Topological entropy of a Lie group automorphism

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Abstract Let \( \varphi \) be an automorphism on a connected Lie group \( G \). Through several \( G \)-subgroups associated to the dynamics of \( \varphi \) we analyze their topological entropy. Assume that \( G \) belongs to the class of finite semisimple center Lie groups which admits a \( \varphi \) invariant Levi subgroup. Then we prove that the topological entropy information of \( \varphi \) is contained in the toral component of the unstable subgroup of \( \varphi \) in the radical of \( G \). We specialize the main result in a couple of interesting situations.

Key words Lie group, automorphism, topological entropy, toral component

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1 Introduction

Let \( G \) be a connected Lie group with Lie algebra \( \mathfrak{g} \). In [3] and [7] was shown that associated to a given endomorphism \( \varphi \) on a connected Lie group \( G \) there are some subgroups of \( G \) that are intrinsically connected with the dynamic behavior of \( \varphi \). We use this set up to analyze the concept of topological entropy introduced

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by Caldas and Patrão in [10]. In fact, in [6] the authors consider the case where
the state space is a connected Lie group. They analyze the topological entropy
of proper endomorphisms on different classes of Lie groups. When the state
space is a nilpotent or linear reductive Lie group they show that the topological
entropy of a proper endomorphism of $G$ coincides with the topological entropy
of its restriction to $T(G)$. Here, $T(G)$ is the maximal connected compact subgroup
of the center $Z_G$ of $G$ called the toral subgroup of $G$. By using the dynamic
subgroups associated with any automorphism $\varphi$ we extend and refine the results
in [6]. In fact, assume that $G$ belongs to the class of finite semisimple center Lie
groups which admits a $\varphi$ invariant Levi subgroup. Therefore, the topological
entropy information of $\varphi$ coincides with the topological entropy of its restriction
to the toral component of the unstable subgroup of $\varphi$ in $R$. Where $R$ denotes
the radical of $G$. In particular, when $G$ is nilpotent or reductive Lie group in
the Harish-Chandra class, we prove that the topological entropy of $\varphi$ coincides
with the topological entropy of its restriction to the unstable subgroup of $T(G)$.
Actually the information given by the entropy is contained in a subgroup smaller
than $T(G)$. We also show that the condition on the existence of invariant Levi
components is far from be trivial.

The paper is structured as follows: In Section 2 we establish the result appear in
[3] relatives to $\mathfrak{g}$-subalgebras and $G$-subgroups containing the dynamic behavior
of a Lie algebra endomorphism and its consequences on the dynamic of the
associated Lie group endomorphism. In Section 3 we use the dynamic subgroups
to analyze the topological entropy of an automorphism. Then, we show our main
results and its consequences in some particular cases.

## 2 The dynamic of a Lie endomorphism

For general fact about Lie theory we refer to [8] and [12]. Let $G$ be a connected
Lie group of dimension $d$ with Lie algebra $\mathfrak{g}$ over a closed field. For a given Lie
groups $G$ and $H$ a continuous map $\varphi : G \to H$ is said to be a homomorphism
if it preserves the group structure. That is, $\varphi(gh) = \varphi(g)\varphi(h)$ for any $g,h \in G$.
If $G = H$ such map is said to be an endomorphism of $G$. Consider an endomorphism $\varphi : G \to G$ and denote by $\phi = (d\varphi)_e : \mathfrak{g} \to \mathfrak{g}$ the corresponding
Lie algebra endomorphism, where as usual $d\varphi$ denotes de derivative of $\varphi$. That
is, $\phi$ is a linear map satisfying $\phi[X,Y] = [\phi X, \phi Y]$ for any $X,Y \in \mathfrak{g}$. Here, $e$
denotes the identity element of $G$.

This section is dedicated to establish some results about Lie algebra endomor-
phism and its consequence on Lie groups endomorphism, appears in [3].

First, for any eigenvalue $\alpha$ of $\phi$ let us consider its generalized eigenspace
\[
\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} : (\phi - \alpha)^n X = 0, \text{ for some } n \geq 1 \}.
\]
If $\beta$ is also an eigenvalue of $\phi$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha\beta}$, where $\mathfrak{g}_{\alpha\beta} = \{0\}$ if $\alpha\beta$ is
not an eigenvalue of $\phi$. Associate to $\phi$ there are several Lie subalgebras that
are intrinsically connected with its dynamic. In fact, let us define the following subsets of \( g \) for any arbitrary \( \phi \)-eigenvalue \( \alpha \):

\[
g_\alpha = \bigoplus_{\alpha \neq 0} g_\alpha, \quad \ell_\alpha = \ker(\phi d), \quad g^+ = \bigoplus_{|\alpha|>1} g_\alpha, \quad g^0 = \bigoplus_{|\alpha|=1} g_\alpha \quad g^- = \bigoplus_{0<|\alpha|<1} g_\alpha.
\]

\[
g^{+0} = g^+ \oplus g^0 \oplus g^- \quad \text{and} \quad g^{-0} = g^- \oplus g^0.
\]

We denote also \( g_\phi = g^+ \oplus g^0 \oplus g^- \) and \( g = g_\phi \oplus \ell_\phi \). It turns out that all these subspaces are Lie subalgebras. Moreover, \( g^+ \) and \( g^- \) are nilpotent. If \( g \) is a real Lie algebra, the algebras above are also well defined.

**2.1 Remark:** We should note that in both cases, i.e., when \( g \) is real or complex, the restriction of \( \phi |_{g_\phi} \) is an automorphism of \( g_\phi \). Furthermore, the restriction of \( \phi \) to the Lie subalgebras \( g^+, g^0 \) and \( g^- \) satisfies the inequalities

\[
|\phi^m(X)| \geq c\mu^{-m}|X| \quad \text{for any} \quad X \in g^+ \quad \text{and} \quad m \in \mathbb{N},
\]

and

\[
|\phi^m(Y)| \leq c^{-1}\mu^m|Y| \quad \text{for any} \quad Y \in g^- \quad \text{and} \quad m \in \mathbb{N},
\]

for some \( c \geq 1 \) and \( \mu \in (0, 1) \). Actually, more is true,

\[
\text{for any} \ a > 0 \quad \text{and} \quad Z \in g^0 \quad \text{it holds that} \quad |\phi^m(Z)|\mu^{a|m|} \to 0 \quad \text{as} \ m \to \pm \infty.
\]

In the sequel any Lie group will be real. A Lie subgroup \( H \subset G \) is said to be \( \phi \)-invariant if \( \phi(H) \subset H \).

The **dynamic subgroups** of \( G \) induced by \( \varphi \) are the Lie subgroups, \( G_\varphi \), \( K_\varphi \), \( G^+, G^0, G^- \), \( G^{+0} \) and \( G^{-0} \) associated with the Lie subalgebras \( g_\varphi \), \( \ell_\varphi \), \( g^+ \), \( g^- \), \( g^{+0} \) and \( g^{-0} \), respectively. The subgroups \( G^+, G^0 \) and \( G^- \) are called the **unstable**, **central** and **stable** subgroups of \( \varphi \) in \( G \), respectively. Before to give in Proposition 2.3, Proposition 2.4 and Theorem 2.6 the main properties of these subgroups proved in [3], let us state a very special topological property of Lie subgroups that will also need in the next sections, [3].

**2.2 Lemma:** Let \( G \) be a Lie group with Lie algebra \( g \). And, \( H \) and \( K \) Lie subgroups of \( G \) with Lie algebras \( h \) and \( k \), respectively such that \( h \oplus k = g \). Then, \( H \) and \( K \) are closed \( \iff \) \( H \cap K \) is a discrete subgroup.

**2.3 Proposition:** It holds:

1. All the dynamic subgroups are \( \varphi \)-invariant
2. The subgroup \( K_\varphi = \ker(\varphi d)_0 \) is normal. Moreover,

\[
G = G_\varphi K_\varphi \quad \text{and} \quad G_\varphi = \text{Im}(\varphi d)
\]
3. The restriction of $\varphi$ is expanding on $G^+$ and contracting on $G^-$

4. If $G_\varphi$ is a solvable Lie group therefore

$$G_\varphi = G^{+0}G^- = G^{-0}G^+ = G^+G^0G^-$$

(1)

5. If $G_\varphi$ is semisimple and $G^0$ is compact, then $G_\varphi = G^0$. In particular, if $G$ is any connected Lie group such that $G^0$ is compact, then $G_\varphi$ has also the decomposition (1).

2.4 Definition: Let $\varphi$ be an endomorphism of the Lie group $G$. We say that $\varphi$ decomposes $G$ if $G_\varphi$ satisfy (1).

2.5 Proposition: Assume that $\varphi$ restricted to $G_\varphi$ is an automorphism in the induced topology of $G$. Then,

1. $G^{+0} \cap G^- = G^+ \cap G^- = G^0 \cap G^- = G^{-0} \cap G^+ = G^+ \cap G^0 = \{e\}$

2. The dynamic subgroups induced by $\varphi$ are closed in $G$

3. For $n \geq d$, $\ker(\varphi^n) = K_\varphi$. In particular, $\ker(\varphi^n)$ is connected.

4. If $G$ is simply connected then $G_\varphi$ and $K_\varphi$ are simply connected. Moreover, the restriction of $\varphi$ to $G_\varphi$ is an automorphism. And, any subgroup induced by an endomorphism $\varphi$ of $G$ is closed.

The main result in [3] shows that the unstable/stable subgroup of a compact $\varphi$-invariant subgroup of $G_\varphi$ is contained in its center. In particular, this implies the decomposition of the group when $G_\varphi$ is compact.

2.6 Theorem: Let $G$ be a Lie group and $\varphi$ an endomorphism of $G$. If $H \subset G_\varphi$ is a $\varphi$-invariant compact subgroup, then $H^+$ and $H^-$ are contained in the center $Z_H$ of $H$. In particular, if $G_\varphi$ is compact the group $G$ is decomposable. Furthermore, assume that $G$ is solvable. Therefore, if $\varphi|_{G^0_\varphi}$ is also an automorphism it follows that any fixed point of $\varphi$ is contained in $G^0$.

3 Topological entropy

In this section we show how the dynamic subgroups induced by any endomorphism can be used in order to get information about their dynamic behavior. We analyze the concept of topological entropy for continuous proper maps on locally compact separable metric spaces as introduced by Caldas and Patrão in [10].

Let $X$ be a topological space and $\xi : X \to X$ a proper map, that is, $\xi$ is a continuous map with the property that its pre-image of any compact set is still
a compact set. An admissible cover of $X$ is a finite open cover $\Psi$ such that any element $A \in \Psi$ has a compact closure or its complement is a compact set. For a given admissible cover $\Psi$ of $X$ and any $n \in \mathbb{N}$ the set

$$\Psi^n := \{ A_0 \cap \xi^{-1}(A_1) \cap \cdots \cap \xi^{-n}(A_n), \ A_1 \in \Psi \}$$

is also an admissible cover of $X$. We denote by $N(\Psi^n)$ the smallest cardinality of all subcovers of $\Psi^n$. The topological entropy of $\xi$ is then defined by

$$h_{\text{top}}(\xi) := \sup_{\Psi} h_{\text{top}}(\xi, \Psi), \quad \text{where} \quad h_{\text{top}}(\xi, \Psi) := \lim_{n \to \infty} \frac{1}{n} \log N(\Psi^n)$$

and $\Psi$ varies among all the admissible subcovers of $X$.

It is worth to notice that when $X$ is a compact space, the top-entropy definition coincide with the Adler-Konheim-MacAndrew topological entropy notion, [1].

For continuous maps on metric spaces, we have also the concept of entropy introduced by Bowen in [5] defined as follows: Let $(X, d)$ be a metric space and $\xi : X \to X$ a continuous map. Given a subset $Y \subset X$, $\varepsilon > 0$ and $n \in \mathbb{N}$ we say that $S \subset X$ is an $(n, \varepsilon)$-spanning set of $Y$ if for every $y \in Y$ there exists $x \in S$ such that

$$d(\xi^i(x), \xi^i(y)) < \varepsilon \quad \text{for} \quad 0 \leq i \leq n.$$ 

We denote by $s(n, \varepsilon, Y)$ the smallest cardinality of any $(n, \varepsilon)$-spanning set of $Y$ and define

$$s(\varepsilon, Y) := \lim_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, Y) \quad \text{and} \quad h(\xi, Y) := \lim_{\varepsilon \to 0^+} s(\varepsilon, Y).$$

The Bowen entropy of $\xi$ with respect to the metric $d$ is defined by

$$h_{d}(\xi) := \sup_{K} h(\xi, K)$$

where $K$ varies among the compact subsets of $X$.

We also mention the concept of $d$-entropy defined by

$$h_{d}^{d}(\xi) := \sup_{Y} h(\xi, Y).$$

In this case $Y$ varies among all the subsets of $X$. It holds that

$$h_{d}(\xi) \leq h_{d}^{d}(\xi).$$

Next, we introduce the concept of an admissible metric. The metric $d$ is admissible if it satisfies the following conditions:

(i) If $\Psi_\delta := \{ B_\delta(x_1), \ldots, B_\delta(x_n) \}$ is a cover of $X$ then for every $\delta \in (a, b)$ with $0 < a < b$, there exists $\delta_0 \in (a, b)$ such that $\Psi_{\delta_0}$ is admissible:
(ii) Every admissible cover of $X$ has a Lebesgue number.

The following result from [10], Proposition 2.2, show that for an admissible metrics the concept of $d$-entropy and topological entropy coincides.

3.1 Proposition: Let $\xi$ be a continuous map on the metric space $(X, d)$. If the metric $d$ is admissible, then

$$h_{\text{top}}(\xi) = h^d(\xi).$$

From here every topological space under consideration will be a locally compact separable metric spaces. Let $X$ be a topological space and denote by $X_\infty$ the one point compactification of $X$. We know that $X_\infty$ is defined as the disjoint union of $X$ with $\{\infty\}$ where $\infty$ is some point that does not belongs to $X$ called the point at the infinite. The topology in $X_\infty$ consists by the former open sets in $X$ and by the sets $U \cup \{\infty\}$, where the complement of $U$ in $X$ is compact. Let $Y$ be a topological space and $\xi : X \rightarrow Y$ a proper map. Define $\tilde{\xi} : X_\infty \rightarrow Y_\infty$ by

$$\tilde{\xi}(x) = \begin{cases} \xi(x), & x \neq \infty_X \\ \infty_Y, & x = \infty_X \end{cases}$$

then, $\tilde{\xi}$ is a continuous map called the extension of $\xi$. The following result (Proposition 2.3 of [10]) assures the existence of an admissible metrics on the topological spaces that we are considering in here.

3.2 Proposition: Let $\xi$ be a proper map of $X$ and consider the metric $d$ in $X$ obtained by the restriction of some metric $\tilde{d}$ on $X_\infty$. It turns out that $d$ is an admissible metric and

$$h^d(\xi) = h^{\tilde{d}}(\tilde{\xi}),$$

where $\tilde{\xi}$ is the extension of $\xi$. In particular, $h_{\text{top}}(\xi) = h_{\text{top}}(\tilde{\xi}).$

Let $Y \subset X$ be a closed subspace of $X$. By taking the closure of $Y$ in $X_\infty$ we obtain that $\text{cl}_{X_\infty}(Y) = Y_\infty$. By Proposition 3.2, it follows that the restriction $d_Y$ of an admissible metric $d$ to $Y$ is an admissible metric in $Y$. Moreover, if $\xi$ is a proper map of $X$ and $Y$ is $\xi$-invariant, we have that $\xi|_{Y_\infty} = \xi|_{Y}$. Therefore, we get the following inequality.

3.3 Proposition: If $\xi : X \rightarrow X$ is a proper map and $Y \subset X$ is a closed $\xi$-invariant subspace, then

$$h_{\text{top}}(\xi|_Y) \leq h_{\text{top}}(\xi).$$

Proof: By considering the extension $\tilde{\xi}$ of $\xi$ and an admissible metric $d$ on $X$, given by the restriction of some metric $\tilde{d}$ on $X_\infty$, we have that

$$h_{\text{top}}(\xi) = h^d(\xi) = h^{\tilde{d}}(\tilde{\xi}) = h^{\tilde{d}}(\tilde{\xi}) \geq h^{\tilde{d}}(\tilde{\xi}, Y_\infty) = h^{\tilde{d}_{Y_\infty}}(\tilde{\xi}|_{Y_\infty})$$
\[ h_{\text{top}} (\xi|_Y) = h^d (\xi|_Y) = h_{\text{top}} (\xi|_Y) \]

as stated. □

The proof of the next two propositions can be found in [6], Proposition 2.2 and [10], Proposition 2.1, respectively.

3.4 Proposition: Let \( X \) and \( Y \) be topological spaces and \( \xi : X \to X \), \( \zeta : Y \to Y \) proper maps. Then

\[ h_{\text{top}} (\xi \times \zeta) = h_{\text{top}} (\xi) + h_{\text{top}} (\zeta) . \]

3.5 Proposition: Let \( X \) and \( Y \) be topological spaces, \( \xi : X \to X \) and \( \zeta : Y \to Y \) two proper maps. If \( f : X \to Y \) is a proper surjective map such that \( f \circ \xi = \zeta \circ f \), then \( h_{\text{top}} (\xi) \geq h_{\text{top}} (\zeta) \).

3.1 Topological entropy of automorphisms

In this section we show that all the information given by the topological entropy of an automorphism \( \varphi \) of \( G \) is contained in the toral part of the unstable subgroup of \( \varphi \) in the radical of \( G \).

3.6 Definition: Let \( G \) be a Lie group and denote by \( Z_G \) its center. The toral component \( T(G) \) of \( G \) is the greatest compact connected Lie subgroup of \( Z_G \).

We start by giving a lower bound for the topological entropy.

3.7 Proposition: If \( \varphi \) is an automorphism of \( G \), then

\[ h_{\text{top}} (\varphi) \geq h_{\text{top}} (\varphi|_{T(G^+)}) \]

where \( T(G^+) \) is the toral component of the subgroup \( G^+ \).

Proof: Since \( \varphi \) is an automorphism, \( G^+ \) is a closed subgroup of \( G \). By Proposition 3.3 we know that \( h_{\text{top}} (\varphi) \geq h_{\text{top}} (\varphi|_{G^+}) \). Furthermore, \( G^+ \) is a connected nilpotent Lie group and \( \varphi|_{G^+} \) is an automorphism, so Theorem 4.3 of [6] assures that \( h_{\text{top}} (\varphi|_{G^+}) = h_{\text{top}} (\varphi|_{T(G^+)}) \). Therefore,

\[ h_{\text{top}} (\varphi) \geq h_{\text{top}} (\varphi|_{G^+}) = h_{\text{top}} (\varphi|_{T(G^+)}) . \]

□

As a consequence of Proposition 3.7 we obtain: in order to check if the topological entropy of any automorphism of \( G \) is not zero, it is enough to restrict it to a considerable smaller subgroup of \( G \). Actually, in the next sections we show that for a vast class of Lie groups the topological entropy of an automorphism is completely determined by its restriction to the toral component of the unstable subgroup of the radical of \( G \).

Furthermore, for an automorphism which turn out \( G \) decomposable, we get
3.8 Theorem: Let $G$ be a connected Lie group and $\varphi$ an automorphism. If $G$ is decomposable, then
$$h_{\text{top}}(\varphi) = h_{\text{top}}(\varphi|_{T(G^+)})$$

Proof: By Proposition 2.5 we obtain
$$G^+ \cap G^0 = G^+ \cap G^- = G^+ \cap G^- = \{e\}$$
Since $G$ is decomposable, the map $f : G^+ \times G^0 \times G^- \to G$ is a homeomorphism that commutes $\varphi$ and $\varphi|_{G^+} \times \varphi|_{G^0} \times \varphi|_{G^-}$. Therefore, by Propositions 3.4 and 3.5 we get that
$$h_{\text{top}}(\varphi) \leq h_{\text{top}}(\varphi|_{G^+}) + h_{\text{top}}(\varphi|_{G^0}) + h_{\text{top}}(\varphi|_{G^-}).$$
Since $\varphi|_{G^+}$ and $\varphi|_{G^0}$ have only eigenvalues with modulo smaller or equal to 1, Corollary 16 of [5] and Theorem 3.2 of [10] implies that $h_{\text{top}}(\varphi|_{G^-}) = h_{\text{top}}(\varphi|_{G^0}) = 0$ and therefore $h_{\text{top}}(\varphi) \leq h_{\text{top}}(\varphi|_{G^+})$. Moreover, since $G^+$ is a connected nilpotent Lie group, we obtain by Theorem 4.3 of [6] that $h_{\text{top}}(\varphi|_{G^+}) = h_{\text{top}}(\varphi|_{T(G^+)})$ which, together with Proposition 3.7 gives us
$$h_{\text{top}}(\varphi) = h_{\text{top}}(\varphi|_{T(G^+)})$$
as we wish to prove. \Box

3.9 Corollary: If $\varphi$ is an automorphism of a solvable Lie group $G$ it follows that
$$h_{\text{top}}(\varphi) = h_{\text{top}}(\varphi|_{T(G^+)})$$
In particular, when $G$ is nilpotent, we get $T(G^+) = T(G)^+$. Proof: By Proposition 2.3, any solvable Lie group $G$ is decomposable and the result follows from Theorem 3.8 above. Let us assume now that $G$ is a nilpotent Lie group. Since $T(G)^+$ is a compact subgroup of $Z_{G^+}$ the inclusion $T(G)^+ \subset T(G^+)$ always hold, and we only have to show that $T(G^+) \subset T(G)^+$. By Proposition 4.1 of [6], the nilpotent Lie group $G/T(G)$ is simply connected and so the compact subgroup $\pi(T(G^+))$ has to be trivial, where
$$\pi : G \to G/T(G)$$
is the canonical projection. Consequently, $T(G^+) \subset \ker(\pi) = T(G)$ showing that $T(G^+) \subset T(G)^+$ and concluding the proof. \Box

3.10 Corollary: If $\varphi$ is an automorphism of a Lie group $G$ and $G^0$ is a compact subgroup, then it follows that
$$h_{\text{top}}(\varphi) = h_{\text{top}}(\varphi|_{T(G^+)})$$
In particular, if $G$ is a connected and compact Lie group $T(G^+) = T(G)^+$. 8
Proof: The first assertion follows from the assumption $G$ is decomposable. In addition, if $G$ is also compact, Proposition 2.6 implies $G^+ \subset Z_G$ showing that $T(G^+) = T(G)^+$ as stated. \hfill \square

3.11 Corollary: If $G$ is a simply connected solvable Lie group and $\varphi$ an automorphism, then
\[ h_{\text{top}}(\varphi) = 0. \]

Proof: In fact, it is well known that any connected subgroup of a simply connected solvable Lie group is simply connected, [13],[8]. Therefore, $G^+$ is simply connected and consequently $T(G^+) = \{e\}$. \hfill \square

### 3.2 Lie groups with finite semisimple center

In [2] the authors introduce the notion of finite semisimple center Lie group $G$ to study the controllability property of a linear control systems on $G$, [4]. In this section we compute the entropy of any automorphisms $\varphi$ for this special class of Lie groups. If $G$ admit a $\varphi$-invariant Levi component, their topological entropy depends just on the unstable subgroup of the radical of $G$.

3.12 Definition: Let $G$ be a connected Lie group. We say that $G$ has finite semisimple center if any semisimple subgroup of $G$ has finite center.

We should notice that there are many groups with such property. For instance, solvable Lie groups and semisimple Lie groups with finite center and also their semi-direct products. Moreover, any reductive Lie group in the Harish-Chandra class has also finite semisimple center.

For a given Lie algebra $\mathfrak{g}$, a **Levi subalgebra** of $\mathfrak{g}$ is a maximal semisimple Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, in the sense, that for any semisimple Lie subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, there exists and inner automorphism $\psi$ such that $\psi(\mathfrak{l}) \subset \mathfrak{s}$. By Theorem 4.1 of [11], any Lie algebra admits a Levi subalgebra. If $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$ a **Levi subgroup** of $G$ is a connected semisimple Lie group $S \subset G$ such that its Lie algebra $\mathfrak{s} \subset \mathfrak{g}$ is a Levi subalgebra.

Let us denote by $r$ for the radical of the Lie algebra $\mathfrak{g}$ and by $R$ the radical of $G$, that is, $R$ is the closed connected Lie subgroup of $G$ with Lie algebra $r$. From Theorem 4.1 of [11] and its corollaries, we have that $G$ decomposes as $G = RS$ with $\dim(S \cap R) = 0$. For groups with finite semisimple center we obtain.

3.13 Proposition: If $G$ is a connected Lie group with finite semisimple center then $R \cap S$ is a discrete finite subgroup of $G$. Thus, any Levi subgroup $S$ of $G$ is closed in $G$.

Proof: Since $S$ is connected, any discrete normal subgroup of $S$ is contained in its center. However, $R$ is normal so $R \cap S$ is also a normal subgroup of $S$. 

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The result follows because $Z_S$ is finite by assumption. Moreover, by Lemma 2.2, $R \cap S$ is discrete if and only if $R$ and $S$ are closed subgroups of $G$. Since any Levi subgroup is conjugated to $S$ (see Corollary 3, Chapter I of [11]) the prove is finish. \hfill \square

Now, we are in a position to establish and prove our main result about the topological entropy of automorphisms.

**3.14 Theorem:** Let $G$ be a Lie group and $\varphi$ a $G$-automorphism. If $G$ has finite semisimple center and admits a $\varphi$ invariant Levi subgroup, then

$$h_{\text{top}}(\varphi) = h_{\text{top}}(\varphi|_{T(R^+)}) ,$$

where $R^+$ is the corresponding unstable component of the radical $R$ of $G$.

**Proof:** Let $S$ be the $\varphi$-invariant Levi subgroup of $G$. Since $\varphi$ is an automorphism we know that $R$ is also $\varphi$-invariant. Since $G$ has the semisimple finite center property, Proposition 3.13 implies that $S$ is a closed subgroup and therefore $R \cap S$ is a discrete finite subgroup of $G$.

Let us consider the product map $f : R \times S \to G$ defined by $(r,s) \mapsto rs$. By Malcev’s Theorem (see [11], Theorem 4.1) we get that $f$ is surjective. On the other hand, since $\varphi$ is an automorphism it follows that $f$ commutes with $\varphi$ and $\varphi|_R \times \varphi|_S$. Moreover, it holds:

1. $f$ is a continuous closed map.

   In fact, if $r_n \to r$ in $R$ and $s_n \to s$ in $S$ then $r_n s_n \to rs$ in $G$, showing that $f$ is continuous. Let us consider two closed subsets $A \subset R$ and $B \subset S$ and assume that $x_n = r_n s_n \to rs$ with $r_n \in A$, $s_n \in B$, $r \in R$ and $s \in S$. By Lemma 6.14 of [13] there exist neighborhoods $0 \in V \subset r$, $0 \in U \subset s$ and $e \in W \subset G$ such that $(X,Y) \mapsto e^X e^Y \in W$ is a diffeomorphism. Since $r_n s_n \to rs$, there exists $n_0 \in \mathbb{N}$ such that $r_n^{-1} r s_n^{-1} \in W$ for $n \geq n_0$ and so, there are sequences $(v_n) \subset e^V$, $(u_n) \subset e^U$ such that $v_n, u_n \to e$ and $r^{-1} r_n s_n^{-1} = v_n u_n$ for $n \geq n_0$. Therefore,

$$v_n^{-1} r_n^{-1} r_n = u_n s^{-1} \in R \cap S .$$

However, $R \cap S$ is a finite discrete subgroup. Thus, without lost of generality we can assume that $v_n^{-1} r_n^{-1} r_n = u_n s^{-1} = l$ for some $l \in R \cap S$. Consequently, $r_n = rv_n l \to rl$ and $s_n = l^{-1} u_n s \to l^{-1} s$. Since $A$ and $B$ are closed subsets, we must have $rl \in A$ and $l^{-1} s \in B$ implying that $rs = rll^{-1} s \in AB$ and showing that $f$ is in fact a closed map.

2. $f$ is a proper map.

   Since $f$ is continuous and closed we just need to show that for any $x \in G$ the set $f^{-1}(x)$ is compact. However, using that $x = rs$ for some $r \in R$ and $s \in S$, it is straightforward to check that

$$f^{-1}(x) = \{ (rl, l^{-1} s), \ l \in R \cap S \} .$$
which shows that \( f^{-1}(x) \) is in fact a finite subset of \( R \times S \).

By Propositions 3.3, 3.4, 3.5 we get

\[
h_{\text{top}}(\varphi|_R) + h_{\text{top}}(\varphi|_S) = h_{\text{top}}(\varphi|_R \times \varphi|_S) \geq h_{\text{top}}(\varphi|_G) \geq h_{\text{top}}(\varphi|_R) .
\]

By Theorem 5.3 of [6] the entropy of any surjective endomorphism of a semisimple Lie group is zero. Therefore, \( h_{\text{top}}(\varphi|_S) = 0 \) which combined with Theorem 3.8 give us

\[
h_{\text{top}}(\varphi) = h_{\text{top}}(\varphi|_R) = h_{\text{top}}(\varphi|_{T(R^+)})
\]

as we wish to prove. \( \square \)

In particular we obtain

**3.15 Corollary:** If \( G \) is a reductive Lie group in the Harish-Chandra class, then the entropy of any automorphism \( \varphi \) of \( G \) satisfies

\[
h_{\text{top}}(\varphi) = h_{\text{top}}\left(\varphi|_{T(G)^+}\right).
\]

**Proof:** In fact, if \( G \) is a reductive Lie group in the Harish-Chandra class, we know that \( G = Z_G G' \) where \( G' \) is the derived subgroup of \( G \). Moreover, \( G' \) is a maximal connected semisimple Lie subgroup of \( G \) and has finite center. Of course, since \( G' \) is certainly \( \varphi \)-invariant we get from our Theorem 3.14 that

\[
h_{\text{top}}(\varphi) = h_{\text{top}}\left(\varphi|_{T(Z_G^+)}\right).
\]

However, \( T(Z_G^+) \) is a compact connected subgroup in \( Z_G \cap G^+ \subset Z_G^+ \) showing that \( T(Z_G^+) \subset T(G)^+ \). Now, since \( T(Z_G^+) \subset Z_G \) we must have that \( T(Z_G^+) \subset T(G) \) and consequently \( T(Z_G^+) = T(G)^+ \) concluding the proof. \( \square \)

For a simply connected solvable Lie group \( G \) we prove in Corollary 3.11 that \( h_{\text{top}}(\varphi) = 0 \), for any \( \varphi \) automorphism. It is possible to extend this result as follows.

**3.16 Corollary:** Let \( G \) be a simply connected Lie group with finite center and \( \varphi \) an automorphism of \( G \). If \( G \) admits a \( \varphi \)-invariant Levi subgroup then

\[
h_{\text{top}}(\varphi) = 0.
\]

In the sequel, we show that the condition on the existence of invariant Levi components is far from be trivial.

**3.17 Definition:** We say that an automorphism \( \varphi \) of \( G \) is **semisimple** if its differential at \( e \in G \) is semisimple. That is, \( (d\varphi)_e \) is diagonalizable as a map of the complexification of \( \mathfrak{g} \).

For this particular class of automorphisms we get.
3.18 Corollary: Let $G$ be a Lie group and $\varphi$ a semisimple automorphism of $G$. If $G$ has the finite semisimple center property, then

$$h_{\text{top}}(\varphi) = h_{\text{top}}(\varphi|_{T(R^+)}) .$$

Proof: By Corollary 5.2 of [9], any group of semisimple automorphisms of a Lie algebra leaves invariant a Levi subalgebra. Therefore, there exists a Levi subalgebra $\mathfrak{s} \subset \mathfrak{g}$ that is invariant by the group of semisimple automorphisms $\{\varphi^n; \ n \in \mathbb{Z}\}$. Hence, the Levi subgroup $S \subset G$ with Lie algebra $\mathfrak{s}$ is $\varphi$-invariant and by Theorem 3.14 we obtain

$$h_{\text{top}}(\varphi) = h_{\text{top}}(\varphi|_{T(R^+)}) .$$

References


