(ii)*.

A $\mathcal{F}$-chain control set is said to be **effective** if it contains an effective $\mathcal{S}$-control set.

When G is a noncompact semisimple Lie group with finite center, it is possible to obtain a refined result (see [2], Proposition 4.7).

Proposition 1. *For each $w \in \mathcal{W}$ there exists an effective $\mathcal{F}$-chain control set $E_{\mathcal{F},\Theta}(w)$ for $\mathcal{S}$ on $\mathbb{F}_\Theta$ such that $D_\Theta(w) \subset E_{\mathcal{F},\Theta}(w)$.*

Moreover, the set $\mathcal{W}_{\mathcal{F}}(\mathcal{S}) = \{w \in \mathcal{W} : E_{\mathcal{F}}(w) = E_{\mathcal{F}}(1)\}$ is a subgroup of $\mathcal{W}$ and

$$\mathcal{W}_{\mathcal{F}}(\mathcal{S})w_1\mathcal{W}_\Theta = \mathcal{W}_{\mathcal{F}}(\mathcal{S})w_2\mathcal{W}_\Theta \Leftrightarrow E_{\mathcal{F},\Theta}(w_1) = E_{\mathcal{F},\Theta}(w_2).$$

The unique subset $\Theta(\mathcal{F}) \subset \Lambda$ such that $\mathcal{W}_{\Theta(\mathcal{F})} = \mathcal{W}_{\mathcal{F}}(S)$ is called the $\mathcal{F}$-**flag type of the semigroup** $\mathcal{S}$. The number of chain control sets in $\mathbb{F}_\Theta$ is equal to the number of elements in the coset space $\mathcal{W}_{\mathcal{F}}(S) \setminus \mathcal{W}/\mathcal{W}_\Theta$. If there is no possible confusion, we denote by $E_\Theta(w)$ the above $\mathcal{F}$-chain control sets on $\mathbb{F}_\Theta$.

By the same Proposition 1 above, for any family of subsets $\mathcal{F}$ of $\mathcal{S}$

$$\Theta(\mathcal{S}) \subset \Theta(\mathcal{F}).$$

Remark 2. The above definitions of $\mathcal{S}$-control sets and $\mathcal{F}$-chain control sets differs from their analogous ones for control systems, mainly by the assumption that both, $\mathcal{S}$-control sets and $\mathcal{F}$-chain control sets, have nonempty interior which is not necessarily true for control systems.

2.2.3. *Flow on flag bundles.* Let X be a compact metric space and $\phi : \mathbb{R} \times X \to X$, $(t, x) \mapsto \phi_t(x)$ be a continuous flow. A compact subset $C \subset X$ is called isolated invariant if $\phi_t(C) \subset C$ for all $t \in \mathbb{R}$ and if there is a neighborhood V of C that satisfies

$$\phi_t(x) \in V \quad \text{for all} \quad t \in \mathbb{R} \quad \Rightarrow \quad x \in C.$$

A **Morse decomposition** of ϕ is a finite collection $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of nonempty pairwise disjoint isolated invariant compact sets satisfying

(A) for all $x \in X$ the ω^* and ω-limit sets $\omega^*(x)$ and $\omega(x)$ are contained in $\bigcup_{i=1}^n \mathcal{M}_i$.

(B) if there are $\mathcal{M}_{j_0}, \ldots, \mathcal{M}_{j_l}$ and $x_1, \ldots, x_l \in X \setminus \bigcup_{i=1}^n \mathcal{M}_i$ with

$$\omega^*(x_i) \subset \mathcal{M}_{j_{i-1}} \quad \text{and} \quad \omega(x_i) \subset \mathcal{M}_{j_i}$$

for $i = 1, \ldots, l$ then $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_l}$.

The elements in a Morse decomposition are called **Morse sets**.

Let G be a connected semisimple noncompact Lie group with finite center and $\pi : Q \to X$ a G-principal bundle, where X is a compact metric space. The Lie group G acts continuously from the right on Q, this action preserves the fibers, and is free and transitive on each fiber. The **flag bundle** $\mathbb{E}_\Theta = Q \times_G \mathbb{F}_\Theta$ with typical fiber $\mathbb{F}_\Theta$ is given by $(Q \times \mathbb{F}_\Theta)/\sim$, where $(q_1, b_1) \sim (q_2, b_2)$ iff there exists $g \in G$ with $q_1 = q_2 \cdot g$ and $b_1 = g^{-1} \cdot b_2$.

Now let $\phi_t : Q \to Q$ be a flow of automorphisms, i.e., $\phi_t(q \cdot g) = \phi_t(q) \cdot g$, and assume that the induced flow on X is chain transitive. For the induced flow $\phi_t^\Theta : \mathbb{E}_\Theta \to \mathbb{E}_\Theta$ we have the following result (see Thm. 9.11 of [3], Thm. 5.2 of [13]).

Theorem 2.4. *There exist $H_\phi \in \mathrm{cl}(\mathfrak{a}^+)$ and a continuous ϕ-invariant map*

$$h : Q \to \mathrm{Ad}(G)H_\phi, \quad h(\phi_t(q)) = h(q),$$

satisfying $h(q \cdot g) = \mathrm{Ad}(g^{-1})h(q),\ q \in Q,\ g \in G$ *such that the induced flow on* $\mathbb{E}_\Theta$ *admits a finest Morse decomposition whose elements are given fiberwise by*

$$\mathcal{M}_\Theta(w)_{\pi(q)} = q \cdot \mathrm{fix}_\Theta(h(q), w).$$

The subset $\Theta(\phi) = \Theta(H_\phi)$ is called the **flag type of the flow** ϕ. The number of the Morse sets in $\mathbb{E}_\Theta$ is equal to the number of elements in the coset space $\mathcal{W}_{\Theta(\phi)} \backslash \mathcal{W} / \mathcal{W}_\Theta$.

3. **The main result.** The aim of this paper is to bring together two areas in which we can analyze the behavior of an invariant control system both, from the control theory and from the semigroup theory point of views. In particular, the main theorem implies that any chain control set of a locally accessible induced system contains a control set with nonempty interior, which is in general not necessarily true (see Example 3.4.2 of [5]). Moreover, when $\Theta(\mathcal{S}) = \Theta(\phi)$ every chain control set is the closure of the only control set with nonempty interior which it contains.

Let G be a noncompact connected semisimple Lie group with finite center and consider the right-invariant system Σ on G. Moreover, assume that Σ is a locally accessible right-invariant system on G, and therefore, that all the induced system Σ_Θ are also locally accessible. From now $\mathcal{S}$ will stand for the semigroup given by $\mathcal{O}^+(1)$, which by the local accessibility condition has nonempty interior in G.

Proposition 2. *A subset* $D \subset \mathbb{F}_\Theta$ *is an effective* $\mathcal{S}$*-control set if and only if it is a control set of* Σ_Θ *with nonempty interior.*

Proof. By Proposition 3.2.4 of [5] and the invariance of Σ we have that any effective $\mathcal{S}$-control set in $\mathbb{F}_\Theta$ is in fact a control set of Σ_Θ with nonempty interior.

Reciprocally, if $D \subset \mathbb{F}_\Theta$ is a control set with nonempty interior of Σ_Θ, we obtain by invariance that D is a $\mathcal{S}$-control set and we only have to show that it is effective.

By the local accessibility assumption, we have that $\mathrm{int}\, D$ is dense in D (see Lemma 3.2.13 of [5]). Moreover, since $\mathrm{int}\, \mathcal{S}$ is dense in $\mathcal{S}$ and $1 \in \mathcal{S}$ we get

$$(\mathrm{int}\, \mathcal{S})^{-1} \cdot x \cap D \neq \emptyset, \quad \text{for any} \quad x \in \mathrm{int}\, D.$$

Therefore, by property (ii) in the definition of control set,

$$(\mathrm{int}\, \mathcal{S})\, x \cap (\mathrm{int}\, \mathcal{S})\, x \neq \emptyset \qquad \text{for any} \qquad x \in \mathrm{int}\, D$$

which implies that D is in fact effective. We thus obtain the desired conclusion. □

Remark 3. We should note that Proposition 2 does not use the fact that the Lie group G is semisimple. Thus the result remains true for any induced system on an arbitrary homogeneous space.

Consider the G-principal bundle $\pi : \mathcal{U} \times G \to \mathcal{U}$, where π is the projection on the first factor and G acts trivially from the right on $\mathcal{U} \times G$ as $(u, g) \cdot h = (u, gh)$. The invariance of Σ shows that $\varphi(t, gh, u) = \varphi(t, g, u)h$ which implies that the control flow ϕ_t is a flow of automorphisms on $\mathcal{U} \times G$. Moreover, the induced flow on $\mathcal{U}$ coincides with the shift flow, which means that it is chain transitive. Therefore, Theorem 2.4 can be applied in order to obtain the Morse sets of the induced control flows ϕ^Θ for any $\Theta \subset \Lambda$.

Let $\mathcal{F}$ be the family of subset of $\mathcal{S}$ given by

$$\mathcal{F} := \left\{ \bigcup_{t > \tau} \mathcal{O}_t^+(e), \quad \tau > 0 \right\}.$$

We are thus led to prove our main result.

Theorem 3.1. *A subset $E \subset \mathbb{F}_\Theta$ is a $\mathcal{F}$-chain control set if and only if it is a chain control set of Σ_Θ.*

Proof. Our proof starts with the observation that for a subset E of $\mathbb{F}_\Theta$ the lift of E is given by

$$\mathcal{E} = \{(u, x) \in \mathcal{U} \times \mathbb{F}_\Theta; \quad \varphi^\Theta(\mathbb{R}, x, u) \subset E\}.$$

By Theorem 4.3.11 of [5] there exists a bijection between the Morse sets of ϕ^Θ and the chain control sets of the system Σ_Θ given by:

If $\mathcal{M}_\Theta$ is a Morse set of ϕ^Θ, then $\pi_2(\mathcal{M}_\Theta)$ is a chain control set of Σ_Θ, where $\pi_2 : \mathcal{U} \times \mathbb{F}_\Theta \to \mathbb{F}_\Theta$ is the projection on the second factor. Reciprocally, if E is a chain control set of Σ_Θ its lift $\mathcal{E}$ is a Morse set of ϕ^Θ.

The remainder of the proof falls naturally into three parts.

Step 1. If $E_\mathcal{F}$ is a $\mathcal{F}$-chain control set in $\mathbb{F}_\Theta$ then its lift $\mathcal{E}_\mathcal{F}$ is contained in a Morse set of ϕ^Θ.

By condition (ii) in the definition of $\mathcal{F}$-chain control sets and by the right-invariance of Σ we have that $E_\mathcal{F}$ is controlled chain transitive. By Theorem 4.3.11 one gets in particular, that the lift $\mathcal{E}_\mathcal{F}$ is chain transitive for the flow ϕ^Θ. Moreover, since $\mathcal{E}_\mathcal{F}$ is certainly ϕ^Θ-invariant, Theorem B.2.26 of [5] applies and $\mathcal{E}_\mathcal{F}$ is contained in some Morse set. In fact the Morse sets are the maximal ϕ^Θ-invariant chain transitive subsets of $\mathcal{U} \times \mathbb{F}_\Theta$.

Step 2. Any effective $\mathcal{F}$-chain control set in $\mathbb{F}_\Theta$ is a chain control set of Σ_Θ.

Let E be an effective $\mathcal{F}$-chain control in $\mathbb{F}_\Theta$ and $D \subset E$ an effective control set. By Proposition 2, D is a control set of Σ_Θ with nonempty interior. By Corollary 4.3.12 of [5] there exists a chain control set E' of Σ_Θ such that $D \subset E'$. By the very definition of $\mathcal{F}$-chain control sets and the fact that $\operatorname{int} D \neq \emptyset$ we get that E' is an effective $\mathcal{F}$-chain control set in $\mathbb{F}_\Theta$ which by maximality implies that $E' \subset E$. By the previous item the inclusion $E \subset E'$ must always happens and so $E = E'$ is a chain control set of Σ_Θ.

Step 3. Any chain control set of Σ_Θ is an effective $\mathcal{F}$-chain control set.

Let E be a chain control set of Σ_Θ. In order to show that E is a $\mathcal{F}$-chain control set, it is enough to show that it contains an effective $\mathcal{S}$-control set.

Let us define $\mathsf{h} : \mathcal{U} \to \operatorname{Ad}(G)H_\phi$ by $\mathsf{h}(u) := h(u, 1)$, where h is the map defined by Theorem 2.4 applied to the control flow ϕ on the principal bundle $\mathcal{U} \times G$. The map h is continuous and

$$\operatorname{Ad}(\varphi(t, e, u))\mathsf{h}(u) = \mathsf{h}(\theta_t u).$$

Moreover, the relation between the Morse sets and the chain control sets together with Theorem 2.4 give us that

$$E = \bigcup_{u \in \mathcal{U}} \operatorname{fix}_\Theta(\mathsf{h}(u), w), \quad \text{for some} \quad w \in \mathcal{W}.$$

Let $g \in \operatorname{int} \mathcal{S}$ and consider $u \in \mathcal{U}$ and $\tau > 0$ such that $g = \varphi(\tau, e, u)$. By extending periodically u to a τ-periodic control function $u^* \in \mathcal{U}$ it turns out that

$$g^n = \varphi(\tau, 1, u)^n = \varphi(n\tau, 1, u^*) \in \operatorname{int} \mathcal{S}.$$

Therefore,

$$g \operatorname{fix}_\Theta(\mathsf{h}(u^*), w) = \operatorname{fix}_\Theta(\operatorname{Ad}(g)\mathsf{h}(u^*), w) = \operatorname{fix}_\Theta(\mathsf{h}(\theta_\tau u^*), w) = \operatorname{fix}_\Theta(\mathsf{h}(u^*), w),$$

consequently $\operatorname{fix}_\Theta(\mathsf{h}(u^*), w)$ is a g-invariant compact set. Hence, there exists a nonempty subset $\Omega \subset \operatorname{fix}_\Theta(\mathsf{h}(u^*), w)$, which is minimal for the g-action in $\mathbb{F}_\Theta$. By

Proposition 2.3 of [14], the set Ω has to be contained in the interior of a $\mathcal{S}$-control set D in $\mathbb{F}_\Theta$. Since $\Omega \subset E$ we must have that $D \subset E$ showing that E is in fact an effective $\mathcal{F}$-chain control set which concludes the proof. $\square$

As a direct consequence of our main result we get.

Corollary 1. *Any chain control set of the system Σ_Θ contains a control set with nonempty interior.*

We have also that the flag type of ϕ and the $\mathcal{F}$-flag type of $\mathcal{S}$ coincides.

Corollary 2. *With the previous assumptions, it holds that*

$$\Theta(\mathcal{F}) = \Theta(\phi).$$

Proof. For any pair of subsets $\Theta_1, \Theta_2 \subset \Lambda$ we know

$$\Theta_1 = \Theta_2 \qquad \text{if and only if} \qquad \mathcal{W}_{\Theta_1} = \mathcal{W}_{\Theta_2}.$$

Therefore, it is enough for us to show that $\mathcal{W}_\mathcal{F}(\mathcal{S}) = \mathcal{W}_{\Theta(\phi)}$.

By Theorem 3.1 the family of the chain control sets of the induced systems in $\mathbb{F}$ and family of $\mathcal{F}$-chain control sets in $\mathbb{F}$ coincides. Consequently

$$|\mathcal{W}_\mathcal{F}(\mathcal{S})\backslash\mathcal{W}| = \left|\mathcal{W}_{\Theta(\phi)}\backslash\mathcal{W}\right| \text{ implying that } |\mathcal{W}_\mathcal{F}(\mathcal{S})| = \left|\mathcal{W}_{\Theta(\phi)}\right|.$$

So, it is enough to show that $\mathcal{W}_\mathcal{F}(\mathcal{S}) \subset \mathcal{W}_{\Theta(\phi)}$.

Since $\pi_2(\mathcal{M}(1))$ is $\mathcal{S}$-invariant, it contains the only $\mathcal{S}$-invariant control set in $\mathbb{F}$, and so $\pi_2(\mathcal{M}(1)) = E(1)$. Let us take an element $w \in \mathcal{W}_\mathcal{F}(\mathcal{S})$. By definition $E(w)$ is the only $\mathcal{F}$-chain control set containing $D(w)$. Moreover, by the choice of w,

$$E(w) = E(1) = \pi_2(\mathcal{M}(1)).$$

As a conclusion,

$$D(w) \subset \bigcup_{u\in\mathcal{U}} \mathrm{fix}(\mathsf{h}(u), 1).$$

Since any other choice of positive Weyl chamber just conjugates the flag types, w.l.o.g. we can assume that

$$\mathrm{int}\, \mathcal{S} \cap \exp \mathfrak{a}^+ \neq \emptyset.$$

It follows that $wb_0 \in D(w)_0$ and there exists $u \in \mathcal{U}$ with $wb_0 \in \mathrm{fix}(\mathsf{h}(u), 1)$. One can check that $wb_0 = kb_0$ where $k \in K$ is such that $\mathrm{Ad}(k)H_\phi = \mathsf{h}(u)$. The equality $wb_0 = kb_0$ implies that $wH_\phi = \mathrm{Ad}(k)H_\phi = \mathsf{h}(u)$. Finally,

$$\mathrm{fix}(\mathsf{h}(u), 1) = \mathrm{fix}(wH_\phi, 1) = (wZ_{H_\phi}w^{-1})wb_0 = \mathrm{fix}(wH_\phi, w) = \mathrm{fix}(\mathsf{h}(u), w).$$

Hence $w \in \mathcal{W}_{\Theta(\phi)}$ implying that $\mathcal{W}_\mathcal{F}(S) \subset \mathcal{W}_{\Theta(\phi)}$ and concluding the proof. $\square$

The above corollary implies in particular that $\Theta(\mathcal{S}) \subset \Theta(\phi)$. We are now interested to see what happens when the equality holds, that is, when any chain control set of Σ_Θ contains exactly one control set with nonempty interior.

Definition 3.2. Let $\Theta \subset \Lambda$ and $w \in \mathcal{W}$. We say that the chain control set $E_\Theta(w)$ is **(uniformly) hyperbolic** if for each $(u, x) \in \mathcal{M}_\Theta(w)$ there exists a decomposition

$$T_x\mathbb{F}_\Theta = E^-_{u,x} \oplus E^+_{u,x}$$

such that the following properties hold:

(H1) $(\mathrm{d}\varphi_{t,u})_x E^\pm_{u,x} = E^\pm_{\phi_t(u,x)}$ for all $t \in \mathbb{R}$ and $(u, x) \in \mathcal{M}_\Theta(w)$.

(H2) There exist constants $c \in (0,1]$, $\lambda > 0$ such that for all $(u,x) \in \mathcal{M}_\Theta(w)$ we have

$$|(\mathrm{d}\varphi_{t,u})_x v| \leq c^{-1}\mathrm{e}^{-\lambda t}|v| \quad \text{for all } t \geq 0,\ v \in E^-_{u,x},$$

and

$$|(\mathrm{d}\varphi_{t,u})_x v| \geq c\mathrm{e}^{\lambda t}|v| \quad \text{for all } t \geq 0,\ v \in E^+_{u,x}.$$

(H3) The linear subspaces $E^\pm_{u,x}$ depend continuously on (u,x), i.e., the projections $\pi^\pm_{u,x} : T_x\mathbb{F}_\Theta \to E^\pm_{u,x}$ along $E^\mp_{u,x}$ depend continuously on (u,x).

In [6] it is shown that any chain control set $E_\Theta(w)$ is partially hyperbolic, that is, the above decomposition of the tangent spaces have a third subspace that corresponds to the tangent space of the submanifold $\mathrm{fix}_\Theta(h(u),w)$. By Theorem 5.6 of [6], hyperbolicity happens if and only if $\langle\Theta(\phi)\rangle \subset w\langle\Theta\rangle$ if and only if for any $u \in \mathcal{U}$ the set $\mathrm{fix}_\Theta(h(u),w)$ consists of only one point. Consequently, if $E_\Theta(w)$ is hyperbolic, Theorem 6.1 of [6] implies that $\mathrm{cl}(D_\Theta(w)) = E_\Theta(w)$.

The concept of uniformly hyperbocity together with the assumption that the flag types agree give us the following strong result about control and chain control sets on flag manifolds.

Theorem 3.3. *$\Theta(\mathcal{S}) = \Theta(\phi)$ if, and only if, $D(1) = E(1)$.*

Proof. Let us first assume that $D(1) = E(1)$. Since $\Theta(\mathcal{S}) \subset \Theta(\phi)$ always holds, we only have to show the opposite inclusion. For any $w \in \mathcal{W}_{\Theta(\phi)}$ it follows that $D(w) \subset E(w) = E(1) = D(1)$ and since two control sets are either disjoints or coincides, we must have $D(w) = D(1)$ showing that $\mathcal{W}_{\Theta(\phi)} \subset \mathcal{W}_{\Theta(\mathcal{S})}$ and consequently $\Theta(\phi) \subset \Theta(\mathcal{S})$.

Let us reciprocally assume that $\Theta(\phi) = \Theta(\mathcal{S})$. By the previous discussion we have that the chain control set $E_{\Theta(\phi)}(1) \subset \mathbb{F}_{\Theta(\phi)}$ is uniformly hyperbolic and so, it is the closure of a control set. Since Theorem 3.1 implies that $D_{\Theta(\phi)}(1) \subset E_{\Theta(\phi)}(1)$ and $D_{\Theta(\phi)}(1)$ is closed, we must have

$$E_{\Theta(\phi)}(1) = \mathrm{cl}\left(D_{\Theta(\phi)}(1)\right) = D_{\Theta(\phi)}(1).$$

Moreover, $\pi^{-1}_{\Theta(\mathcal{S})}\left(D_{\Theta(\mathcal{S})}(1)\right) = D(1)$ by Theorem 4.3 of [14] and by Proposition 8.12 of [3], together with the relation between chain control sets and Morse sets, $\pi^{-1}_{\Theta(\phi)}\left(E_{\Theta(\phi)}(1)\right) = E(1)$ where $\pi_{\Theta(\mathcal{S})} : \mathbb{F} \to \mathbb{F}_{\Theta(\mathcal{S})}$ and $\pi_{\Theta(\phi)} : \mathbb{F} \to \mathbb{F}_{\Theta(\phi)}$ are the canonical projections. Therefore if $\Theta(\mathcal{S}) = \Theta(\phi)$ we have that

$$E(1) = \pi^{-1}_{\Theta(\phi)}\left(E_{\Theta(\phi)}(1)\right) = \pi^{-1}_{\Theta(\mathcal{S})}\left(D_{\Theta(\mathcal{S})}(1)\right) = D(1)$$

concluding the proof. □

Since the flag types of the reversed time system are given by $\Theta(\mathcal{S}^*) = -w_0\Theta(\mathcal{S})$ and $\Theta(\phi)^* = -w_0\Theta(\phi)$ ([3]) the equality $\Theta(\mathcal{S}) = \Theta(\phi)$ holds if and only if $\mathrm{cl}(D(w_0)) = E(w_0)$. Next we show that the equality of the flag types implies the equality above for any control control set, not just the invariant ones. In order to do that, we need to introduce some concepts.

For any $w \in \mathcal{W}$, the **domain of attraction** of $D(w)$ is the subset of $\mathbb{F}$ given by

$$\mathcal{A}(D(w)) = \{x \in \mathbb{F},\ gx \in D(w) \quad \text{for some} \ \ g \in \mathcal{S}\}.$$

The following lemma relates the domain of attraction and the core of an effective control set.

Lemma 3.4. *For any $w \in \mathcal{W}$ it holds that*

$$D(w)_0 = \mathcal{A}(D(w)) \cap \mathcal{A}^*(D^*(w)),$$

where $D^*(w)$ is the unique $\mathcal{S}^{-1}$-control set in $\mathbb{F}$ whose core is $D^*(w)_0 = D(w)_0$ and $\mathcal{A}^*(D^*(w))$ its domain of attraction.

Proof. Since $D(w)_0 = D^*(w)_0$ we already have that

$$D(w)_0 \subset \mathcal{A}(D(w)) \cap \mathcal{A}^*(D^*(w)).$$

For any $x \in \mathcal{A}(D(w))$ there exists $g \in \mathcal{S}$ such that $gx \in D(w)$. By using condition (ii) in the definition of $\mathcal{S}$-control sets we get that $D(w)_0 \subset D(w) \subset \mathrm{cl}(\mathcal{S}gx)$. Since $D(w)_0$ is an open set and $\mathrm{int}\,\mathcal{S}$ is dense in $\mathcal{S}$, we obtain

$$D(w)_0 \cap \mathrm{int}\,\mathcal{S}gx \neq \emptyset.$$

Analogously, if $x \in \mathcal{A}^*(D^*(w))$ we get that $D^*(w)_0 \cap \mathrm{int}\,\mathcal{S}^{-1}(g'x) \neq \emptyset$ for some $g' \in \mathcal{S}^{-1}$. Therefore, if $x \in \mathcal{A}(D(w)) \cap \mathcal{A}^*(D^*(w))$ there exists $h' \in \mathcal{S}^{-1}$ and $h \in \mathcal{S}$ such that $h'g'x, hgx \in D(w)_0$, since $D(w)_0 = D^*(w)_0$. Being that in $D(w)_0$ we have controllability, there exists $k \in \mathcal{S}$ such that

$$khgx = h'g'x \Rightarrow ((h'g')^{-1}khg)x = x.$$

However, since $\mathrm{int}\,\mathcal{S}$ is $\mathcal{S}$-invariant we get that $(h'g')^{-1}khg \in \mathrm{int}\,\mathcal{S}$ showing that $x \in D(w)_0$ and concluding the proof. □

By Theorem 6.3 of [12], the above domain of attraction can be characterized as follows: For a finite sequence $\alpha_1, \ldots, \alpha_n$ in Λ let us denote by $s_1, \ldots, s_n$ the reflections with respect these roots and consider $P_i := P_{\{\alpha_i\}}$. The corresponding flag manifold is $\mathbb{F}_i = G/P_i$ and we denote by π_i the canonical projection of $\mathbb{F}$ onto $\mathbb{F}_i$. Moreover, for a given subset X of $\mathbb{F}$ we denote by $\gamma_i(X) := \pi_i^{-1}\pi_i(X)$ the operation of exhausting the subset X with the fibers of π_i.

Now, take $w \in \mathcal{W}$ and consider the reduced expressions $w = r_m \cdots r_1$ and $w_0 w = s_n \cdots s_1$ with exhausting maps γ'_i and γ_j, respectively. It holds that

$$\mathcal{A}(D(w)) = \gamma_1 \cdots \gamma_n (D(w_0)) \quad \text{and} \quad \mathcal{A}^*(D^*(w)) = \gamma'_1 \cdots \gamma'_m (D(1)) \tag{2}$$

Concerning chain control sets, from Propositions 9.9 and 9.10 of [3] and the fact that $\pi_2(\mathcal{M}(w)) = E(w)$ it is straightforward to see that

$$E(w) \subset \gamma'_1 \cdots \gamma'_m (E(1)) \cap \gamma_1 \cdots \gamma_n (E(w_0)). \tag{3}$$

We can now prove the following result.

Theorem 3.5. *It holds that $\Theta(\phi) = \Theta(\mathcal{S})$ if and only if*

$$\mathrm{cl}(D_\Theta(w)) = E_\Theta(w), \tag{4}$$

for any $\Theta \subset \Lambda$, $w \in \mathcal{W}$.

Proof. By Theorem 3.3 we just need to show that equality (4) holds for any $\Theta \subset \Lambda$ and any $w \in \mathcal{W}$. However, for any $\Theta \subset \Lambda$, $w \in \mathcal{W}$ it holds that

$$\pi_\Theta(D(w)_0) = D_\Theta(w)_0 \quad \text{and} \quad \pi_\Theta(E(w)) = E_\Theta(w).$$

Therefore, it is missing to show the equality for the control sets on the maximal flag bundle $\mathbb{F}$.

If $\Theta(\phi) = \Theta(\mathcal{S})$, Theorem 3.3 implies that $D(1) = E(1)$ and also that $\mathrm{cl}(D(w_0)) = E(w_0)$. By (2) we get

$$\mathcal{A}^*(D^*(w)) = \gamma'_1 \cdots \gamma'_m (D(1)) = \gamma'_1 \cdots \gamma'_m (E(1)).$$

Moreover, since $\pi_i : \mathbb{F} \to \mathbb{F}_i$ is a continous open map between compact topological spaces, we have $\gamma_i(\mathrm{cl}(X))) = \mathrm{cl}(\gamma_i(X))$ for any subset X of $\mathbb{F}$, which together with (2) implies that

$$\mathrm{cl}\left(\mathcal{A}(D(w))\right) = \mathrm{cl}\left(\gamma_1 \cdots \gamma_n \left(D(w_0)\right)\right) = \gamma_1 \cdots \gamma_n(E(w_0))).$$

Using (3) we obtain

$$E(w) \subset \mathcal{A}^*(D^*(w)) \cap \mathrm{cl}\left(\mathcal{A}(D(w))\right).$$

Now, by Lemma 3.4 we get $\mathcal{A}^*(D^*(w)) \cap \mathcal{A}(D(w)) = D(w)_0$ and so

$$E(w) \subset \mathcal{A}^*(D^*(w)) \cap \mathrm{cl}\left(\mathcal{A}(D(w))\right) \subset \mathrm{cl}(D(w)_0) = \mathrm{cl}(D(w)) \subset E(w)$$

concluding the proof. □

4. **Examples.** In this section we give some examples where the condition on the equality of the flag types is satified.

Bilinear systems on sl(2)**:** In [1] the authors described all the possibilities of the control sets of a bilinear system on the projective space $\mathbb{P}$. In what follows we will use their description to show a case where we always have $\Theta(\phi) = \Theta(\mathcal{S})$.

Let us consider the family of bilinear control systems on $\mathbb{R}^2$ defined by

$$(\Sigma^\rho) \qquad \dot{x} = (X + uY)x, \qquad u \in U_\rho = [-\rho, \rho]$$

where $X, Y \in \mathrm{sl}(2)$. Following [1], if $\det[X,Y] \neq 0$ it turns out that $\mathrm{span}\{X, [X,Y]\} = \mathrm{sl}(2)$ and the systems Σ^ρ satisfies in particular the Lie algebra rank condition.

Let us still denote by Σ^ρ the induced system on $\mathbb{P}$. Under the assumptions that $\det X < 0$ and $\det[X,Y] > 0$, Theorem 5.2 of [1] assures that the map $\rho \mapsto D_\rho$ is continuous in the Hausdorff metric. Here, D^ρ denote the positive invariant control set on the induced system Σ^ρ on $\mathbb{P}$. Moreover, Lemma 4.7.4 (ii) of [5] implies that the continuity points of the map $\rho \mapsto D^\rho$ are exactly the points where $D^\rho = E^\rho$ is a chain control set.

On the other hand, sl(2) acts transitively on $\mathbb{P}$ so $\mathbb{P} = \mathrm{sl}(2)/P$ is a homogeneous space. A simple calculation shows that, in fact P is a minimal parabolic subgroup of G, implying that $\mathbb{P}$ is the maximal flag manifold $\mathbb{F}$ of sl(2). Therefore, $D^\rho = \mathrm{cl}\, D^\rho = E^\rho$ which means by Theorem 3.3 that $\Theta(\phi^\rho) = \Theta(\mathcal{S}^\rho)$ for any $\rho > 0$.

Control systems with small control range: In this example we show that, for invariant control systems with small control range we also have the equality of the flag types.

Let G be a noncompact connected semisimple Lie group and consider the invariant system

$$(\Sigma) \qquad \dot{g}(t) = X_0(g(t)) + \sum_{i=1}^{m} u_i(t) X_i(g(t)), \quad u \in \mathcal{U}.$$

Assume that

$$\mathrm{span}\{X_0, \mathrm{ad}(X_0)^k X_i, k \in \mathbb{N}, i = 1, \ldots, m\} = \mathfrak{g}.$$

and consider the induced system on $\mathbb{F}$. By Corollary 4.7.6 (ii) of [5], there exists $\rho^* > 0$ such that the induced system Σ^ρ on $\mathbb{F}$ satisfies $D^\rho(1) = E^\rho(1)$ for all up to at most countably many values of $\rho \in (0, \rho^*)$. Here Σ^ρ is the induced system on $\mathbb{F}$ whose set of control functions is $\mathcal{U}^\rho = L^\infty(\mathbb{R}, \rho\Omega)$. Therefore, by Theorem 3.3 we get that $\Theta(\phi^\rho) = \Theta(\mathcal{S}^\rho)$ for all up to at most countably many values of $\rho \in (0, \rho^*)$.

In particular, if X_0 is an element of a positive Weyl chamber $\mathfrak{a}^+$ for some Iwasawa decomposition, the origin $b_0 = 1 \cdot P \in \mathbb{F}$ is an hyperbolic fixed point of the uncontrolled system. Therefore, for $\rho > 0$ small enough we have that E^ρ is a hyperbolic chain control set. By Theorem 4.6 of [6] we get that $\Theta(\phi) = \emptyset$ and so $\Theta(\mathcal{S}) \subset \Theta(\phi) = \emptyset$ implying the equality.

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