



Control sets of linear systems on semi-simple Lie groups

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Abstract

In this paper we study the main properties of control sets with nonempty interior of linear systems on semisimple Lie groups. We show that, unlike the solvable case, linear systems on semisimple Lie groups may have more than one control set with nonempty interior and that they are contained in right translations of the one around the identity.

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1. Introduction

In this paper we study the control sets of the so-called *linear systems* on semisimple Lie groups, which are controlled systems

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$$(\Sigma) \quad \dot{g} = \mathcal{X}(g) + \sum_{j=1}^m u_j Y^j(g),$$

where \mathcal{X} is a linear vector field, that is a vector field whose flow is a one-parameter group of automorphisms, and the Y^j 's are right-invariant.

They appear as natural extensions of linear systems on Euclidean spaces, but they are actually deeply nonlinear in the non-Abelian case. In particular on semi-simple Lie groups their behaviour is closer to the one of invariant systems than to the one of linear systems on Euclidean spaces. Our interest for these systems is threefold. On the one hand they are the more natural systems that respect the Lie structure after the invariant ones. On the other one it has been proven in [18] that all control systems that generate a finite dimensional Lie algebra are diffeomorphic to a linear system on a Lie group or a homogeneous space. In third place they may be used as approximation to certain nonlinear systems. Moreover semi-simple Lie groups, compact or noncompact, are the state spaces of many applications, mechanical systems ([7]) in particular, but also in other fields (see [25] for instance).

Since restricted inputs are the relevant ones in practice the inputs we consider herein are assumed to take their values in a bounded subset of \mathbb{R}^m . However many systems are not controllable for restricted inputs, even if they are for unbounded ones. It is for instance the case of linear systems on Euclidean spaces. In this setting the control sets, that are roughly speaking the maximal regions on which the system is controllable, appear as fundamental objects.

They have attracted some interest (see for instance [6], [9], [10], [16], [22], [23], [28] and the book [11]), but to our knowledge control sets of linear systems on semi-simple Lie groups do not appear in the literature. Controllability of these systems for unbounded inputs has been studied in [19] where it is shown that they are closely related to right-invariant ones, and there is a series of papers about invariant systems on semi-simple Lie groups (see for instance [21], [26]) that culminate at [15] (see also the bibliography of that paper). These papers deal with unrestricted inputs and their proofs make use of extension technics and root systems.

On the other hand several topological properties of control sets were proven in [4] for linear systems on solvable Lie groups. By using the close relationship between the dynamics of the drift and the behaviour of the control system (see [3] and [12]) the authors were able to prove boundedness and uniqueness of the control sets. The picture is very different on semi-simple Lie groups. Indeed more than one control set with nonempty interior may exist and they are contained in right translations of the existent control set around the identity (see Theorems 3.4 and 3.5). Moreover, the existence of an *invariant control set* implies global controllability (Theorem 3.7), showing how semisimplicity strongly influences the behaviour of the control system. It is worth noticing that this last result is a true consequence of the semisimple structure and that an analogous fact has been proven for global controllability. Indeed on a semi-simple Lie group controllability of a linear system from the identity implies controllability (see [19]), but this statement is wrong on nilpotent or solvable groups, counter-example are known (see [13]).

Just like the classical papers about invariant systems we use the Lie machinery to prove our main results, that are the previously quoted Theorems 3.4, 3.5, and 3.7. However our method is different from the one of [15] for instance. Instead of extension technics and root systems we mainly make use of parabolic subgroups and projection on the quotient by the centralizer of the hyperbolic part of the derivation associated to the linear vector field, which allows us to relate the control sets with nonempty interior with the ones of the quotient. We also use some special properties of the subsemigroups of semisimple groups with finite center in the proof of Theorem 3.7.

The paper is structured as follows: Section 2 introduces the main definitions and principal properties concerning control systems, control sets, linear vector fields and linear systems on Lie groups. Since our work is devoted to semisimple Lie groups, we also provide in Section 2.2 a small subsection about semisimple theory in order to make the paper self-contained. Section 3 contains the main results concerning control sets with nonempty interior of a linear system on a connected semisimple Lie group. We show that all the possible control sets with nonempty interior of the system are contained in the right translations of the control set around the identity. The particular case where the drift of the system has trivial nilpotent part, these right translations are precisely the control sets of the system. In this section we also show that for linear systems on semisimple Lie groups, the only possible invariant control set is the whole group. Section 4 is devoted to illustrating the paper with an example in $Sl(2)$.

Notations. Let G be a connected Lie group. We denote by e the identity element of G . For any element $g \in G$, the maps L_g and R_g stand for the left and right translations in G , respectively. By $C_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g$ we denote the conjugation of G and by i we denote the inversion of G , that is, the diffeomorphism defined by $g \in G \mapsto g^{-1} \in G$. By $Aut(G)$ we denote the group of automorphisms of G . If $(\varphi_t)_{t \in \mathbb{R}} \subset Aut(G)$ is a 1-parameter subgroup, its orbit from g is the subset $\mathcal{O}(g, \varphi) = \{\varphi_t(g), t \in \mathbb{R}\}$. We say that a subset $B \subset G$ is φ -invariant if $\varphi_t(B) \subset B$ for any $t \in \mathbb{R}$. If $f : M \rightarrow N$ is a differentiable map between smooth manifolds, we denote by $(df)_x$ the differential of f at the point $x \in M$ and by f_* the differential of f at any given point.

2. Preliminaries

2.1. Control systems

In this section we introduce control systems. For more on the subject the reader may consult the standard references [1], [11] and [20].

Let M be a d -dimensional smooth manifold. A *control system* on M is a family of ordinary differential equations

$$\dot{x}(t) = f(x(t), u(t)), \quad u \in \mathcal{U}, \tag{2.1}$$

where $f : M \times \mathbb{R}^m \rightarrow TM$ is a smooth map and

$$\mathcal{U} := \{u : \mathbb{R} \rightarrow \mathbb{R}^m; u \text{ is measurable with } u(t) \in \Omega \text{ a.e.}\},$$

is the set of the *admissible control functions*, with Ω a bounded subset of \mathbb{R}^m such that $0 \in \text{int } \Omega$. For any $x \in M$ and $u \in \mathcal{U}$ we denote by $\phi(t, x, u)$ the unique solution of (2.1) with initial value $x = \phi(0, x, u)$. We use $\phi_{t,u}$ to denote the diffeomorphism $x \in M \mapsto \phi(t, x, u) \in M$. Given $u_1, u_2 \in \mathcal{U}$ and $t_1, t_2 > 0$ we have that

$$\phi(t_1, \phi(t_2, x, u_2), u_1) = \phi(t_1 + t_2, x, u)$$

where $u = u_1 * u_2 \in \mathcal{U}$ is the *concatenation* of u_1 and u_2 define by

$$u(t) = \begin{cases} u_1(t), & t \in [0, t_1] \\ u_2(t - t_1), & t \in (t_1, t_1 + t_2] \end{cases}$$

The set of points *reachable from x at time exactly $\tau > 0$* , the set of points *reachable from x up to time $\tau > 0$* and the *reachable set from x* are respectively denoted by

$$\mathcal{A}_\tau(x) := \{\varphi(\tau, x, u), u \in \mathcal{U}\}, \quad \mathcal{A}_{\leq \tau}(x) := \bigcup_{t \in (0, \tau]} \mathcal{A}_t(x) \quad \text{and} \quad \mathcal{A}(x) := \bigcup_{t > 0} \mathcal{A}_t(x).$$

By $\mathcal{A}_\tau^*(x)$, $\mathcal{A}_{\leq \tau}^*(x)$ and $\mathcal{A}^*(x)$ we denote the corresponding sets for the time-reversed system. We say that the system (2.1) is *locally accessible from x* if $\text{int} \mathcal{A}_{\leq \tau}(x)$ and $\text{int} \mathcal{A}_{\leq \tau}^*(x)$ are nonempty for all $\tau > 0$. The system is said to be *locally accessible* if it is locally accessible from any $x \in M$. A sufficient condition for local accessibility is the Lie algebra rank condition (LARC). It is satisfied if the Lie algebra \mathcal{L} generated by the vector fields $x \in M \mapsto f_u(x) := f(x, u)$, for $u \in \Omega$, satisfies $\mathcal{L}(x) = T_x M$ for all $x \in M$ (see for instance [11, Theorem A.4.4]).

Definition 2.1. A subset $D \subset M$ is a control set of (2.1) if it is maximal w.r.t. set inclusion with the following properties:

- (i) For each $x \in D$ there is $u \in \mathcal{U}$ with $\phi(\mathbb{R}_+, x, u) \subset D$.
- (ii) $D \subset \text{cl} \mathcal{A}(x)$ for all $x \in D$.

Following [11, Proposition 3.2.4.], any subset D of M with nonempty interior that is maximal with the property (ii) of Definition 2.1 is a control set.

The next result states the main properties of control sets with nonempty interior. Its proof can be found at [11, Lemma 3.2.13].

Lemma 2.2. *Let D be a control set of (2.1) with nonempty interior. It holds:*

- (i) *If the system is locally accessible from all $x \in \text{cl} D$, then D is connected and $\text{cl} \text{int} D = \text{cl} D$;*
- (ii) *If $y \in \text{int} D$ is locally accessible, then $y \in \mathcal{A}(x)$ for all $x \in D$;*
- (iii) *If the system is locally accessible from all $y \in \text{int} D$, then $\text{int} D \subset \mathcal{A}(x)$ for all $x \in D$ and for every $y \in \text{int} D$ one has*

$$D = \text{cl} \mathcal{A}(y) \cap \mathcal{A}^*(y).$$

In particular, exact controllability holds on $\text{int} D$.

We say that a control set D is *positively-invariant* (resp. *negatively-invariant*) if $\phi_{t,u}(D) \subset D$ for any $u \in \mathcal{U}$ and $t > 0$ (resp. $t < 0$).

2.2. Semisimple theory

Standard references for the theory of semisimple Lie groups are Duistermat-Kolk-Varadarajan [14], Helgason [17], Knapp [24] and Warner [29]. In the sequel, we only provide a brief review of the concepts used in this paper.

Let G be a connected semisimple non-compact Lie group G with finite center and Lie algebra \mathfrak{g} . We choose a Cartan involution $\zeta : \mathfrak{g} \rightarrow \mathfrak{g}$ and denote by $B_\zeta(X, Y) = -C(X, \zeta(Y))$ the associated inner product, where $C(X, Y) = \text{tr}(\text{ad}(X) \text{ad}(Y))$ is the Cartan-Killing form. If \mathfrak{k} and \mathfrak{s} stand,

respectively, for the eigenspaces of ζ associated with 1 and -1 , the *Cartan decompositions* of \mathfrak{g} and G are given, respectively, by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \quad \text{and} \quad G = KS, \quad \text{where} \quad K = \exp \mathfrak{k} \quad \text{and} \quad S = \exp \mathfrak{s}.$$

Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and denote by Π the set of roots for this choice. If $\mathfrak{n} := \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$, where Π^+ is the set of positive roots and

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \text{ad}(H)X = \alpha(H)X, \forall H \in \mathfrak{a}\}$$

is the root space associated with $\alpha \in \Pi$, the *Iwasawa decompositions* of \mathfrak{g} and G are given, respectively, by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \quad \text{and} \quad G = KAN, \quad \text{where} \quad N = \exp \mathfrak{n} \quad \text{and} \quad A = \exp \mathfrak{a}.$$

Let $\mathfrak{a}^+ = \{H \in \mathfrak{a}; \alpha(H) > 0, \alpha \in \Pi^+\}$ be the positive *Weyl chamber* associated with the above choices and consider $H \in \text{cl } \mathfrak{a}^+$. The eigenspaces of $\text{ad}(H)$ in \mathfrak{g} are given by $\mathfrak{g}_\alpha, \alpha \in \Pi$ and $\mathfrak{g}_0 = \ker \text{ad}(H)$. The centralizer of H in \mathfrak{g} is given by

$$\mathfrak{z}_H := \sum_{\alpha \in \Pi \cup \{0\}; \alpha(H)=0} \mathfrak{g}_\alpha$$

and the centralizer in \mathfrak{k} by $\mathfrak{k}_H := \mathfrak{k} \cap \mathfrak{z}_H$. They are, respectively, the Lie algebra of the centralizer of H in G , $Z_H := \{g \in G : \text{Ad}(g)H = H\}$, and in K , $K_H = K \cap Z_H$. Since G has finite center, the *centralizer* M of \mathfrak{a} in K is a compact subgroup of G . This fact, together with the equality $Z_H = M(Z_H)_0$ implies that Z_H has a finite number of connected components. The finite subgroup $\Gamma = Z_H / (Z_H)_0$ parametrizes the connected components of Z_H and hence, $(Z_H)_\gamma$ will stand for the connected component of Z_H related with $\gamma \in \Gamma$.

The negative and positive nilpotent subalgebras of type H are given by

$$\mathfrak{n}_H := \sum_{\alpha \in \Pi; \alpha(H) > 0} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}_H^- := \sum_{\alpha \in \Pi; \alpha(H) < 0} \mathfrak{g}_\alpha.$$

The parabolic subalgebra and the negative parabolic subalgebra of type H are given, respectively, by

$$\mathfrak{p}_H := \sum_{\alpha \in \Pi \cup \{0\}; \alpha(H) \geq 0} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{p}_H^- := \sum_{\alpha \in \Pi \cup \{0\}; \alpha(H) \leq 0} \mathfrak{g}_\alpha.$$

At the group level, $N_H = \exp(\mathfrak{n}_H)$ and $N_H^- = \exp(\mathfrak{n}_H^-)$ stand for the connected nilpotent Lie subgroup associated with \mathfrak{n}_H and \mathfrak{n}_H^- , respectively. The parabolic subgroups P_H and P_H^- are, respectively, the normalizer of \mathfrak{p}_H and \mathfrak{p}_H^- in G . It holds that N_H is a normal subgroup of P_H and the same is true for N_H^- and P_H^- . In particular, it holds that

$$P_H = Z_H N_H = N_H Z_H \quad \text{and} \quad P_H^- = Z_H N_H^- = N_H^- Z_H.$$

Moreover, the set

$$U_H := P_H N_H^- = N_H Z_H N_H^- = N_H P_H^- \tag{2.2}$$

is an open and dense subset of G .

2.3. Linear vector fields on semisimple Lie groups

A vector field \mathcal{X} on a connected Lie group G is said to be *linear* if its flow $(\varphi_t)_{t \in \mathbb{R}}$ is a 1-parameter subgroup of $\text{Aut}(G)$. Associated to any linear vector field \mathcal{X} there is a derivation \mathcal{D} of \mathfrak{g} defined by the formula

$$\mathcal{D}Y = -[\mathcal{X}, Y](e), \text{ for all } Y \in \mathfrak{g}.$$

The relation between φ_t and \mathcal{D} is given by the formula

$$(d\varphi_t)_e = e^{t\mathcal{D}} \text{ for all } t \in \mathbb{R}, \tag{2.3}$$

where here $e^{t\mathcal{D}}$ stands for the matrix exponential of \mathcal{D} . In particular, it holds that

$$\varphi_t(\exp Y) = \exp(e^{t\mathcal{D}}Y), \text{ for all } t \in \mathbb{R}, Y \in \mathfrak{g}.$$

Let G be semisimple and consider \mathcal{X} to be a linear vector field on G . If \mathcal{D} stands for the derivation associated with \mathcal{X} , the fact that \mathfrak{g} is a semisimple Lie algebra implies that \mathcal{D} is inner, that is, there exists $X \in \mathfrak{g}$ such that $\mathcal{D} = -\text{ad}(X)$.

By equation (2.3) we get that

$$\varphi_t(\exp Y) = \exp(e^{t\mathcal{D}}Y) = \exp(e^{-t \text{ad}(X)}Y) = C_{e^{tX}}(\exp Y),$$

where the minus sign on the above formula is related with the choice of right-invariant vector fields. Since G is connected, we conclude that $\varphi_t = C_{e^{tX}}$ and consequently that

$$\mathcal{X} = X + i_*X. \tag{2.4}$$

Following [17, Chapter 9, Lemma 3.1] the *Jordan* decomposition of an element $X \in \mathfrak{g}$ is the commuting decomposition $X = E + H + N$ where $H \in \text{cl}(\mathfrak{a}^+)$, $E \in \mathfrak{k}_H$ and $\text{ad}(N)$ is nilpotent. In particular, the Lie subalgebras \mathfrak{n}_H , \mathfrak{z}_H and \mathfrak{n}_H^- coincide, respectively, with the sum of the real generalized eigenspaces of \mathcal{D} associated with the eigenvalues with positive, zero and negative real parts.

We call the elements E , H and N obtained from the Jordan decomposition of X the *elliptic*, *hyperbolic* and *nilpotent* parts of \mathcal{X} , respectively. Moreover, the Jordan decomposition of X implies that the flow of \mathcal{X} is given by the commutative product

$$\varphi_t = C_{e^{tX}} = C_{e^{tE}} \circ C_{e^{tH}} \circ C_{e^{tN}}.$$

A simple calculation shows that N_H , N_H^- and Z_H are φ -invariant.

Remark 2.3. Expression (2.4) gives us an explicit relation between linear and invariant vector fields that are independent of any identification of \mathfrak{g} with the Lie algebra of right or left invariant vector fields. Moreover, it shows that a linear vector field cannot be invariant.

2.4. Linear systems on semisimple Lie groups

A linear system on a connected Lie group G is a family of ordinary differential equations of the form

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^m u_j(t) Y^j(g(t)), \quad u = (u_1, \dots, u_m) \in \mathcal{U} \tag{\Sigma_G}$$

where \mathcal{X} is a linear vector field, $Y^j, j = 1, \dots, m$ are right-invariant vector fields and $u(t) \in \Omega$. The solutions of linear systems are related to the flow of \mathcal{X} by the formula

$$\phi(t, g, u) = \phi(t, e, u)\varphi_t(g) = L_{\phi(t,e,u)}(\varphi_t(g)).$$

Let us denote by $\mathcal{A}_\tau, \mathcal{A}_\tau^*, \mathcal{A}_{\leq\tau}, \mathcal{A}_{\leq\tau}^*, \mathcal{A}$ and \mathcal{A}^* the sets $\mathcal{A}_\tau(e), \mathcal{A}_\tau^*(e), \mathcal{A}_{\leq\tau}(e), \mathcal{A}^*(e)_{\leq\tau}, \mathcal{A}(e)$ and $\mathcal{A}^*(e)$, respectively.

The next proposition states the main properties of the reachable sets of linear systems (see for instance [19, Proposition 2]).

Proposition 2.4. *With the previous notations it holds:*

1. $\mathcal{A}_{\leq\tau} = \mathcal{A}_\tau$;
2. $\mathcal{A}_\tau(g) = \mathcal{A}_\tau\varphi_\tau(g)$;
3. $\mathcal{A}_{\tau_1+\tau_2} = \mathcal{A}_{\tau_1}\varphi_{\tau_1}(\mathcal{A}_{\tau_2}) = \mathcal{A}_{\tau_2}\varphi_{\tau_2}(\mathcal{A}_{\tau_1})$.

The next result shows that the set \mathcal{A} is invariant by right translations of elements whose φ -orbits are contained in \mathcal{A} ([12, Lemma 3.1]).

Lemma 2.5. *Let $g \in \mathcal{A}$ and assume that $\mathcal{O}(g, \varphi) \subset \mathcal{A}$. Then $\mathcal{A} \cdot g \subset \mathcal{A}$.*

Let G to be a connected semisimple Lie group with finite center. By using the notations introduced in Section 2.2 for the semisimple case, Theorem 3.9 of [3] gives us a strict relation between the subgroups P_H and P_H^- and the linear system Σ_G as follows.

Theorem 2.6. *Let G be a connected semisimple Lie group with finite center. If \mathcal{A} is open, then $(P_H)_0 \subset \mathcal{A}$ and $(P_H^-)_0 \subset \mathcal{A}^*$.*

Next we extend the above results relating \mathcal{A} with the other connected components of P_H .

Proposition 2.7. *Let G be a connected semisimple Lie group with finite center and assume that \mathcal{A} is open. For any $\gamma \in \Gamma$ we have that*

$$(Z_H)_\gamma \cap \mathcal{A} \neq \emptyset \implies (P_H)_{\gamma^n} \subset \mathcal{A}, \text{ for any } n \in \mathbb{Z}.$$

Proof. Let $x \in (Z_H)_\gamma \cap \mathcal{A}$. Since $(Z_H)_\gamma = x(Z_H)_0$ we get that any $z \in (P_H)_\gamma$ can be written as $z = xg$ with $g \in (P_H)_0$ and hence

$$\mathcal{O}(g, \varphi) \subset (P_H)_0 \subset \mathcal{A} \implies z = xg \in \mathcal{A} \cdot g \subset \mathcal{A}.$$

Let us notice that $(P_H)_{\gamma^n} = ((P_H)_\gamma)^n$ and then, if $(P_H)_{\gamma^n} \subset \mathcal{A}$ we have by the φ -invariance of $(P_H)_{\gamma^n}$ and Lemma 2.5 that $(P_H)_{\gamma^{n+1}} \subset \mathcal{A}$. Since Γ is finite, the above is true for any $n \in \mathbb{Z}$ showing the result. \square

The next technical lemma will be useful ahead.

Lemma 2.8. *Let $\gamma \in \Gamma$, $x \in (Z_H)_\gamma$ and assume that \mathcal{A} is open.*

- (i) *It holds that $\mathcal{A}(x) \subset \mathcal{A} \cdot x$;*
- (ii) *If $\mathcal{O}(x, \varphi) \subset \mathcal{A}(x)$ then $\mathcal{A} \cdot x \subset \mathcal{A}(x)$.*

Proof. (i) For any $z \in \mathcal{A}(x)$ consider $\tau > 0$ and $u \in \mathcal{U}$ with $z = \phi(\tau, x, u)$. By the φ -invariance of $(Z_H)_\gamma$, there exists $l \in (Z_H)_0$ such that $\varphi_\tau(x) = lx$ and hence

$$z = \phi(\tau, x, u) = \phi(\tau, e, u)\varphi_\tau(x) = \phi(\tau, e, u)lx \implies z \in \mathcal{A} \cdot lx \subset \mathcal{A} \cdot x,$$

where for the inclusion we used that $\mathcal{O}(l, \varphi) \subset (Z_H)_0 \subset \mathcal{A}$.

(ii) Let $z \in \mathcal{A}$ and write it as $z = \phi(\tau, e, u)$ for some $\tau > 0$ and $u \in \mathcal{U}$. Then

$$zx = \phi(\tau, e, u)x = \phi(\tau, e, u)\varphi_\tau(\varphi_{-\tau}(x)) \in \phi_{\tau,u}(\mathcal{O}(x, \varphi)) \subset \phi_{\tau,u}(\mathcal{A}(x)) \subset \mathcal{A}(x)$$

and by the arbitrariness of $z \in \mathcal{A}$ we get $\mathcal{A} \cdot x \subset \mathcal{A}(x)$ concluding the proof. \square

3. Control sets of linear systems on semisimple Lie groups

According to the Levi Theorem (see [24, Theorem B.2]), any finite-dimensional Lie algebra is the semidirect product of a solvable ideal and a semisimple subalgebra. On the other hand, in [4] the authors obtain several results of control sets for solvable Lie algebras. Furthermore, in [2] there exists a characterization of the controllability property for linear control systems on nonnilpotent, solvable three-dimensional Lie groups. In order to have a global picture, in this section we study the main properties of control sets on semisimple Lie groups.

Let G be a connected noncompact semisimple Lie group with finite center and Σ_G a linear system on G . In the whole section we will assume that the reachable set \mathcal{A} of Σ_G is open. The openness of \mathcal{A} is equivalent to the existence of some $\tau > 0$ such that $e \in \text{int } \mathcal{A}_\tau$ (see [11, Lemma 4.5.2]). In particular, if \mathcal{A} is open the system is locally accessible (see Theorem 3.3 of [8]) and \mathcal{A}^* is also open. There is an easily checkable algebraic condition that ensures, in particular, the openness of \mathcal{A} . Precisely, we say that the linear system Σ_G satisfies the *ad-rank* condition if

$$\{\mathcal{D}^j Y^i, i = 1, \dots, m, j \in \mathbb{N}\} = \mathfrak{g}.$$

In other words, the system satisfies the *ad-rank* condition if \mathfrak{g} is the smallest \mathcal{D} -invariant subspace containing Y^1, \dots, Y^m . Under the *ad-rank* condition, Theorem 3.5 of [8] assures that $e \in \text{int } \mathcal{A}_\tau$ for any $\tau > 0$, which certainly implies that \mathcal{A} is open. These conditions are however not equivalent as showed by the Example 5 in Section 7.3 of [19].

Let

$$\mathcal{C}_1 := \text{cl}(\mathcal{A}) \cap \mathcal{A}^*.$$

Since, by Corollary 4.5.11 of [11], there exists of a control set containing the identity in its interior, we have by Lemma 2.2 that \mathcal{C}_1 is a control set Σ_G . Our aim in this section is to show that any other control set of Σ_G with nonempty interior is related with \mathcal{C}_1 by right translations.

Let us denote by H the hyperbolic part of the linear vector field \mathcal{X} , drift of the system Σ_G . By considering the homogeneous space $G/(Z_H)_0$ we have, by the φ -invariance of $(Z_H)_0$, a well defined system $\Sigma_{G/(Z_H)_0}$ on $G/(Z_H)_0$ induced by the linear system Σ_G (see [18, Proposition 4]). We aim to show that there is a strict relation between the control sets of Σ_G and the ones of $\Sigma_{G/(Z_H)_0}$.

If $\pi : G \rightarrow G/(Z_H)_0$ is the canonical projection and ψ_t the flow induced by \mathcal{X} on $G/(Z_H)_0$, we have that

$$\psi_t \circ \pi = \pi \circ \varphi_t.$$

Moreover, if \mathcal{L}_g stands for the left translation in $G/(Z_H)_0$ given by $x \in G/(Z_H)_0 \mapsto gx \in G/(Z_H)_0$ we have that the solutions of $\Sigma_{G/(Z_H)_0}$ satisfy

$$\Phi(t, \pi(g), u) = \mathcal{L}_{\phi(t,e,u)}(\psi_t(\pi(g))) = \pi(\phi(t, g, u)). \tag{3.5}$$

In particular, for any $g \in G$

$$\pi(\mathcal{A}(g)) = \mathcal{A}_H(\pi(g)) \quad \text{and} \quad \pi(\mathcal{A}^*(g)) = \mathcal{A}_H^*(\pi(g)),$$

are the reachable sets from $\pi(g)$ for the induced system.

The next result shows that \mathcal{C}_1 behaves well when projected to $G/(Z_H)_0$.

Theorem 3.1. *The projection of the control set \mathcal{C}_1 to $G/(Z_H)_0$ is a control set of the system $\Sigma_{G/(Z_H)_0}$. Moreover, it satisfies $\pi^{-1}(\pi(\mathcal{C}_1)) = \mathcal{C}_1$.*

Proof. Since π is an open map, it holds that $\pi(\mathcal{C}_1)$ has nonempty interior. Moreover, by (3.5) we have that

$$\text{for all } g \in \mathcal{C}_1, \quad \pi(\mathcal{C}_1) \subset \pi(\text{cl}(\mathcal{A}(g))) \subset \text{cl}(\pi(\mathcal{A}(g))) = \text{cl}(\mathcal{A}_H(\pi(g))),$$

and therefore, $\pi(\mathcal{C}_1)$ is contained in a control set D_1 for the control system $\Sigma_{G/(Z_H)_0}$.

The result is proved if we show that $\pi^{-1}(D_1) \subset \mathcal{C}_1$. However, since $\mathcal{C}_1 \subset \pi^{-1}(D_1)$ it is enough to show that

$$\pi^{-1}(D_1) \subset \text{cl}(\mathcal{A}(g)) \quad \text{for any } g \in \pi^{-1}(D_1).$$

Let then $g_1, g_2 \in \pi^{-1}(\text{int } D_1)$. Denote by $o = e \cdot (Z_H)_0$. Since exact controllability holds on $\text{int } D_1$ and $o \in \text{int } D_1$, there exist $t_1, t_2 > 0$ and $u_1, u_2 \in \mathcal{U}$ such that

$$\Phi(t_1, \pi(g_1), u_1) = o \quad \text{and} \quad \Phi(t_2, o, u_2) = \pi(g_2) \iff \phi(t_1, g_1, u_1) = l_1 \quad \text{and} \quad \phi(t_2, l_2, u_2) = g_2$$

for some $l_1, l_2 \in (Z_H)_0$. On the other hand, the openness of \mathcal{A} implies $(Z_H)_0 \subset \mathcal{A} \cap \mathcal{A}^* \subset \mathcal{C}_1$ and hence there exist $t_3 > 0$ and $u_3 \in \mathcal{U}$ such that

$$\begin{aligned} \phi(t_3, l_1, u_3) = l_2 &\implies g_2 = \phi(t, g_1, u), \\ \text{where } t = t_1 + t_2 + t_3 > 0 &\text{ and } u = (u_1 * u_2) * u_3 \in \mathcal{U}. \end{aligned}$$

Therefore, $\pi^{-1}(\text{int } D_1) \subset \mathcal{A}(g)$ for any $g \in \pi^{-1}(\text{int } D_1)$. Using the fact that $\text{int } D_1$ is dense in D_1 and the maximality of \mathcal{C}_1 we get the desired result. \square

We define now a group of homeomorphisms in $G/(Z_H)_0$. For any $\gamma \in \Gamma$ let us define the map $f_\gamma : G/(Z_H)_0 \rightarrow G/(Z_H)_0$ by $g \cdot o \in G/(Z_H)_0 \mapsto gl \cdot o \in G/(Z_H)_0$, where $l \in Z_H$ satisfies $\gamma = l \cdot o$. Since $(Z_H)_0$ is a normal subgroup of Z_H the map f_γ is a well defined homeomorphism of $G/(Z_H)_0$ satisfying:

- (i) $(f_\gamma)^{-1} = f_{\gamma^{-1}}$ for all $\gamma \in \Gamma$;
- (ii) $f_{\gamma_1\gamma_2} = f_{\gamma_2} \circ f_{\gamma_1}$ for $\gamma_1, \gamma_2 \in \Gamma$;

Lemma 3.2. *For any $\gamma \in \Gamma$ it holds that*

$$\Phi_{t,u} \circ f_\gamma = f_\gamma \circ \Phi_{t,u} \text{ for any } t \in \mathbb{R}, u \in \mathcal{U}, \gamma \in \Gamma.$$

In particular, $\psi_t \circ f_\gamma = f_\gamma \circ \psi_t$ for any $t \in \mathbb{R}$ and $\gamma \in \Gamma$.

Proof. Since $\varphi_t((Z_H)_\gamma) = (Z_H)_\gamma$ for any $\gamma \in \Gamma$ it holds that

$$\begin{aligned} \Phi(t, f_\gamma(g \cdot o), u) &= \phi(t, e, u)\varphi_t(gl) \cdot o = \phi(t, e, u)\varphi_t(g)\varphi_t(l) \cdot o \\ &= \phi(t, e, u)\varphi_t(g)l \cdot o = f_\gamma(\phi(t, e, u)\varphi(g) \cdot o) = f_\gamma(\Phi(t, g \cdot o, u)) \end{aligned}$$

and the result follows. \square

A direct consequence of Lemma 3.2 is that $D_\gamma := f_\gamma(D_1)$ is a control set with nonempty interior of $\Sigma_{G/(Z_H)_0}$, where $D_1 = \pi(\mathcal{C}_1)$ is the projection of the control set of Σ_G containing the identity. The set

$$\Gamma_0 := \{\gamma \in \Gamma; D_\gamma = D_1\}$$

is a subgroup of Γ and the map ξ given by $\Gamma_0\gamma \in \Gamma_0 \setminus \Gamma \mapsto D_\gamma$ is a well-defined injective map, since

$$\Gamma_0\gamma_1 = \Gamma_0\gamma_2 \iff \gamma_2\gamma_1^{-1} \in \Gamma_0 \iff D_1 = D_{\gamma_2\gamma_1^{-1}} \iff D_{\gamma_1} = D_{\gamma_2}.$$

The next result shows that the control sets of a linear system are related with the control sets D_γ for $\gamma \in \Gamma$.

Lemma 3.3 (Fundamental Lemma). *If \mathcal{C} is a control set of Σ_G with nonempty interior then*

$$\mathcal{C} \subset \pi^{-1}(D_\gamma), \text{ for some } \gamma \in \Gamma.$$

Proof. Let $\mathcal{C} \subset G$ be a control set with nonempty interior of Σ_G . By equation (2.2) the set

$$U_H = \bigcup_{\gamma \in \Gamma} (P_H)_\gamma N_H^-$$

is an open and dense subset of G and hence $\text{int}\mathcal{C} \cap (P_H)_{\gamma_1} N_H^- \neq \emptyset$ for some $\gamma_1 \in \Gamma$. There exist $g \in (P_H)_{\gamma_1}, h \in N_H^-$ with $gh \in \text{int}\mathcal{C}$. Since in $\text{int}\mathcal{C}$ we have exact controllability, there exist $\tau > 0$ and $u \in \mathcal{U}$ such that

$$\phi(n\tau, gh, u) = gh, \text{ for each } n > 0.$$

If ϱ stands for a left invariant Riemannian metric on G we get

$$\varrho(gh, \phi(n\tau, g, u)) = \varrho(\phi(n\tau, gh, u), \phi(n\tau, g, u)) = \varrho(\varphi_{n\tau}(h), e).$$

Since $N_H^- = \exp \mathfrak{n}_H^-$ and $\text{ad}(X)|_{\mathfrak{n}_H^-}$ has only eigenvalues with negative real part, we have that $\varphi_{n\tau}(h) \rightarrow e$ as $n \rightarrow +\infty$. On the other hand, the fact that $g \in (P_H)_{\gamma_1}$ implies that $\phi(n\tau, g, u) \in \mathcal{A} \cdot l_1$ for each $n > 0$, where $l_1 \in (Z_H)_{\gamma_1}$. Hence, $gh \in \text{cl}(\mathcal{A} \cdot l_1)$ implying that $\text{int}\mathcal{C} \cap \mathcal{A} \cdot l_1 \neq \emptyset$. Let then $x \in \text{int}\mathcal{C} \cap \mathcal{A} \cdot l_1$ and $y \in \text{int}\mathcal{C}$. By exact controllability there exists $t > 0$ and $u \in \mathcal{U}$ with $y = \phi_{t,u}(x)$. If we write $\varphi_t(l_1) = ml_1$ with $m \in (Z_H)_{\gamma_1}$ we get

$$y = \phi_{t,u}(x) \in \phi_{t,u}(\mathcal{A} \cdot l_1) = \phi_{t,u}(\mathcal{A}) \cdot \varphi_t(l_1) \subset \mathcal{A} \cdot ml_1 \subset \mathcal{A} \cdot l_1$$

where for the last inclusion we used Lemma 2.5. By the arbitrariness of $y \in \text{int}\mathcal{C}$ we conclude that

$$\text{int}\mathcal{C} \subset \mathcal{A} \cdot l_1.$$

By arguing analogously for U_H^{-1} we assure the existence of $\gamma_2 \in \Gamma$ such that $\text{int}\mathcal{C} \subset \mathcal{A}^* \cdot l_2$ where $l_2 \in (Z_H)_{\gamma_2}$. Therefore,

$$\text{int}\mathcal{C} \subset \mathcal{A} \cdot l_1 \cap \mathcal{A}^* \cdot l_2, \text{ where } l_i \in (Z_H)_{\gamma_i}, i = 1, 2. \tag{3.6}$$

In particular, $\mathcal{A} \cdot l_1 \cap \mathcal{A}^* \cdot l_2 \neq \emptyset$. Let then $x \in \mathcal{A} \cdot l_1 \cap \mathcal{A}^* \cdot l_2$ and consider $t_1, t_2 > 0$ and $u_1, u_2 \in \mathcal{U}$ such that

$$x = \phi_{t_1, u_1}(e) \cdot l_1 \text{ and } \phi_{t_2, u_2}(xl_2^{-1}) = e \implies e = \phi_{t_2, u_2}(\phi_{t_1, u_1}(e)l_1l_2^{-1}) = \phi_{t, u}(e)\varphi_{t_2}(l_1l_2^{-1}),$$

where $t = t_1 + t_2$ and $u = u_1 * u_2$. Therefore, $\varphi_{t_2}(l_2l_1^{-1}) = \phi_{t, u}(e) \in \mathcal{A}$ implying that $(Z_H)_{\gamma_2\gamma_1^{-1}} \cap \mathcal{A} \neq \emptyset$. By Proposition 2.7 we get

$$(P_H)_{(\gamma_1\gamma_2^{-1})} = (P_H)_{(\gamma_2\gamma_1^{-1})^{-1}} \subset \mathcal{A} \implies (Z_H)_{\gamma_1} \cap \mathcal{A} \cdot l_2 \neq \emptyset. \tag{3.7}$$

Since, $\pi(\mathcal{A} \cdot l_i) = \mathcal{A}_H(\gamma_i)$ and $\pi(\mathcal{A}^* \cdot l_i) = \mathcal{A}_H^*(\gamma_i)$, equation (3.7) implies

$$\gamma_1 \in \pi(\mathcal{A} \cdot l_2) = \mathcal{A}_H(\gamma_2) \implies \mathcal{A}_H(\gamma_1) \subset \mathcal{A}_H(\gamma_2)$$

and using equation (3.6) we get

$$\pi(\text{int}\mathcal{C}) \subset \pi(\mathcal{A} \cdot l_1) \cap \pi(\mathcal{A}^* \cdot l_2) \subset \mathcal{A}_H(\gamma_1) \cap \mathcal{A}_H^*(\gamma_2) \subset \mathcal{A}_H(\gamma_2) \cap \mathcal{A}_H^*(\gamma_2) \subset D_{\gamma_2}$$

concluding the proof. \square

Now we can prove our main result.

Theorem 3.4. *If \mathcal{C} is a control set with nonempty interior of Σ_G then*

$$\mathcal{C} \subset R_l(\mathcal{C}_1), \text{ for some } l \in Z_H,$$

that is, any control set of Σ_G with nonempty interior is contained in the right translation of the control set containing the identity element.

Proof. In fact, for any $l \in (Z_H)_\gamma$, it holds that $\pi \circ R_l = f_\gamma \circ \pi$ and consequently

$$\pi^{-1}(D_\gamma) = R_l(\mathcal{C}_1).$$

In fact,

$$\begin{aligned} x \in \pi^{-1}(D_\gamma) &\iff \pi(x) \in D_\gamma = f_\gamma(D_1) \iff f_{\gamma^{-1}}(\pi(x)) \in D_1 \\ &\iff \pi(R_{l^{-1}}(x)) \in D_1 = \pi(\mathcal{C}_1) \iff R_{l^{-1}}(x) \in \pi^{-1}(\pi(\mathcal{C}_1)) = \mathcal{C}_1 \iff x \in R_l(\mathcal{C}_1) \end{aligned}$$

and the result follows from Lemma 3.3. \square

The next result shows that for linear system whose drift has trivial nilpotent part the right translations of \mathcal{C}_1 coincides with the control sets of Σ_G . This case is particularly important because generically linear vector fields have trivial nilpotent part since it is true as soon $\alpha(H) \neq 0$ for $\alpha \in \Pi$.

Theorem 3.5. *If \mathcal{X} has trivial nilpotent part, then*

$$R_l(\mathcal{C}_1) = \pi^{-1}(D_\gamma)$$

are all the control sets with nonempty interior of Σ_G . In this case Σ_G admits exactly $|\Gamma|/|\Gamma_0|$ control sets with nonempty interior.

Proof. Since we already have that any control set with nonempty interior is contained in $\pi^{-1}(D_\gamma)$ for some $\gamma \in \Gamma$ it is enough to show that, if the nilpotent part of \mathcal{X} is trivial, we actually have that $\pi^{-1}(D_\gamma)$ is a control set of Σ_G for any $\gamma \in \Gamma$.

By the assumption on the linear vector field we have that the flow of the linear vector field restricted to Z_H is given by $\varphi_t|_{Z_H} = C_{e^t E}$. Since $E \in \mathfrak{k}_H$ and K is compact, we get that $\{e^{tE}, t \in \mathbb{R}\} \subset K$ is bounded and hence, if $x \in Z_H$ we have that $\text{cl}(\mathcal{O}(x, \varphi))$ is a compact subset. Therefore, for any $\gamma \in \Gamma$ and any $x \in (Z_H)_\gamma$ we obtain by Corollary 4.5.11 of [11] that there exists a control set \mathcal{C}_x with nonempty interior such that

$$\text{cl}(\mathcal{O}(x, \varphi)) \subset \text{int} \mathcal{C}_x.$$

In particular $\mathcal{C}_x = \text{cl}(\mathcal{A}(x)) \cap \mathcal{A}^*(x)$ but $\mathcal{O}(x, \varphi) \subset \mathcal{A}(x)$ implies by item (ii) of Lemma 2.8 that $\mathcal{A}(x) = \mathcal{A} \cdot x$ for any $x \in (Z_H)_\gamma$. On the other hand, for any $x, y \in (Z_H)_\gamma$ there is $z \in (Z_H)_0$ with $zx = y$ which gives us

$$\mathcal{A}(x) = \mathcal{A} \cdot x \subset \mathcal{A} \cdot zx = \mathcal{A} \cdot y = \mathcal{A}(y)$$

and hence $\mathcal{A}(x) = \mathcal{A}(y)$. Analogously, for any $x, y \in (Z_H)_\gamma$ we get $\mathcal{A}^*(x) = \mathcal{A}^*(y)$ implying that $\mathcal{C}_x = \mathcal{C}_y$. Moreover,

$$\text{int} D_\gamma = \mathcal{A}_H(\gamma) \cap \mathcal{A}_H^*(\gamma) \implies \pi^{-1}(\text{int} D_\gamma) = \bigcup_{x,y \in (Z_H)_\gamma} \mathcal{A} \cdot x \cap \mathcal{A} \cdot y = \mathcal{A}(x) \cap \mathcal{A}^*(x) \subset \text{int} \mathcal{C}_x.$$

Using that $\pi(\mathcal{C}_x) \subset D_\gamma$ and that $\text{int} D_\gamma$ is dense in D_γ we have $\mathcal{C}_x = \pi^{-1}(D_\gamma)$ as stated. \square

Remark 3.6. By following the idea of the proof of the above theorem, it is not hard to show that $\pi^{-1}(D_\gamma)$ is a control set as soon as $(Z_H)_\gamma$ possesses a fixed or a periodic point for the flow of \mathcal{X} , even when the nilpotent part of \mathcal{X} is not trivial.

We finish this section by showing that the existence of an invariant control set is equivalent to the controllability of Σ_G .

Theorem 3.7. *The only positively-invariant (resp. negatively-invariant) control set of Σ_G with nonempty interior is G .*

Proof. Our proof is divided in two steps:

Step 1. $\mathcal{A} = G$ if and only if $\mathcal{A}^* = G$.

Since both cases are analogous we will only show that $\mathcal{A} = G \implies \mathcal{A}^* = G$. Recall that G being semisimple the derivation $\mathcal{D} = -\text{ad}(\mathcal{X})$ is inner and equal to $-\text{ad}(X)$ for some right-invariant vector field X . We can consequently define the associated invariant system Σ_I by:

$$\dot{g}(t) = X(g(t)) + \sum_{j=1}^m u_j(t) Y^j(g(t)), \quad u = (u_1, \dots, u_m) \in \mathcal{U}. \tag{\Sigma_I}$$

The solutions ϕ^I of Σ_I and ϕ of Σ_G are related by

$$\phi^I(t, g, u) = R_{e^{tX}}(\phi(t, g, u)), \quad \text{for any } t \in \mathbb{R}, g \in G, u \in \mathcal{U}. \tag{3.8}$$

An easy proof of equation (3.8) can be found in [19, Proposition 8]. As consequence the reachable set at time $t \geq 0$ from the identity for Σ_I is $\mathcal{S}_t = \mathcal{A}_t \exp(tX)$.

The reachable set from the identity $\mathcal{S} = \bigcup_{t \geq 0} \mathcal{S}_t$ is a semigroup with nonempty interior. It is said to be left reversible (resp. right reversible) if $\mathcal{S}\mathcal{S}^{-1} = G$ (resp. $\mathcal{S}^{-1}\mathcal{S} = G$). Following Theorem 6.7 of [27], if G is a connected semisimple Lie group with finite center then G itself is the only subsemigroup with nonempty interior which is left or right reversible.

Assume then that $\mathcal{A} = G$. The first thing to show is that $\mathcal{S} = G$. Let $g \in G$. Since $\mathcal{A} = G$ there exists $t \geq 0$ such that $g \in \mathcal{A}_t = \mathcal{S}_t e^{-tX}$. This implies $\mathcal{S} \exp(-\mathbb{R}_+ X) = G$. However, since $\exp(-\mathbb{R}_+ X) \subset \mathcal{S}^{-1}$ we obtain $\mathcal{S}\mathcal{S}^{-1} = G$ and hence $\mathcal{S} = G$.

We can now prove that $\mathcal{A}^* = G$. Let $g \in G$. There exists $t \geq 0$ such that $g^{-1} \in \mathcal{S}_t$ or equivalently $g^{-1} e^{-tX} \in \mathcal{A}_t$. Consequently:

$$e^{-tX} = g^{-1} e^{-tX} e^{tX} g e^{-tX} = g^{-1} e^{-tX} \varphi_t(g) \in \mathcal{A}_t \varphi_t(g) = \mathcal{A}_t(g).$$

But $\mathcal{A} = G$ and there exists $s > 0$ and $u \in \mathcal{U}$ such that $\exp(tX) = \phi_{s,u}(e)$, so that:

$$e = e^{tX} \varphi_s(e^{-tX}) = \phi_{s,u}(e) \varphi_s(e^{-tX}) = \phi_{s,u}(e^{-tX}) \in \phi_{s,u}(\mathcal{A}(g)) \subset \mathcal{A}(g),$$

and hence $g \in \mathcal{A}^*$ concluding the proof.

Step 2. If Σ_G admits a positively-invariant (resp. negatively-invariant) control set \mathcal{C} with nonempty interior then $\mathcal{C} = G$.

In fact, let us assume that \mathcal{C} is a positively-invariant control set of Σ_G with nonempty interior. By Theorem 3.3 we get that $\pi(\mathcal{C}) \subset D_\gamma$ for some $\gamma \in \Gamma$. Since $\pi(\text{int}\mathcal{C}) \subset \text{int}D_\gamma$ and exact controllability holds on $\text{int}D_\gamma$, we can always build an periodic orbit passing for a given point in $\text{int}D_\gamma$ and intersecting $\pi(\text{int}\mathcal{C})$ which by the positively-invariance of \mathcal{C} implies that $\text{int}D_\gamma \subset \pi(\mathcal{C})$ and hence that D_γ is positively-invariant. Since $D_1 = f_{\gamma^{-1}}(D_\gamma)$ we get that D_1 is also positively-invariant and by Theorem 3.1, the same holds for \mathcal{C}_1 . In particular, \mathcal{C}_1 is closed which by Theorem 3.6 of [4] implies that $\mathcal{A}^* = G$ and by the previous step that $\mathcal{A} = G$ which implies the result. \square

Remark 3.8. It is important to remark that Step 1 on the previous proof was first stated in [19] but for unrestricted inputs.

4. Example

Let us consider the three-dimensional semisimple Lie group $\text{Sl}(2)$ of the 2×2 matrices with determinant equal to one. Its Lie algebra is given by $\mathfrak{sl}(2)$, the set of the 2×2 matrices with zero trace.

Denote any element $h \in \text{Sl}(2)$ by $h = (v_1, v_2)$, where $v_1, v_2 \in \mathbb{R}^2$ satisfies $\langle v_1^*, v_2 \rangle = 1$. Here v_1^* is the orthogonal vector of v_1 obtained by a counter-clockwise rotation of $\pi/2$. If $A := \{g \in \text{Sl}(2); g \text{ is diagonal with positive entries}\}$ we have the diffeomorphism

$$\xi : \text{Sl}(2)/A \rightarrow S^1 \times \mathbb{R} \text{ defined by } \xi(v_1, v_2) = \left(\frac{v_1}{|v_1|}, \langle v_1, v_2 \rangle \right).$$

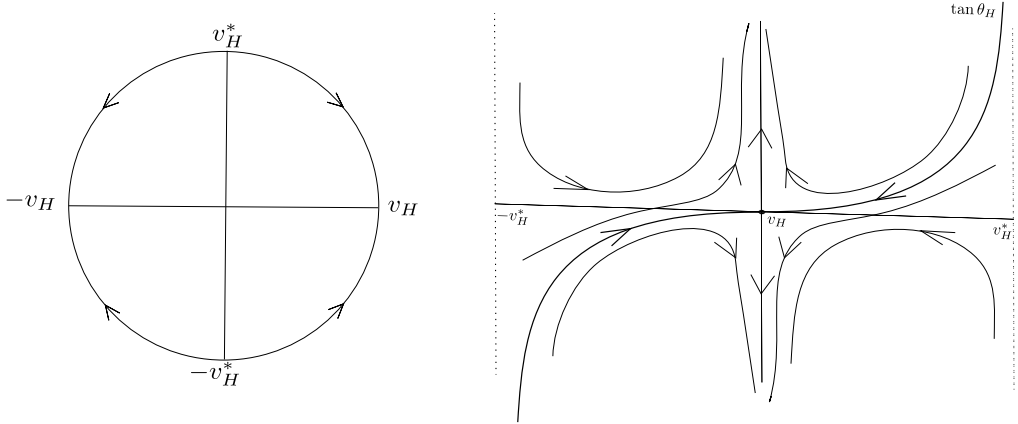


Fig. 1. Dynamical behaviour of ϕ_1 and ϕ_2 .

Note that

$$\xi(g(v_1, v_2)) = \pi(gv_1, gv_2) = \left(\frac{gv_1}{|gv_1|}, \langle gv_1, gv_2 \rangle \right), \text{ for any } g \in \text{Sl}(2)$$

and hence, the flow of any right-invariant vector field e^{tX} , $X \in \mathfrak{sl}(2)$, induces on $S^1 \times \mathbb{R}$ the flow

$$(t, (v, x)) \in \mathbb{R} \times (S^1 \times \mathbb{R}) \mapsto \left(\frac{e^{tX}v}{|e^{tX}v|}, \langle e^{tX}v, e^{tX}(xv + v^*) \rangle \right) \tag{4.9}$$

where we used that $\xi(v, xv + v^*) = (v, x)$.

A simple calculation shows that the flow (4.9) is associated with the vector field

$$f_X(v, x) = \left(Xv - \langle Xv, v \rangle v, \langle Xv, xv + v^* \rangle + \langle v, X(xv + v^*) \rangle \right).$$

We are interested in the dynamical behaviour of the flow of f_H where $H \in \mathfrak{sl}(2)$ has a pair of distinct real eigenvalues. For such case, there is a basis $\{v_H, v_H^*\}$ of eigenvalues of H that we always assume ordered such that v_H is associated with the positive eigenvalue.

For such case, the first component of the solution is given by

$$\phi_1(t, (v, x)) = \frac{e^{tH}v}{|e^{tH}v|}$$

and its dynamical behaviour in the circle is given as in the picture ahead (see Fig. 1 left-hand side).

The second component of the solution of f_H is then

$$\phi_2(t, (v, x)) = \cos^2 \theta_H(v) e^{2t\lambda_H} (x - \tan \theta_H(v)) + \sin^2 \theta_H(v) e^{-2t\lambda_H} (x + \cot \theta_H(v))$$

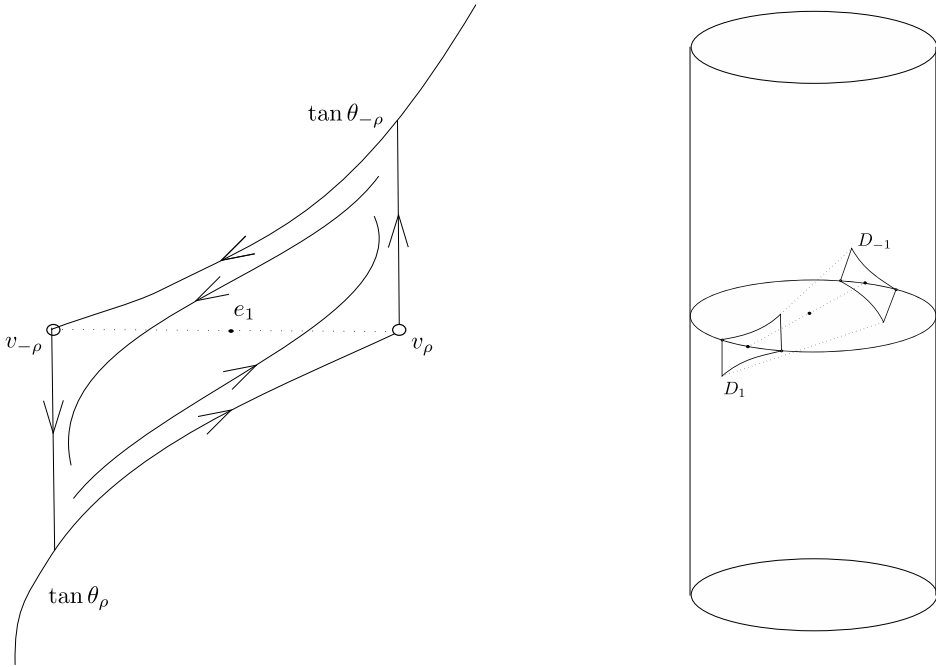


Fig. 2. The control sets of $\Sigma_{\text{Sl}(2)/A}$ associated with $\Gamma = \{\pm 1\}$.

where λ_H is the positive eigenvalue of H and $\theta_H(v)$ is defined by $\cos \theta_H(v) = \langle v_H, v \rangle_H$ where $\langle \cdot, \cdot \rangle_H$ is the inner product that makes $\{v_H, v_H^*\}$ orthogonal. The dynamical behaviour of ϕ_2 is described on the right-hand side of Fig. 1.

Let $H, Z \in \mathfrak{sl}(2)$, $\rho > 0$ and assume that

1. H is a nonzero diagonal matrix;
2. $H_u := H + uZ$ has a pair of distinct real eigenvalues for any $u \in [-\rho, \rho]$;
3. $\{Z, [H, Z], [H, [H, Z]]\}$ is a basis for $\mathfrak{sl}(2)$.⁴

The linear system

$$\dot{g} = \mathcal{X}(g) + uZ, \quad u \in [-\rho, \rho], \tag{\Sigma_{\text{Sl}(2)}}$$

where $\mathcal{X} = H + i_*H$. The previous conditions imply that $\Sigma_{\text{Sl}(2)}$ cannot be controllable (see [5, Theorem 5.2]) and, since the associated derivation is $\mathcal{D} = -\text{ad}(H)$, it satisfies the ad-rank condition and hence \mathcal{A} is open. Moreover, $Z_H = A \cup -A$ and hence we have to analyze the induced system $\Sigma_{\text{Sl}(2)/A}$ on $\text{Sl}(2)/A$.

On the other hand, the fact that e^{tH} belongs to A for all $t \in \mathbb{R}$ implies that $\Sigma_{\text{Sl}(2)/A}$ coincides with the control system on $\text{Sl}(2)/A$ induced by Σ_I , the associated invariant system. Using that the piecewise constant control functions are dense in \mathcal{U} allow us to reduce our work to the analysis

⁴ A pair of matrices satisfying the conditions are, for instance, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 1 \\ 1/2 & -1 \end{pmatrix}$.

of how the concatenations of e^{tH_u} for $u \in [-\rho, \rho]$ acts on $S^1 \times \mathbb{R}$. Following [11], Chapter 6, the system on S^1 given by ϕ_1 has, for small $\rho > 0$, four control sets where the one containing e_1 is the closed interval on S^1 given by $[v_{-\rho}, v_\rho]$, where $v_u := v_{H_u}$, $u \in [-\rho, \rho]$ is the attractor of e^{tH_u} . The control set D_1 that contains $\pi(e) = e_1$ is then described in the left-hand side of Fig. 2. Since $\Gamma = Z_H/(Z_H)_0 = \{\pm 1\}$ implies that $f_{-1}(v, x) = (-v, x)$ we have $D_{-1} = f_{-1}(D_1)$ (see Fig. 2).

By Theorem 3.5 we have that the control sets with nonempty interior of $\Sigma_{\text{SI}(2)}$ are given by $\pi^{-1}(D_1)$ and $\pi^{-1}(D_{-1})$.

Remark 4.1. Notice that there are control sets around the points $(e_2, 0)$ and $(-e_2, 0)$ who are still related by the map f_{-1} . This fact implies that the induced control system on $G/(Z_H)_0$ can have more control sets than the ones given by D_γ , $\gamma \in \Gamma$.

References

- [1] A. Agrachev, Y. Sachkov, *Control Theory from a Geometric Viewpoint*, Springer-Verlag, Berlin, 2004.
- [2] V. Ayala, A. Da Silva, On the characterization of the controllability property for linear control systems on nonnilpotent, solvable three-dimensional Lie groups, *J. Differ. Equ.* 266 (2019) 1–25.
- [3] V. Ayala, A. Da Silva, Controllability of linear control systems on Lie groups with semisimple finite center, *SIAM J. Control Optim.* 55 (2) (2017) 1332–1343.
- [4] V. Ayala, A. Da Silva, G. Zsigmond, Control sets of linear systems on Lie groups, *Nonlinear Differ. Equ. Appl.* 24 (8) (2017).
- [5] V. Ayala, L.A.B. San Martin, Controllability of two-dimensional bilinear systems: restricted controls and discrete-time, *Proyecciones* 18 (2) (1999) 207–233.
- [6] C.J. Braga Barros, L.A.B. San Martin, On the number of control sets on projective spaces, *Syst. Control Lett.* 29 (1) (1996) 21–26.
- [7] A.M. Bloch, *Nonholonomic Mechanics and Control*, Springer-Verlag, New-York, 2003.
- [8] V. Ayala, J. Tirao, Linear control systems on Lie groups and controllability, in: *Symposia in Pure Mathematics*, vol. 64, American Mathematical Society, 1999, pp. 47–64.
- [9] F. Colonius, W. Du, Hyperbolic control sets and chain control sets, *J. Dyn. Control Syst.* 7 (1) (2001) 49–59.
- [10] F. Colonius, W. Kliemann, On control sets and feedback for nonlinear systems, *IFAC Proc. Vol.* 25 (13) (1992) 275–282.
- [11] F. Colonius, W. Kliemann, *The Dynamics of Control*, Birkhäuser, 2000.
- [12] A. Da Silva, Controllability of linear systems on solvable Lie groups, *SIAM J. Control Optim.* 54 (1) (2016) 372–390.
- [13] M. Dath, P. Jouan, Controllability of linear systems on low dimensional nilpotent and solvable Lie groups, *J. Dyn. Control Syst.* 22 (2) (2016) 207–225.
- [14] J.J. Duistermat, J.A.C. Kolk, V.S. Varadarajan, Functions, flows and oscillatory integrals on flag manifolds, *Compos. Math.* 49 (3) (1983) 309–398.
- [15] R. El Assoudi, J.P. Gauthier, I. Kupka, On subsemigroups of semisimple Lie groups, *Ann. Inst. Henri Poincaré* 13 (1) (1996) 117–133.
- [16] T. Gayer, Control sets and their boundaries under parameter variation, *J. Differ. Equ.* 201 (1) (2004) 177–200.
- [17] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, 1978.
- [18] Ph. Jouan, Equivalence of control systems with linear systems on Lie groups and homogeneous spaces, *ESAIM Control Optim. Calc. Var.* 16 (2010) 956–973.
- [19] Ph. Jouan, Controllability of linear systems on Lie groups, *J. Dyn. Control Syst.* 17 (4) (2011) 591–616.
- [20] V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, 1996.
- [21] V. Jurdjevic, I. Kupka, Controllability of right invariant systems on semi-simple Lie groups and their homogeneous spaces, *Ann. Inst. Fourier* 31 (4) (1981) 151–179.
- [22] C. Kawan, Invariance entropy of control sets, *SIAM J. Control Optim.* 49 (2) (2011) 732–751.
- [23] C. Kawan, On the structure of uniformly hyperbolic chain control sets, *Syst. Control Lett.* 90 (2016) 71–75.
- [24] A.W. Knap, *Lie Groups Beyond an Introduction*, second ed., Birkhäuser, Berlin, 2004.

- [25] U. Ledzewicz, H. Schattler, Optimal control for a two compartment model for cancer chemotherapy with quadratic objective, in: *Proceeding of Control 2002, Aveiro, Portugal*, pp. 241–246.
- [26] F.S. Leite, P.E. Crouch, Controllability on classical Lie groups, *Math. Control Signals Syst.* 1 (1988) 31–42.
- [27] L.A.B. San Martin, P. Tonelli, Semigroup actions on homogeneous spaces, *Semigroup Forum* 50 (1995) 59–88.
- [28] D. Szolnoki, Viability kernels and control sets, *ESAIM Control Optim. Calc. Var.* 5 (2002) 175–185.
- [29] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups I*, Springer-Verlag, 1972.