obtained by a reflection through the plane \( s \) of symmetric matrices (spanned by \( \{ H, S \} \)). Therefore \( B^\perp \) is the reflection through \( s \) of the plane orthogonal to \( B \) with respect to \( \langle \cdot, \cdot \rangle \).

From the above proposition we get the following description of the trajectories.

**Proposition 4.2.** The trajectories in \( C \) are:

1. If \( B \in C_{\text{int}} \) : ellipses around \( C \).
2. If \( B \in C \) : points in the ray of \( C \) orthogonal to \( B \) or the parabolas \( \{(B, Z) = c\} \cap C \).
3. If \( B \in C_{\text{ext}} \) : the two rays in \( B^\perp \cap C \setminus \{0\} \) or the hyperbolas \( \{(B, Z) = c\} \cap C \).

In particular if \( B \in C_{\text{ext}} \) then it has different real eigenvalues and the eigenvectors projected in \( P \) — identified with the set \([C^+]\) of rays of \( C^+ \) — are given by the intersection of \( B^\perp \) with \( C^+ \). From this geometry it is possible to detect also the attractor and the repeller for \( B \) in \( P \). In fact, if \( B \) is diagonalizable then \( B = \text{Ad}(g)(cH) \) for some \( c > 0 \) and \( g \in \text{Sl}(2) \), where

\[
H = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

The attractor and the repeller for \( H \) in \( P \) are \( Z \) and \( W \) respectively, where

\[
Z = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \quad W = \begin{pmatrix}
0 & 0 \\
-1 & 0
\end{pmatrix}.
\]

Note that the basis \( \{Z, H, W\} \) has the same orientation as the canonical basis \( \{S, H, A\} \). Also, \( Z, W \in C^+ \). Since \( \det \text{Ad}(g) = 1 \), it follows that the basis formed by the attractor of \( B \), \( B \) and the repeller is positively oriented w.r.t. \( \{S, H, A\} \). From this it is clear which is the attractor and the repeller of \( B \).

**Proposition 4.3.** Take \( B \in C_{\text{ext}} \). Under the identification of \( P \) with the set of rays of \( C^+ \), the fixed points of \( B \) in \( P \) are the rays in \( B^\perp \cap C^+ \). Let \( Z \) and \( W \) in \( C^+ \) correspond to the attractor and repeller respectively. Then the basis \( \{Z, B, W\} \) has the same orientation as the standard basis \( \{S, H, A\} \).
Now, we consider a segment
\[ \sigma = \{ B + uC : u \in [a,b] \} , \]
where \([a,b] \subset R\) is an interval. We assume that \(\sigma\) is contained in \(C_{\text{ext}}\) and look at the set of attractors and repellers in \(P\) of the matrices in \(\sigma\).

According to the above proposition these fixed points are given by the intersection with \(C\) of the plane \(W^\perp\) with \(W \in \sigma\).

Suppose that \(Z \neq 0\) spans the line \(\sigma^\perp\). In what follows we assume the generic situation in which \(Z \notin C\). The planes \(W^\perp\), \(W \in \sigma\), are better visualized in case \(\sigma\) is put in one of the following normal forms.

1. Suppose that \(Z \in C_{\text{int}}\). Then there exists \(g \in \text{Sl}(2)\) such that \(gZg^{-1}\) is skew-symmetric. Since the conjugation preserves the trace form, we can assume without loss of generality that
\[ Z = A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \]

In this case \(\sigma\) is contained in the subspace \(s\) of symmetric matrices. Therefore is \(W \in \sigma\), \(W^\perp\) is the plane spanned by \(Z\) and \(W^\perp \cap s\).

2. In case \(Z \in C_{\text{ext}}\), it is conjugate to a symmetric matrix. So we can assume that
\[ Z = S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \]

In this case \(\sigma\) is contained in the plane spanned by \(\{H, A\}\). In this case \(W \in \sigma\) has the form
\[ W = \begin{pmatrix} x & -y \\ y & -x \end{pmatrix} \]
with \(\det W = -x^2 + y^2 < 0\). Thus \(W^\perp\) is the plane spanned by \(S\) and
\[ \begin{pmatrix} y & -x \\ x & -y \end{pmatrix} \in C_{\text{int}}. \]

From this picture of \(W^\perp\), \(W \in \sigma\), the set of attractors and repellers are easily given.
Proposition 4.4. Let σ be a segment contained in \( C_{\text{ext}} \). A \( W \in \sigma \) has an attractor and one repeller in \( P \). Making \( W \) runs through \( \sigma \), denote by \( A(\sigma) \) and \( R(\sigma) \) respectively the set of attractors and repellers thus obtained. Then \( A(\sigma) \) and \( R(\sigma) \) are nonoverlapping intervals in \( P \).

Furthermore, \( A(\sigma) \) is invariant under the one-parameter semigroup

\[
\exp(tW), \quad t \geq 0
\]

for all \( W \in \sigma \).

**Proof:** The first statement follows by directly from the above description of \( W^t \), \( W \in \sigma \) and the identification of \( P \) with \([C^+]\). As to the invariance of \( A(\sigma) \), note that any interval in \( P \) containing the attractor of \( W \) but not its repeller is invariant under \( \exp(tW) \), \( t \geq 0 \). \( \square \)

5. Restricted controls

In this section we consider bilinear systems with restricted controls. The objective is to extend to this case the controllability result given for unrestricted controls. Instead of looking plainly at controllability we consider a bilinear system with varying control range

\[
\dot{x} = (X + uY)x \quad \quad \quad u \in U^p = [-\rho, \rho], \rho \geq 0.
\]

The problem is to detect the values of \( \rho \) for which the corresponding system is controllable. Actually, we consider the bifurcation scenario posed by Colonius and Kliemann [2], in what regards the transitivity properties of the system in the projective line \( P \).

In order to state precisely this scenario we note that it is assumed, as above that \( \text{tr}X = \text{tr}Y = 0 \) and that \( \det[X,Y] \neq 0 \). These conditions ensure that the Lie algebra generated by \( X \) and \( Y \) is \( \text{sl}(2) \). Since \( \det[X,X+uY] = u^2 \det[X,Y] \) this condition also ensures that the system group is \( \text{Sl}(2) \) for any value of \( \rho > 0 \).

Denote by \( S_\rho \) the semigroup of the system (5.1\(^p\)), that is, the system (5.1) with control range \( U^p \). Since \( \text{int}S_\rho \neq \emptyset, \rho > 0 \), the control sets \( C_\rho^\pm \subset P \) are well defined. Clearly \( S_{\rho_1} \subset S_{\rho_2} \) if \( \rho_1 \leq \rho_2 \). This implies that the maps \( \chi^\pm: \rho \mapsto C_\rho^\pm \) are increasing. Consider in particular \( \chi^+ \) which associates with \( \rho \) the invariant control set of (5.1\(^p\)) in \( P \). We look at its continuity properties when the set of compact of subsets of \( P \) is endowed with the Hausdorff topology. It was proved in [2], Chapter 3, that \( \chi^+ \) is
lower semicontinuous (see [2], Appendix B, for the definition.) With this in mind we define the bifurcation points of \((5.1^\rho)\).

**Definition 5.1.** A \(\rho > 0\) is said to be a bifurcation point of \((5.1^\rho)\) if \(\chi^+\) is not continuous at \(\rho\) (with respect to the Hausdorff topology on compact subsets.)

**Remark:** Before proceeding let us mention that the above continuity concept is the right one for the general theory in [2]. However, in our context we do not need to consider the Hausdorff topology in \(P\). In fact, the invariant control set \(C^+_\rho\) is connected because \(S_\rho\) is connected. Since \(P\) is one-dimensional \(C^+_\rho\) is an interval in \(P\), and hence given by its endpoints, so that instead of the Hausdorff topology on all compact sets we can look at the simpler topology given by pairs of points in \(P\).

We look now at the bifurcation points of \((5.1^\rho)\). Recall that \(\{X,Y\}\) generates \(sl(2)\) if and only if \(\text{det}[X,Y] \neq 0\).

**Theorem 5.2.** Suppose that \(\text{det}[X,Y] \neq 0\). Then the controllability of the system \((5.1^\rho)\) with restricted control range \(U^\rho = [-\rho, \rho]\) is given by the relative position of the segment

\[
\sigma_\rho = \{X + uY : u \in U^\rho\}
\]

as follows:

1. If \(\text{det} X \geq 0\) then \(\sigma_\rho \cap C_{\text{int}} \neq \emptyset\) and the system is controllable for any \(\rho > 0\), that is, \(\chi^+(\rho) = P\) for all \(\rho > 0\).

2. If \(\text{det} X < 0\), there are the possibilities:

   (a) If \(\text{det}[X,Y] < 0\) then the line \(X + uY, u \in R\), crosses the interior of \(C\) and \((5.1^\rho)\) is controllable if and only if \(\sigma_\rho \cap C_{\text{int}} \neq \emptyset\). The only bifurcation point is \(\rho^* = \inf \{\rho : \sigma_\rho \cap C_{\text{int}} = \emptyset\}\).

   (b) In case \(\text{det}[X,Y] > 0\) the system is not controllable for any \(\rho > 0\). Furthermore, \(\chi^+\) is continuous in \((0, +\infty)\).

The rest of this section is devoted to the case by case proof of this theorem.

1) If \(\text{det} X > 0\) then the eigenvalues of \(X\) are complex. Hence is no compact proper subset of \(P\) invariant under \(\exp(tX)\), \(t > 0\). This implies that \((5.1^\rho)\) is controllable for any \(\rho > 0\), that is, \(\chi^+(\rho) = P\) for all \(\rho > 0\).
II) If \( \det X = 0 \), the system is also controllable for any \( \rho > 0 \). In fact, \( u = 0 \) is a real root of \( \det (X + uY) \). Hence if \( \det Y \neq 0 \) the discriminant of the quadratic polynomial is \( -\det [X, Y] \geq 0 \). Since we are assuming that \( \{X, Y\} \) generates \( \mathfrak{s}l(2) \), it follows that \( \det [X, Y] < 0 \). This implies that for any \( \rho > 0 \) there exists \( u_0 \in [-\rho, \rho] \) such that \( X + u_0 Y \) belongs to the interior of \( C \). Therefore (5.1\( ^\rho \)) is controllable for any \( \rho > 0 \). On the other hand if \( \det Y = 0 \) then

\[
\det (X + uY) = -(XY + YX)u + \det X.
\]

But \( XY + YX \neq 0 \), as follows from (2.3). This implies again that \( \det (X + u_0 Y) > 0 \) for some \( u_0 \in [-\rho, \rho] \) for all \( \rho > 0 \), ensuring that the system is controllable.

III) If \( \det X < 0 \) then there are two cases to be considered:

1. The unrestricted system is controllable (\( \det [X, Y] < 0 \)). This holds if and only if the straight line \( X + uY, u \in R \), meets \( C_{int} \). Since \( \det X < 0, X \in C_{ext} \). Hence there exists \( \rho_* > 0 \) such that the segment \( X + uY, u \in [-\rho_*, \rho_*] \) is the smallest one meeting \( C \). In other words, \( X + \rho_* Y \) or \( X - \rho_* Y \) is the first hitting \( C \) of the line \( X + uY \), starting from \( X \).

We claim that (5.1\( ^\rho \)) is controllable if \( \rho > \rho_* \) and not controllable otherwise.

To see this note first that if \( \rho > \rho_* \) then some point \( X + u_0 Y \) belongs to the interior of the double cone \( C \) for some \( u_0 \in [-\rho, \rho] \), so that the system is controllable, because no proper subset of \( P \) is invariant under \( X + u_0 y \).

Now, suppose that \( \rho < \rho_* \). Let \( Z \neq 0 \) be orthogonal (w.r.t. the trace form) to the plane \( \pi \) spanned by \( X \) and \( Y \). Then \( Z \in C_{ext} \) because \( \pi \) intersects the interior of \( C \) and the plane orthogonal to any line in \( C_{int} \) is contained in the exterior of \( C \). Therefore by Proposition 4.4, the set \( A(\sigma_\rho) \) of attractors of the matrices in the segment

\[
\sigma_\rho = \{X + uY : u \in [-\rho, \rho]\},
\]

is invariant under the semigroup generated by (5.1\( ^\rho \)). Therefore the system is not controllable. Moreover, the invariance of \( A(\sigma_\rho) \) together with the fact that it is contained in the invariant control set of (5.1\( ^\rho \)) (see Proposition 3.1) implies that \( \chi^+(\rho) = A(\sigma_\rho) \). Therefore, the
characterization of \( A(\sigma_\rho) \) by intersections of planes with \( C^+ \) shows
that \( \chi^+ \) is continuous at \( \rho \).

Finally, for \( \rho = \rho_* \) the system is not controllable. This can be seen
either by the lower semicontinuity of \( \chi^+ (\rho) \) is proper in case
\( \rho < \rho^* \) or directly, as follows: Suppose without loss of generality
that \( X + \rho_* Y \) is the first hitting in \( C \). The intersection with \( C^+ \) of
\( (X + \rho_* Y)^+ \) is the ray defined by \( X + \rho_* Y \) and the interval of the
attractors for \( X + uY, u \in [-\rho_*, \rho_*] \) ends in this ray. So as above the
invariant control set is the closure of the interval of the attractors,
and the system is not controllable.

Since the system is controllable for \( \rho > \rho_* \) it follows that \( \rho_* \) is the
only point of discontinuity.

2. The unrestricted system is not controllable (\( \det [X, Y] > 0 \)).

Let \( Z \neq 0 \) span the line orthogonal to the plane spanned \( X \) and \( Y \).
Then \( Z \in C_{\text{int}} \) and the segment \( \sigma_\rho \) defined by the system is contained
in \( C_{\text{ext}} \) for all \( \rho > 0 \). Applying Proposition 4.4 we see as above that
the invariant control set of \( (5.1^p) \) is \( A(\sigma_\rho) \) the set of attractors of
the matrices in \( \sigma_\rho \). Therefore \( \chi^+ (\rho) = A(\sigma_\rho) \) implying that \( \chi^+ \) is
continuous at every \( \rho > 0 \). A similar argument shows that \( \chi^- \) is
continuous as well.

6. Discrete-time

In this section we consider the discrete-time version of (1.1), namely

\[
x_{n+1} = \exp (X + uY) x_n
\]

where \( X \) and \( Y \) are \( 2 \times 2 \) matrices. Analogous to the continuous-time
case we can consider the control range \( U \) to be unrestricted \( (U = R) \) or
restricted \( (U^p = [-\rho, \rho]) \). Our purpose here is to show that controllability
is given again by the intersection of \( X + uY \) with the cone \( C \) of nilpotent
matrices.

The semigroup generated by system (6.1) is defined by

\[
S_d = \{ \exp (X + u_1 Y) \cdots \exp (X + u_k Y) : k \geq 1 \}
\]

where \( u_k \) varies in the control range. Also, the group of the system \( G \) is the
group generated by \( S_d \). The system is said to be controllable if \( R^2 \setminus \{0\} \) if
for any pair of nonzero vectors \( x, y \) there exists \( g \in S_d \) such that \( gx = y \).
In the sequel we consider only systems whose group is $\text{SL}(2)$. As happens to the continuous-time case there exists a simple criterion for ensuring that $G = \text{SL}(2)$. First, we must have $\text{tr}X = \text{tr}Y = 0$ to have that $S \subset \text{SL}(2)$. On the other hand, if this condition holds, then the fact that $\text{SL}(2)$ is a simple group implies that $\text{int} S \neq \emptyset$ in case the Lie algebra generated by $X$ and $Y$ is $\text{sl}(2)$ (see 8, Section 4.) Therefore we have

**Proposition 6.1.** Suppose that $\text{tr}X = \text{tr}Y = 0$ and $\det[X,Y] \neq 0$. Then the semigroup $S$ generated by the system (6.1) is contained in $\text{SL}(2)$ and has nonempty interior in $\text{SL}(2)$.

For systems satisfying the assumptions of this proposition we can apply the results about control sets to analyze controllability. In particular, the system is controllable if and only if $S = \text{SL}(2)$ which in turn holds if and only if $S$ is transitive in $\mathbb{P}$.

In the following lemma we check the system is controllable if $X + uY$ crosses the interior of $\mathcal{C}$.

**Lemma 6.2.** Consider the system (6.1) with control range $U^p = [−\rho, \rho]$. Assume that $X + u_0Y$ belongs to the interior of $\mathcal{C}$ for some $u_0 \in U^p$. Then the system is controllable.

**Proof:** Put $Z_0 = X + u_0Y$. By assumption $\det(Z_0) > 0$. Moreover, the eigenvalues of $X + uY$ are $\pm \sqrt{\det(X + uY)}$. Then we can change $u_0$ slightly and assume without loss of generality that the eigenvalues of $Z_0$ are $\pm \sqrt{-1}$ with $\varepsilon/\pi$ irrational. This being so, put $g_0 = \exp(Z_0)$. Then for any $x \in \mathbb{P}$, the orbit $\{g_k^k x : k \geq 0\}$ is dense in $\mathbb{P}$. Clearly, $g_k^k \in S_d$ if $u \geq 0$. Hence $S$ does not leave invariant any compact subset $C \subset \mathbb{P}$, showing that the invariant control set in $\mathbb{P}$ is not proper, that is, $S_d$ is transitive in $\mathbb{P}$.

Reciprocally, assume that the segment $X + uY$, $u \in U^p$, does not meet the interior of $\mathcal{C}$. Then the discrete-time control system (6.1) is not controllable. In fact, by definition of these semigroups it follows that $S_d \subset S_c$. Now, segment $X + uY$, $u \in U^p$, does not meet the interior of $\mathcal{C}$, then $S_c$ is a proper semigroup, as was proved before. Therefore $S_d \neq \text{SL}(2)$, showing that the discrete-time system is not controllable. Summarizing, we have

**Theorem 6.3.** Given the discrete-time system (6.1) with restricted control range $U^p$. Assume that $\text{tr}X = \text{tr}Y = 0$ and $\det[X,Y] \neq 0$. Then the system is controllable if and only if the segment $X + uY$, $u \in U^p$, crosses the interior $\mathcal{C}_\text{int}$ of $\mathcal{C}$. This geometric condition holds for some $\rho$ only if $\det[X,Y] < 0$. 
References


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