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The dynamic of a Lie group endomorphism

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Abstract: For a given endomorphism φ on a connected Lie group G this paper studies several subgroups of G that are intrinsically connected with the dynamic behavior of φ .

Keywords: Lie group, Dynamic, Endomorphism

MSC: 20K30, 22E15

1 Introduction

In [1] was shown that associated to a given continuous flow of automorphisms on a connected Lie group G there are dynamical subgroups of G that are intrinsically connected with the behavior of the flow. The author shows there that only by looking at such subgroups one can get information about the controllability of any control system whose drift generates a 1-parameter flow of automorphisms. In the present paper we extend such results by showing that for any G -endomorphism, one can also define such subgroups and they still share many of the properties of the continuous case.

On the other hand, we use the results of this article to study the notion of entropy in our forthcoming paper "Topological entropy of Lie group automorphisms".

The paper is structured as follows. In Section 2 we introduce the subalgebras induced by an arbitrary endomorphism ϕ on the Lie algebra \mathfrak{g} . Then, we show that \mathfrak{g} decomposes in a dynamical way. In Section 3 we prove that the \mathfrak{g} -decompositions can be carried on to a connected Lie group. And the associated endomorphism φ of G allows us to associate to φ subgroups that contains most of its dynamical behavior. In the sequence, we establish the main properties of such subgroups. At the end we show an example on the Euclidean Lie group \mathbb{R}^d and on $Sl(n, \mathbb{R})$, the group of real matrices of order 2 and trace 0.

2 Lie algebra endomorphisms

The aim of this section is to introduce the Lie subalgebras induced by a \mathfrak{g} -endomorphism and show their main properties. For general facts on Lie algebras we use the reference [2].

Let \mathfrak{g} be a Lie algebra of dimension d and assume that $\phi : \mathfrak{g} \rightarrow \mathfrak{g}$ is an endomorphism of \mathfrak{g} . That is, ϕ is a linear map satisfying

$$\phi[X, Y] = [\phi X, \phi Y] \text{ for any } X, Y \in \mathfrak{g}.$$

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Proposition 2.1. Let \mathfrak{g} be a Lie algebra over a closed field and $\phi : \mathfrak{g} \rightarrow \mathfrak{g}$ an endomorphism. For any eigenvalue α of ϕ let us consider its generalized eigenspace given by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}; (\phi - \alpha)^n X = 0, \text{ for some } n \geq 1\}.$$

If β is also an eigenvalue of ϕ then

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha\beta}, \quad (1)$$

where $\mathfrak{g}_{\alpha\beta} = \{0\}$ if $\alpha\beta$ is not an eigenvalue of ϕ .

Proof. In order to decomposes the ϕ -eigenspace \mathfrak{g}_λ in its Jordan components, we consider r linear independent vectors $Z_1, \dots, Z_r \in \mathfrak{g}_\lambda$ such that

$$\phi(Z_j) = \lambda Z_j + Z_{j-1}, \quad j = 1, \dots, r \quad \text{with } Z_0 = 0.$$

To prove the proposition it is enough to show the following:

$$\text{if } \{X_1, \dots, X_n\} \subset \mathfrak{g}_\alpha \text{ and } \{Y_1, \dots, Y_m\} \subset \mathfrak{g}_\beta$$

are linearly independent sets, hence

$$[X_i, Y_j] \subset \mathfrak{g}_{\alpha\beta}, \quad i = 1, \dots, n; \quad j = 1, \dots, m.$$

The proof is done by induction on the sum $i + j$. In fact, since

$$\begin{aligned} \phi[X_i, Y_j] &= [\phi X_i, \phi Y_j] = [\alpha X_i + X_{i-1}, \beta Y_j + Y_{j-1}] \\ &= \alpha\beta[X_i, Y_j] + \alpha[X_i, Y_{j-1}] + \beta[X_{i-1}, Y_j] + [X_{i-1}, Y_{j-1}] \end{aligned}$$

we obtain

$$(\phi - \alpha\beta)[X_i, Y_j] = \alpha[X_i, Y_{j-1}] + \beta[X_{i-1}, Y_j] + [X_{i-1}, Y_{j-1}]. \quad (2)$$

If $i = j = 1$ we get $(\phi - \alpha\beta)[X_1, Y_1] = 0$ which implies $[X_1, Y_1] \in \mathfrak{g}_{\alpha\beta}$. Let us assume that the result holds for $i + j < n$ and let $i + j = n$. By the induction hypothesis, every term in the right-side of equation (2) is in $\mathfrak{g}_{\alpha\beta}$ which implies

$$(\phi - \alpha\beta)[X_i, Y_j] \in \ker((\phi - \alpha\beta)^n) \text{ for some } n \geq 1.$$

Consequently,

$$(\phi - \alpha\beta)^{n+1}[X_i, Y_j] = 0$$

showing that $[X_i, Y_j] \in \mathfrak{g}_{\alpha\beta}$ and concluding the proof. \square

In the sequel we prove a primary decomposition for any \mathfrak{g} -automorphism.

Proposition 2.2. Let ϕ be an automorphism of \mathfrak{g} and consider its Jordan decomposition

$$\phi = \phi^S \phi^N = \phi^N \phi^S$$

with ϕ^S semisimple and ϕ^N unipotent. Then ϕ^S and ϕ^N are also \mathfrak{g} -automorphisms.

Proof. Without lost of generality we can assume that the field of the scalars is algebraically closed. To prove that ϕ^S is an automorphism, it is enough to show that

$$\phi^S([X, Y]) = [\phi^S(X), \phi^S(Y)]$$

for every couple of basis elements.

Since \mathfrak{g} is decomposed in generalized eigenspaces of ϕ it is enough to show that ϕ^S satisfies the property of automorphisms for $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$ and α, β eigenvalues of ϕ . From Proposition 1, $[X, Y] \in \mathfrak{g}_{\alpha\beta}$. On the other hand, since the eigenspaces of ϕ and ϕ^S coincide, we get

$$\phi^S([X, Y]) = \alpha \cdot \beta[X, Y] \text{ and } [\phi^S(X), \phi^S(Y)] = [\alpha X, \beta Y] = \alpha \cdot \beta[X, Y]$$

showing that ϕ^S is in fact an automorphism. Therefore,

$$\phi^N = (\phi^S)^{-1} \phi$$

is also an automorphism ending the proof. □

Let \mathfrak{g} be a Lie algebra over a closed field. Proposition 2.2 allows to associate to any endomorphism ϕ of \mathfrak{g} several Lie subalgebras that are intrinsically connected with its dynamics. In fact, let us define the following subsets of \mathfrak{g} where α is an arbitrary ϕ -eigenvalue

$$\begin{aligned} \mathfrak{g}_\phi &= \bigoplus_{\alpha=0} \mathfrak{g}_\alpha, & \mathfrak{k}_\phi &= \ker(\phi^d) \\ \mathfrak{g}^+ &= \bigoplus_{|\alpha|>1} \mathfrak{g}_\alpha, & \mathfrak{g}^0 &= \bigoplus_{\alpha:|\alpha|=1} \mathfrak{g}_\alpha & \mathfrak{g}^- &= \bigoplus_{0<|\alpha|<1} \mathfrak{g}_\alpha, \\ \mathfrak{g}^{+,0} &= \mathfrak{g}^+ \oplus \mathfrak{g}^0 & \text{and} & & \mathfrak{g}^{-,0} &= \mathfrak{g}^- \oplus \mathfrak{g}^0. \end{aligned}$$

Also, we denote by $\mathfrak{g}_\phi = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$ and $\mathfrak{g} = \mathfrak{g}_\phi \oplus \mathfrak{k}_\phi$. By the property (1) is easy to see that all these subspaces are in fact Lie subalgebras. Moreover, \mathfrak{g}^+ and \mathfrak{g}^- are nilpotent.

If \mathfrak{g} is a real Lie algebra, the algebras above are well defined. In fact, let us denote by $\bar{\mathfrak{g}}$ the complexification of \mathfrak{g} . By considering the $\bar{\mathfrak{g}}$ -endomorphism $\bar{\phi}$ induced by ϕ we can define the subalgebras

$$\bar{\mathfrak{g}}_{\bar{\phi}}, \quad \bar{\mathfrak{k}}_{\bar{\phi}}, \quad \bar{\mathfrak{g}}^*, \quad \text{where } * = +, 0, -.$$

Moreover, since all the mentioned $\bar{\mathfrak{g}}$ -subalgebras are invariant by complex conjugation, they are also the complexification of the following ϕ -invariant subalgebras of \mathfrak{g}

$$\mathfrak{g}_\phi = \bar{\mathfrak{g}}_{\bar{\phi}} \cap \mathfrak{g}, \quad \mathfrak{k}_\phi = \bar{\mathfrak{k}}_{\bar{\phi}} \cap \mathfrak{g}, \quad \text{and} \quad \mathfrak{g}^* = \bar{\mathfrak{g}}^* \cap \mathfrak{g}$$

with \mathfrak{g}^+ and \mathfrak{g}^- nilpotent Lie subalgebras. We should notice that the equality $\mathfrak{k}_\phi = \ker(\phi^d)$ is still true.

Remark 2.3. *In the real or complex case the restriction of $\phi|_{\mathfrak{g}_\phi}$ is an automorphism of \mathfrak{g}_ϕ . Furthermore, the restriction of ϕ to the Lie subalgebras \mathfrak{g}^+ , \mathfrak{g}^0 and \mathfrak{g}^- satisfies the inequalities*

$$|\phi^m(X)| \geq c\mu^{-m}|X| \text{ for any } X \in \mathfrak{g}^+ \text{ and } m \in \mathbb{N},$$

and

$$|\phi^m(Y)| \leq c^{-1}\mu^m|Y| \text{ for any } Y \in \mathfrak{g}^- \text{ and } m \in \mathbb{N},$$

for some $c \geq 1$ and $\mu \in (0, 1)$.

Furthermore, for all $a > 0$ and $Z \in \mathfrak{g}^0$ it holds that

$$|\phi^m(Z)|\mu^{a|m|} \rightarrow 0 \text{ as } m \rightarrow \pm\infty.$$

In the sequel we prove that any linear map commuting with two endomorphisms preserves their associated decompositions.

Proposition 2.4. *For $i = 1, 2$, let $\phi_i : \mathfrak{g}_i \rightarrow \mathfrak{g}_i$ an endomorphism of the Lie algebra \mathfrak{g}_i over a closed field. Assume that $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a surjective linear map such that $f \circ \phi_1 = \phi_2 \circ f$. Hence, it holds*

$$\begin{aligned} f(\mathfrak{g}_{\phi_1}) &= \mathfrak{g}_{\phi_2}, & f(\mathfrak{k}_{\phi_1}) &= \mathfrak{k}_{\phi_2}, \\ f(\mathfrak{g}_1^+) &= \mathfrak{g}_2^+, & f(\mathfrak{g}_1^0) &= \mathfrak{g}_2^0 & \text{and} & & f(\mathfrak{g}_1^-) &= \mathfrak{g}_2^-. \end{aligned}$$

Proof. Let α be an eigenvalue of ϕ_1 and $X \in \mathfrak{g}_\alpha$. There exists $n \geq 1$ such that $(\phi_1 - \alpha)^n X = 0$. By the commutating property, we get

$$(\phi_2 - \alpha)^n f(X) = f((\phi_1 - \alpha)^n X) = f(0) = 0.$$

Consequently, $f((\mathfrak{g}_1)_\alpha) \subset (\mathfrak{g}_2)_\alpha$, where $(\mathfrak{g}_2)_\alpha = \{0\}$ if α is not an eigenvalue of ϕ_2 . In particular we obtain

$$f(\mathfrak{g}_{\phi_1}) \subset \mathfrak{g}_{\phi_2}, \quad f(\mathfrak{k}_{\phi_1}) \subset \mathfrak{k}_{\phi_2}, \quad f(\mathfrak{g}_1^+) \subset \mathfrak{g}_2^+, \quad f(\mathfrak{g}_1^0) \subset \mathfrak{g}_2^0 \quad \text{and} \quad f(\mathfrak{g}_1^-) \subset \mathfrak{g}_2^-.$$

Since for $i = 1, 2$, $\mathfrak{g}_i = \mathfrak{g}_{\phi_i} \oplus \mathfrak{k}_{\phi_i}$ and f is a surjective linear map, we must have

$$f(\mathfrak{g}_{\phi_1}) = \mathfrak{g}_{\phi_2} \quad \text{and} \quad f(\mathfrak{k}_{\phi_1}) = \mathfrak{k}_{\phi_2}.$$

By the restriction of f to \mathfrak{g}_{ϕ_1} we recover all the equalities ending the proof. □

Proposition 2.4 is still true for the real case.

Corollary 2.5. *For $i = 1, 2$, let \mathfrak{g}_i be real algebras and $\phi_i : \mathfrak{g}_i \rightarrow \mathfrak{g}_i$ an endomorphism. If $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a surjective linear map such that $f \circ \phi_1 = \phi_2 \circ f$, the same equalities as in Proposition 2.4 hold.*

Proof. The proof follows by considering the complexification of \mathfrak{g}_i , $i = 1, 2$ and the complex extensions of ϕ_1, ϕ_2 and f . Then, we apply Proposition 2.4. □

3 Lie group endomorphisms

In the sequel all the Lie groups considered are real. For given Lie groups G, H a continuous map $\varphi : G \rightarrow H$ is said to be a **homomorphism** if it preserves the group structure. That is,

$$\varphi(gh) = \varphi(g)\varphi(h) \quad \text{for any } g, h \in G.$$

If $G = H$ such map is said to be an **endomorphism** of G .

Our aim here is to show that associated with any endomorphism of a connected Lie group G there are connected Lie subgroups which contain most of the dynamic information of the endomorphism. Throughout the paper we always assume the Lie groups and their subgroups are connected.

Definition 3.1. *Let G, H be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$, respectively, and $\varphi : G \rightarrow H$ an homomorphism. If there are constants $c \geq 1$ and $\mu \in (0, 1)$ such that*

$$|(d\varphi)_e^m X| \leq c^{-1} \mu^m |X|, \quad \text{for any } m \in \mathbb{Z}^+, X \in \mathfrak{g}$$

*we say that φ is **contracting**. On the other hand, if*

$$|(d\varphi)_e^m X| \geq c \mu^{-m} |X|, \quad \text{for any } m \in \mathbb{Z}^+, X \in \mathfrak{g}$$

*the homomorphism φ is said to be **expanding**.*

Next, we characterize some general topological property of Lie subgroups that will be needed in the next sections.

Lemma 3.2. *Let G be a Lie group with Lie algebra \mathfrak{g} and, H and K Lie subgroups of G with Lie algebras \mathfrak{h} and \mathfrak{k} , respectively such that $\mathfrak{h} \oplus \mathfrak{k} = \mathfrak{g}$. Then,*

$$H \text{ and } K \text{ are closed} \Leftrightarrow H \cap K \text{ is a discrete subgroup.}$$

Proof. If H and K are closed subgroups then $H \cap K$ is also a closed Lie subgroup. As \mathfrak{g} decomposes into a direct sum of the corresponding Lie subalgebras, it follows that $\dim(H \cap K) = 0$. Hence, the result follows.

Reciprocally, assume that $H \cap K$ is a discrete subgroup of G . By Proposition 6.7 of [3] and also by the hypothesis on \mathfrak{h} and \mathfrak{k} , there exist open neighborhoods U, V and W with

$$0 \in U \subset \mathfrak{h}, 0 \in V \subset \mathfrak{k} \text{ and } e \in W \subset G$$

such that the map

$$f : U \times V \rightarrow W \text{ defined by } f(X, Y) = e^X e^Y$$

is a diffeomorphism.

Without loss of generality, we can assume that W is small enough in order to obtain

$$W \cap (H \cap K) = \{e\}.$$

In particular, if $g = xy$ where $x \in e^U \subset H, y \in e^V \subset K$ and $g \in W \cap H$, we get

$$K \ni y = x^{-1}g \in H \Rightarrow y \in W \cap (H \cap K) = \{e\}.$$

Thus, $H \cap W = e^U = f(U \times \{0\})$. Therefore, $H \cap W$ is closed in W since $U \times \{0\}$ is closed in $U \times V$. As a consequence

$$H \cap W = \text{cl}(H) \cap W.$$

Hence, H has nonempty interior in $\text{cl}(H)$ which only happens if $H = \text{cl}(H)$, showing that H is in fact a closed subgroup of G . Analogously, it is possible to prove that K is a closed subgroup of G as stated. \square

Definition 3.3. Let φ be an endomorphism of a Lie group G . A Lie subgroup $H \subset G$ is said to be φ -invariant if $\varphi(H) \subset H$.

If $H \subset G$ is a φ -invariant Lie subgroup, the restriction $\varphi|_H$ is an endomorphism of H in the induced topology. Let us consider a Lie group G and $\varphi : G \rightarrow G$ a continuous endomorphism. In order to avoid cumbersome notations, from here we write $\phi = (d\varphi)_e$. The **dynamical subgroups** of G induced by φ are the Lie subgroups, $G_\varphi, K_\varphi, G^+, G^0, G^-, G^{+,0}$ and $G^{-,0}$ associated with the Lie subalgebras $\mathfrak{g}_\varphi, \mathfrak{k}_\varphi, \mathfrak{g}^+, \mathfrak{g}^0, \mathfrak{g}^-, \mathfrak{g}^{+,0}$ and $\mathfrak{g}^{-,0}$, respectively.

The subgroups G^+, G^0 and G^- are called the **unstable, central** and **stable** subgroups of φ in G , respectively. The following result sets the main properties of these subgroups.

Proposition 3.4. It holds:

1. All the dynamical subgroups are φ -invariant
2. There exists a natural number d such that the subgroup $K_\varphi = \ker(\varphi^d)_0$ is normal. Moreover,

$$G = G_\varphi K_\varphi \text{ and } G_\varphi = \text{Im}(\varphi^d)$$

3. The restriction of φ is expanding on G^+ and contracting on G^-
4. If G_φ is a solvable Lie group it holds that

$$G_\varphi = G^{+,0} G^- = G^{-,0} G^+ = G^+ G^0 G^- \tag{3}$$

5. If G_φ is semisimple and G^0 is compact, then $G_\varphi = G^0$. Therefore, if G is any connected Lie group such that G^0 is compact, then G_φ has also the decomposition (3).

Proof. 1. It is well known that the following diagram is commutative,

$$\begin{array}{ccccc}
 & & \phi = (d\varphi)_e & & \\
 & & \rightarrow & & \\
 \mathfrak{g} & & & & \mathfrak{h} \\
 \exp_{\mathfrak{g}} \downarrow & \leftrightarrow & & & \downarrow \exp_{\mathfrak{h}} \\
 G & & \varphi & & H
 \end{array}$$

Since the Lie subgroups are connected, their φ -invariance follows directly from the ϕ -invariance of their own Lie algebras.

2. K_φ and $\ker(\varphi^d)_0$ are connected Lie subgroups with the same Lie algebra

$$\mathfrak{k}_\varphi = \ker(\phi^d).$$

So, the desired equality follows.

Moreover, since $\ker(\varphi^d)$ is a normal subgroup of G , its connected component of the identity K_φ is also normal. In particular, the product $G_\varphi K_\varphi$ is a connected subgroup of G with Lie algebra $\mathfrak{g}_\varphi \oplus \mathfrak{k}_\varphi = \mathfrak{g}$. Therefore, by uniqueness we get $G = G_\varphi K_\varphi$. From this G -decomposition and the φ -invariance of G_φ we obtain

$$\text{Im}(\varphi^d) = \varphi^d(G) = \varphi^d(G_\varphi)\varphi^d(K_\varphi) \subset G_\varphi.$$

On the other hand, since ϕ restricted to \mathfrak{g}_φ is an automorphism, it turns out that

$$e^X = e^{\phi^d(\phi|_{\mathfrak{g}_\varphi}^{-d}(X))} = \varphi^d\left(e^{\phi|_{\mathfrak{g}_\varphi}^{-d}(X)}\right) \in \text{Im}(\varphi^d), \quad \text{for all } X \in \mathfrak{g}_\varphi.$$

Consequently, $G_\varphi \subset \text{Im}(\varphi^d)$ which concludes the proof

3. Follows directly by the definition of G^+ and G^- and by Remark 2.3.

4. For the decomposition $G = G_\varphi K_\varphi$ one can easily show that

$$G^{+,0} = G^+ G^0 = G^0 G^+.$$

Thus, $G^{-,0} = G^- G^0 = G^0 G^-$. Hence, in order to prove the result it is enough to show that

$$G_\varphi = G^{+,0} G^-.$$

We prove it by induction on the dimension of G_φ .

i) If $\dim(G_\varphi) = 1$ the group G_φ is Abelian and the result is certainly true

ii) Let us assume that the result holds for any endomorphism φ such that G_φ is solvable with $\dim(G_\varphi) < n$.

iii) Consider a φ -endomorphism of G with G_φ solvable and $\dim(G_\varphi) = n$.

The assumption of G_φ solvable implies that there exists a nontrivial closed normal Lie subgroup B_φ of G_φ which is Abelian and φ -invariant, (see for instance the proof in Proposition 2.9 of [1]). By considering the homogeneous space $H_\varphi = G_\varphi/B_\varphi$ we obtain a connected solvable Lie group H_φ such that

$$\dim(H_\varphi) = \dim(G_\varphi) - \dim(B_\varphi) < n.$$

Moreover, the canonical projection $\pi : G_\varphi \rightarrow H_\varphi$ induces on H_φ a well-defined surjective endomorphism $\tilde{\varphi}$ given by $\tilde{\varphi}(\pi(g)) = \pi(\varphi(g))$.

By the induction hypothesis we obtain $H_\varphi = H^{+,0} H^-$. By taking derivative

$$\tilde{\phi} \circ (d\pi)_e = (d\pi)_e \circ \phi.$$

Therefore, Proposition 2.4 and the fact that all the subgroups are connected give us

$$\pi(G^{+,0}) = H^{+,0} \text{ and } \pi(G^-) = H^-.$$

Consequently,

$$H_\varphi = \pi(G^{+,0} G^-) \text{ and so } G_\varphi = G^{+,0} G^- B_\varphi.$$

The Lie subgroup B_φ is Abelian, hence $B_\varphi = B^{+,0} B^-$ with $B^{+,0} \subset G^{+,0}$ and $B^- \subset G^-$. But, B is also normal, so

$$G = G^{+,0} G^- B_\varphi = G^{+,0} B_\varphi G^- = G^{+,0} B^{+,0} B^- G^- = G^{+,0} G^-$$

which ends the proof of item 4.

5. Let us start by proving that the second assertion is implied by the first one. We know that R_φ is φ -invariant. As before, we obtain an induced surjective endomorphism $\tilde{\varphi}$ on G_φ/R_φ such that

$$(G_\varphi/R_\varphi)^0 = \pi(G^0), \text{ where } \pi : G_\varphi \rightarrow G_\varphi/R_\varphi$$

is the canonical projection.

But, G_φ/R_φ is semisimple and $\pi(G^0)$ is compact, therefore

$$\pi(G^0) = (G_\varphi/R_\varphi)^0 = G_\varphi/R_\varphi \text{ hence } G_\varphi = G^0 R_\varphi.$$

Moreover, R_φ is a solvable Lie subgroup which by item 4. decomposes as $R_\varphi = R^{+,0}R^-$. Finally,

$$G_\varphi = G^0 R_\varphi = G^0 R^{+,0} R^- \subset G^{+,0} G^- \subset G_\varphi$$

as stated.

Now, assume that G_φ is semisimple and G^0 is a compact subgroup. Since $\phi|_{\mathfrak{g}_\varphi}$ is an automorphism, Theorem 5.4 of [4] implies that there exists $k \in \mathbb{N}$ such that $\phi|_{\mathfrak{g}_\varphi}^k = \text{Ad}(g)$ for some $g \in G_\varphi$. It follows that

$$\mathfrak{g}_{\text{Ad}(g)}^+ = \mathfrak{g}^+, \quad \mathfrak{g}_{\text{Ad}(g)}^0 = \mathfrak{g}^0 \quad \text{and} \quad \mathfrak{g}_{\text{Ad}(g)}^- = \mathfrak{g}^-.$$

Now, because G_φ is semisimple, there exists an Iwasawa decomposition $G_\varphi = KAN$ and elements $a \in A$, $u \in K$ and $n \in N$ such that

$$\text{Ad}(g) = \text{Ad}(u)\text{Ad}(a)\text{Ad}(n)$$

with $\text{Ad}(a)$ hyperbolic, $\text{Ad}(n)$ unipotent and $\text{Ad}(u)$ elliptic commuting matrices (see Chapter IX, Lemma 7.1 of [4]). Therefore,

- i) $\mathfrak{g}^+ = \mathfrak{g}_{\text{Ad}(g)}^+$ is the sum of eigenspaces with positive eigenvalues of $\text{Ad}(a)$
- ii) $\mathfrak{g}^- = \mathfrak{g}_{\text{Ad}(g)}^-$ is the sum of eigenspaces with negative eigenvalues of $\text{Ad}(a)$, and
- iii) $\mathfrak{g}^0 = \mathfrak{g}_{\text{Ad}(g)}^0 = \ker(\text{Ad}(a))$.

Furthermore, the subgroup A is a simply connected Abelian Lie group and $A \subset G^0$. By the compactness hypothesis of G^0 we must have $a = e$. So, $\mathfrak{g}^+ = \mathfrak{g}^- = \{0\}$ implying that $G^0 = G$ as stated. □

Definition 3.5. Let φ be an endomorphism of the Lie group G . We say that φ **decomposes** G if G_φ satisfy (3), i.e.,

$$G_\varphi = G^{+,0}G^- = G^{-,0}G^+ = G^+G^0G^-.$$

Let us assume that φ restricted to G_φ is in fact an automorphism. From 2.3 we get that for any right (left) invariant Riemannian metric ϱ

$$\varrho(\varphi^n(x), e) \leq c^{-1}\mu^n \varrho(x, e), \quad \text{for any } x \in G^-, n \in \mathbb{N}, \text{ and} \tag{4}$$

$$\varrho(\varphi^n(y), e) \geq c\mu^{-n} \varrho(y, e), \quad \text{for any } y \in G^+, n \in \mathbb{N}. \tag{5}$$

Moreover, for any $a > 0$,

$$\varrho(\varphi^n(z), e) \mu^{a|n|} \rightarrow 0, \quad n \rightarrow \pm\infty \quad \text{for any } z \in G^0. \tag{6}$$

These facts bring topological consequences on the induced subgroups.

Proposition 3.6. Suppose that φ restricted to G_φ is an automorphism in the induced topology of G . Then,

1. $G^{+,0} \cap G^- = G^+ \cap G^- = G^0 \cap G^- = G^{-,0} \cap G^+ = G^+ \cap G^0 = \{e\}$
2. The dynamical subgroups induced by φ are closed in G
3. For $n \geq d$, $\ker(\varphi^n) = K_\varphi$. In particular, $\ker(\varphi^n)$ is connected.

Proof. Since other cases are analogous, we just show $G^{-,0} \cap G^+ = \{e\}$

Let $y \in G^{-,0} \cap G^+$, $x \in G^-$ and $z \in G^0$ such that $y = xz$. The right invariance of the metric gives

$$\varrho(\varphi^n(y), e) = \varrho(\varphi^n(x)\varphi^n(z), e) \leq \varrho(\varphi^n(x), e) + \varrho(\varphi^n(z), e).$$

Since $y \in G^+$ and $x \in G^-$, from (5) and (4), it follows that

$$c\mu^{-n}\varrho(y, e) \leq \varrho(\varphi^n(z), e) + c^{-1}\mu^n\varrho(x, e).$$

Hence,

$$\varrho(y, e) \leq c^{-1}\varrho(\varphi^n(z), e)\mu^n + c^{-2}\mu^{2n}\varrho(x, e).$$

Because $z \in G^0$, equation (6) implies that in the last inequality, each term on the right hand goes to zero as $n \rightarrow +\infty$. Therefore,

$$\varrho(y, e) = 0 \Rightarrow G^{-,0} \cap G^+ = \{e\}$$

as desired.

2. For $n \in \mathbb{N}$ we know that

$$G_\varphi \cap \ker(\varphi^n) = \ker(\varphi|_{G_\varphi}^n). \tag{7}$$

By the assumption, $\varphi|_{G_\varphi}$ is an automorphism. From that we get $G_\varphi \cap K_\varphi = \{e\}$. Then, Proposition 3.2 implies that G_φ is closed in G . Using again Proposition 3.2 and item 1., we also obtain that G^+ , G^0 , G^- , $G^{+,0}$ and $G^{-,0}$ are closed subgroups of G_φ . As a consequence, they are also closed subgroups of G .

3. Let $n \geq d$, $x \in \ker(\varphi^n)$ and consider the decomposition $x = gk$ with $g \in G_\varphi$ and $k \in K_\varphi$ given by item 2. of Proposition 3.4. Hence,

$$G_\varphi \ni g = xk^{-1} \in \ker(\varphi^n)K_\varphi \subset \ker(\varphi^n).$$

Therefore, (7) implies $x = k \in K_\varphi$, concluding the proof. □

In the sequel we prove that some strong topological property of G are also maintained by φ .

Proposition 3.7. *Let φ be an endomorphism of a simply connected Lie group G . Then, G_φ and K_φ are simply connected. Moreover, the restriction of φ to G_φ is an automorphism.*

Proof. By Proposition III.3.17 of [5] both, the subgroup $\ker(\varphi^d)$ and the quotient $G/\ker(\varphi^d)$ are simply connected, for any $n \geq d$. Since the application

$$G/K_\varphi \rightarrow G/\ker(\varphi^d)$$

is a covering map, Proposition 6.12 of [6] implies that $K_\varphi = \ker(\varphi^d)$.

Moreover, from the decomposition $G = G_\varphi K_\varphi$ we obtain that $\varphi^d : G \rightarrow G_\varphi$ is a surjective continuous homomorphism. Thus, by the canonical isomorphism theorem it follows that G_φ and $G/\ker(\varphi^d)$ are isomorphic, showing in particular that G_φ is simply connected.

Knowing that ϕ restricted to \mathfrak{g}_φ is an automorphism and G_φ is simply connected, we must have that φ restricted to G_φ is an automorphism, ending the proof. □

Corollary 3.8. *Let G be a simply connected Lie group. Then, any subgroup induced by an endomorphism φ of G is closed.*

The next result shows that the unstable/stable subgroup of a compact φ -invariant subgroup of G_φ is contained in its center. This implies the decomposition of the group when G_φ is compact.

Theorem 3.9. *Let G be a Lie group and φ an endomorphism of G . If $H \subset G_\varphi$ is a φ -invariant compact subgroup, then $H^+, H^- \subset Z_H$ the center of H . In particular, if G_φ is compact G is decomposable.*

Proof. Since H is a compact subgroup it is in particular reducible and so $H = Z_H H'$, where H' is the derivated subgroup. Since both, H' and Z_H are φ -invariant subgroups and H' is semisimple, item 5. of Proposition 3.4 implies that $(H')^0$ and by the φ -invariance, H^+ and H^- are subsets of Z_H .

If G_φ is compact, we get

$$G'_\varphi \subset G^0 \text{ and so } G_\varphi = Z_{G_\varphi} G^0.$$

Since Z_{G_φ} is solvable subgroup, item 4. of Proposition 3.4 implies that Z_{G_φ} is contained in $G^{+,0}G^-$ which gives us the desired conclusion. \square

For the special case of solvable Lie groups more is true. In fact,

Theorem 3.10. *Let G be a solvable Lie group and φ an endomorphism of G . If $\varphi|_{G_\varphi}$ is an automorphism, then any fixed point of φ is contained in G^0 .*

Proof. Since $\varphi|_{G_\varphi}$ is an automorphism we know that $G_\varphi \cap K_\varphi = \{e\}$. Therefore, the decomposition of $x \in G$ as $x = gk$ with $g \in G_\varphi$ and $k \in K_\varphi$ is unique. Thus, $x = gk$ is a fixed point of φ if and only if g and k are fixed points of φ . Since $\varphi^d(k) = e$ we must have $k = e$. So, we only have to analyze the case where $g \in G_\varphi$ is a fixed point.

By Proposition 3.4 item 4., we know that

$$g = g_1 g_2 g_3 \text{ with } g_1 \in G^+, g_2 \in G^0 \text{ and } g_3 \in G^-$$

Moreover, by Proposition 3.6 item 1. and the φ -invariance of the subgroups it turns out that g is a fixed point of φ if and only if g_i is a fixed point of φ for $i = 1, 2, 3$. However, since $g_1 \in G^+$, from the equation (5) we obtain

$$\varrho(g_1, e) = \varrho(\varphi^n(g_1), e) \geq c\mu^{-n}\varrho(g_1, e), \text{ for any } n \in \mathbb{N}$$

which happens if and only if $g_1 = e$.

In the same way, by using the fact $g_3 \in G^-$ is a fixed point and the equation (4), we get that $g_2 = e$ showing that $x = g_2 \in G^0$ as we stand. \square

Examples

Example 3.11. *Take $G = \mathbb{R}^d$, $A \in \mathfrak{gl}(d)$ a $d \times d$ matrix and the endomorphism φ_A of G given by $\varphi_A(x) = Ax$. In this case, the subgroups induced by φ_A are given by sums of the eigenspaces of A .*

Example 3.12. *Consider $G = Sl(n, \mathbb{R})$, the group of the invertible matrices with determinant equals to one. If $A = \text{diag}(a_1 > \dots > a_d)$ is a matrix with trace equal to zero we can induce the automorphism $\varphi_A : G \rightarrow G$ defined by*

$$\varphi_A(B) = e^A B e^{-A}$$

where e^A is the exponential of the square matrix A .

An easy calculation shows that in this case

$$G^+ = \{B \in G : B \text{ is upper triangular with } 1\text{'s in the main diagonal}\},$$

$$G^- = \{B \in G : B \text{ is lower triangular with } 1\text{'s in the main diagonal}\}$$

and

$$G^0 = \{B \in G; B \text{ is diagonal}\}.$$

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