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OPTIMALITY ON HOMOGENEOUS SPACES, AND THE ANGLE SYSTEM ASSOCIATED WITH A BILINEAR CONTROL SYSTEM*

V. AYALA†, J. C. RODRÍGUEZ‡, AND L. A. B. SAN MARTIN§

Abstract. Let G be a Lie group. In order to study optimal control problems on a homogeneous space G/H, we identify its cotangent bundle T^*G/H as a subbundle of the cotangent bundle of G. Next, this identification is used to describe the Hamiltonian lifting of vector fields on G/H induced by elements in the Lie algebra $\mathfrak{g}$ of G. As an application, we consider a bilinear control system Σ in $\mathbb{R}^2$ whose matrices generate $\mathfrak{sl}(2)$. Through the Pontryagin maximum principle, we analyze the time-optimal problem for the *angle system* $\mathbb{P}\Sigma$ defined by the projection of Σ onto the projective line $\mathbb{P}^1$. We compute some examples, and in particular we show that the bang-bang principle does not need to be true.

Key words. optimal time, Pontryagin maximum principle, bilinear control systems, Cartan–Killing form, real projective line

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1. Introduction.

Let G be a Lie group and H a closed subgroup. In order to study optimal control problems on a homogeneous space G/H, we identify the cotangent bundle of G/H as a subbundle of the cotangent bundle of G. Next, we use the mentioned identification to describe the Hamiltonian lifting of vector fields on G/H induced by elements in the Lie algebra $\mathfrak{g}$ of G.

As an application of the construction we consider a bilinear control system Σ in $\mathbb{R}^2$ given by the family of differential equations:

$$\dot{x}(t) = (A + uB)x(t), \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{R}^2, \tag{1}$$

where A, $B \in \mathfrak{sl}(2)$ are real 2×2 matrices of trace zero and

$$u \in \mathcal{U} = \{u : \mathbb{R} \to [-1, 1], u \text{ locally integrable}\}$$

is the set of the restricted admissible controls. We assume that Σ satisfies the Lie algebra rank condition, LARC, which means that the Lie algebra generated by A and B coincides with $\mathfrak{sl}(2)$.

The *angle system* $\mathbb{P}\Sigma$ is defined by projection of Σ onto the projective line

$$\mathbb{P}\Sigma : \qquad \dot{s}(t) = h(A, s(t)) + u(t)h(B, s(t)), \qquad s \in \mathbb{P}^1, \tag{2}$$

where $h(X, s) = (X - s^T X s I)s$, with I the identity matrix and $u \in \mathcal{U}$.

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†Departamento de Matemáticas, Universidad Católica del Norte, Casilla 1280, Antofagasta–Chile and Instituto de Ciências Exatas, Universidade Federal do Amazonas, Manaus AM, Brasil (vayala@ucn.cl). This research was partially supported by Proyectos FONDECYT 1060981, 7080232, and Programa AMAZONAS SENIOR Processo Número: 507/2007, FAPEAM.

‡Departamento de Matemáticas, Universidad Católica del Norte, Casilla 1280, Antofagasta–Chile (jrodriguez@ucn.cl). This research supported by CONICYT Proyecto D-21070955.

§Instituto de Matemática Estatística e Ciências da Computação, Universidade Estadual de Campinas, Cx. Postal 6065, 13081-970 Campinas SP, Brasil (smartin@ime.unicamp.br). This research supported by CNPq grant 305513/2003-6 and FAPESP grant 07/06896-5.

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For this class of control systems we consider the following time-optimal control problem. If q_b and q_e denote the initial and final points on the projective line, respectively, then

find a trajectory of $\mathbb{P}\Sigma$ *steering* q_b *to* q_e *at minimum time.*

Through the Pontryagin maximum principle (PMP), the time-optimal problem for single input control systems has been studied by many authors; see, for instance, [2], [3], [6], [12], [13], and [14].

Following [1] we identify through the Cartan–Killing form, the cotangent bundle of the projective line as a cone in $\mathbb{R}^3$; see section 2. Next, we describe on the cone the trajectories of the lifted vector fields corresponding to the dynamics of the control system on $\mathbb{P}^1$. We analyze the extremals as solutions of a pseudo-Hamiltonian system. They are trajectories of the adjoint system staying on the cone $\mathcal{C}^+$, which corresponds to the elements of $\mathcal{C}$ with positive three coordinates. Therefore, every time-optimal trajectory of the projected system (2) on the real projective line $\mathbb{P}^1$ is obtained by radial projection.

1.1. Generalities on the cotangent bundle $T^*(G/H)$. Let G be a Lie group and $H \subset G$ a closed Lie subgroup. It is well known that $G/H = \{zH : z \in G\}$ is an analytic differentiable manifold. The elements in the cotangent bundle

$$T^*(G/H) = \bigcup_{z\in G} T^*_{zH}(G/H)$$

can be represented as elements in the cotangent bundle $T^*(G)$ of G as follows.

Consider the canonical projection $\pi : G \to G/H$ defined by $\pi(z) = zH$. Then, for any $z \in G$, its differential

$$d\pi_z : T_zG \to T_{\pi(z)}(G/H) \text{ has the transpose}$$

$$(d\pi_z)^* : T^*_{\pi(z)}(G/H) \to T^*_zG, \text{ given by } (d\pi_z)^*(\alpha) = \alpha \circ d\pi_z.$$

Since the linear map $d\pi_z$ is surjective, it follows that $(d\pi_z)^*$ is injective. In particular, for each $z \in G$ the cotangent space $T^*_{zH}(G/H)$ is isomorphic to its image under $(d\pi_z)^*$. That is,

$$T^*_{zH}(G/H) \cong (d\pi_z)^* T^*_{zH}(G/H) \subset T^*_zG.$$

Therefore, the elements of the cotangent bundle $T^*(G/H)$ can be represented by elements of T^*G.

Recall that if V and W are two finite-dimensional vector spaces and $T : V \to W$ is an onto linear map with $T^* : W^* \to V^*$ its transpose. Then, the image of T^* is the annihilator of $\ker T$; that is, Im $T^* = (\ker T)^0$, where

$$(\ker T)^0 = \{\lambda \in V^* : \forall v \in \ker T, \lambda(v) = 0\}.$$

In the sequel we identify T^*G with $\mathfrak{g}^* \times G$ where $\mathfrak{g} = T_1G$ is the Lie algebra of G and $\mathfrak{g}^*$ its dual. The identification is made through the G-translations. More precisely, to the pair $(\lambda, z) \in \mathfrak{g}^* \times G$ we associated the element of T^*_zG defined by $\lambda \circ dR_{z^{-1}} \in T^*_zG$, where R_z denotes the right translation by $z \in G$. Its inverse is given by $\mu \in T^*_zG \longmapsto \left(d(R_z)^*_1(\mu), z\right) \in \mathfrak{g}^* \times G$.

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PROPOSITION 1.1. *Through the previous identification, the annihilator* $(\ker d\pi_z)^0$ *is identified to the annihilator of* $\mathrm{Ad}(z)\mathfrak{h}$*, where* $\mathfrak{h}$ *denotes the Lie algebra of the Lie subgroup* H *of* G.

Proof. In fact, the tangent space of H at the point z is $(dL_z)_1\mathfrak{h}$. Therefore,

$$(\ker d\pi_z)^0 = ((dL_z)_1\mathfrak{h})^0.$$

In particular, the right translation identification shows that the annihilator is

$$(dL_z)_1\mathfrak{h} = (dR_{z^{-1}})_z(dL_z)_1\mathfrak{h}, \quad \text{i.e., } (\ker d\pi_z)^0 = (\mathrm{Ad}\,(z)\,\mathfrak{h})^0\,,$$

ending the proof. ∎

A particular case that will be exploited below is when G is a semisimple Lie group. In this case it is well known that the Cartan–Killing form $\mathcal{K}$ of its Lie algebra $\mathfrak{g}$ is nondegenerate and defines an isomorphism between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$, $X \in \mathfrak{g} \longmapsto \mathcal{K}(X,\cdot) \in \mathfrak{g}^*$ (see [9]). Through this isomorphism the annihilator of a subspace $V \subset \mathfrak{g}$ is identified to the orthogonal

$$V^{\perp} = \{X \in \mathfrak{g} : \forall Y \in V, \mathcal{K}(X,Y) = 0\}$$

of V with respect to the Cartan–Killing form. In particular, Proposition 1.1 admits the following version.

PROPOSITION 1.2. *Suppose* $\mathfrak{g}$ *is semisimple. Let us denote by* $\mathfrak{h}$ *the Lie algebra of* H*. Through the previous identifications we have*

(i) $(\ker d\pi_z)^0 \cong (\mathrm{Ad}\,(z)\,\mathfrak{h})^{\perp}$,

(ii) $(\mathrm{Ad}\,(z)\,\mathfrak{h})^{\perp} \cong \mathrm{Ad}\,(z)\,\mathfrak{h}^{\perp}$.

1.2. Hamiltonian lifting on homogeneous spaces. Let X be a vector field on a differentiable manifold N. There exists a lifting X^* of X on the cotangent bundle T^*N, with flow X_t^* given by

$$X_t^*(\alpha) = \alpha \circ (dX_{-t})_{X_t(x)} = (dX_{-t})^*_{X_t(x)}(\alpha)\,, \tag{3}$$

where X_t denotes the flow of X and $\alpha \in T_x^*N$.

It is well known that X^* is a Hamiltonian vector field. The corresponding Hamiltonian function $H_{X^*} : T^*N \to \mathbb{R}$ is defined by

$$H_{X^*}(\alpha) = \alpha\,(X\,(p\,(\alpha)))\,, \qquad \alpha \in T^*N,$$

where $p : T^*N \to N$, $\alpha \longmapsto p(\alpha)$ is the fiber bundle projection. H_{X^*} is a fiberwise linear functional on T_xN.

Next we consider the homogeneous space G/H, and we denote by $\mathfrak{g}$ and $\mathfrak{h}$, the Lie algebras of G and H, respectively. Each $X \in \mathfrak{g}$ induces a vector field on G/H, denoted by $\widetilde{X}$. Its flow is given by $\widetilde{X}_t\,(x) = \exp\,(tX)\,x$, $x \in G/H$.

In the following we use the previous identifications to describe the lifting $\widetilde{X}^*$ of the vector fields $\widetilde{X}$ on the homogeneous space G/H induced by $X \in \mathfrak{g}$.

PROPOSITION 1.3. *Let* $\mathfrak{h} \subset \mathfrak{g}$ *be the Lie algebra of* H *and* $X \in \mathfrak{g}$*. Through the previous identifications, the vector field* $\widetilde{X}^*$ *on the cotangent bundle* $T^*(G/H)$ *is given by*

$$\widetilde{X}^*\,(\lambda) = -\mathrm{ad}\,(X)^*\,(\lambda) = -\lambda \circ \mathrm{ad}\,(X)$$

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and its flow is

$$\widetilde{X}_t^* (\lambda) = \mathrm{Ad} (\exp (-tX))^* \lambda.$$

Proof. First, we consider the identification of T^*G/H with its image via $d\pi^*$. Formula (3) yields

$$\widetilde{X}_t^* (\alpha) = \left(d\widetilde{X}_{-t}\right)^*_{\widetilde{X}_t(x)} (\alpha) .$$

Applying the transpose $d\pi^*_{X_t(z)}$ to this equality we get

$$d\pi^*_{X_t(z)} \circ \widetilde{X}_t^* = d\pi^*_{X_t(z)} \circ \left(d\widetilde{X}_{-t}\right)^*_{\widetilde{X}_t(x)} = d\left(\widetilde{X}_{-t} \circ \pi\right)^*_{X_t(z)} .$$

For each $g \in G$ we denote by L_g the left translation on G. Since the projection π commutes with left translations, i.e., $\pi \circ L_g = L_g \circ \pi$, we obtain

$$d\pi^*_{(\exp tX)(z)} \circ \widetilde{X}_t^* = d\left(\pi \circ L_{\exp(-tX)}\right)^*_{(\exp tX)z} = d\left(L_{\exp(-tX)}\right)^*_{(\exp tX)z} \circ d\pi_z^*.$$

Therefore, if $\alpha \in T^*_{zH} (G/H)$ is identified with $\mu \in T_z^*G$, the vector

$$X_t^* (\alpha) \in T^*_{\exp(tX)zH} (G/H)$$

is also identified with

$$d\left(L_{\exp(-tX)}\right)^*_{(\exp tX)z} (\mu) .$$

By applying the linear map $d\left(R_{\exp(tX)}\right)^*_1$ we have

$$\begin{aligned} &d\left(R_{\exp(tX)}\right)^*_1 \circ d\left(L_{\exp(-tX)}\right)^*_{\exp(tX)z} \\ &= d\left(L_{\exp(-tX)} \circ R_{\exp(tX)}\right)^*_1 \circ (dR_z)_1^* = \mathrm{Ad} (\exp (-tX))^* \circ (dR_z)_1^*. \end{aligned}$$

On the other hand, Proposition 1.1 shows that

$$(d\pi)_z^*(T^*_{zH} (G/H)) = (\ker d\pi_z)^0.$$

So, if $\alpha \in T^*_{zH} (G/H)$ is identified with $\lambda \in \mathfrak{g}^*$, then $X_t^*\alpha \in T^*_{\exp(tX)zH} (G/H)$ is identified with $\mathrm{Ad} (\exp (-tX))^* \lambda \in \mathfrak{g}^*$. ∎

In the semisimple case we identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$, to get the following statement.

PROPOSITION 1.4. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ the Lie algebra of H. Then, the vector field $\widetilde{X}^*$ on $T^* (G/H)$ induced by $X \in \mathfrak{g}$ is given by*

$$\widetilde{X}^* (Y) = \mathrm{ad} (X) (Y) \text{ with flow } \widetilde{X}_t^* (Y) = \mathrm{Ad} (\exp tX) Y.$$

Proof. We know that $Y \in \mathfrak{g}$ corresponds to $\mathcal{K}(Y, \cdot) \in \mathfrak{g}^*$. Since $\mathcal{K}(\mathrm{Ad} (g) Y, Z) = \mathcal{K}(Y, \mathrm{Ad} \left(g^{-1}\right) Z)$, the result follows. ∎

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2. The cotangent bundle $T^*(\mathbb{P}^1)$ as a cone. As usual we denote by SL(2) the Lie group of the Lie algebra $\mathfrak{sl}(2)$, the set of 2×2 real matrices with determinant 1. In this particular case, the Cartan–Killing form $\mathcal{K}$ is a multiple of the trace form and is given by

$$\mathcal{K}(X, Y) = k \operatorname{tr}(XY) = k \langle X, Y \rangle ,$$

where $k \neq 0$ is a constant. Furthermore, consider the corresponding quadratic form $Q(X) = \langle X, X \rangle = \operatorname{tr}(X^2)$ and denote the zero set of Q by

$$\mathcal{C} = \{X \in \mathfrak{sl}(2) : Q(X) = 0\}.$$

In particular, $\mathcal{C}$ is invariant under conjugation by invertible matrices; i.e., $gZg^{-1} \in \mathcal{C}$ if $Z \in \mathcal{C}$. The set $\mathcal{C} - \{0\}$ has two connected components. We distinguish them by putting $\mathcal{C}^+$ and $\mathcal{C}^-$ for the one which contains the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

respectively. Each of these components is an orbit under conjugation of SL(2). For instance, to see that SL(2) acts transitively on $\mathcal{C}^+$, note that the rotation group turns around $\mathcal{C}^+$ while the group of diagonal matrices is transitive along the ray of upper triangular matrices in $\mathcal{C}^+$. Furthermore, we note that $\mathcal{C}$ is the set of nilpotent matrices in $\mathfrak{sl}(2)$ while $\mathcal{C} - \{0\}$ is the set of rank one 2×2 matrices having trace zero.

We recall that if u and v are orthogonal vectors in $\mathbb{R}^2$, taking them as 2×1 matrices we have that $u^t v$ belongs to $\mathcal{C}$. Moreover, $u^T v \in \mathcal{C}^+$ if $\{u, v\}$ is positively oriented with respect to the standard basis of $\mathbb{R}^2$. Thus, the map

$$\phi : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} -y & x \end{pmatrix}$$

induces a map from $\mathbb{P}^1$ into the set $[\mathcal{C}^+]$ of rays of $\mathcal{C}^+$ as $x \in \mathbb{P}^1 \to \mathbb{R}^+ x$. It turns out that the action of SL(2) on $[\mathcal{C}^+]$ is equivalent to the action of SL(2) in the projective line $\mathbb{P}^1$, through the map ϕ. Next, an algebraic picture of the cone is given.

PROPOSITION 2.1. *The cone $\mathcal{C}$ is identified with the cotangent bundle $T^*(\mathbb{P}^1)$.*

Proof. Consider the parabolic subgroup $H \subset G = \mathrm{SL}(2)$ defined by the matrices of the form

$$\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}$$

with $a \in \mathbb{R} - \{0\}$. Obviously the G-action on $\mathbb{P}^1$ is transitive and the stabilizer of the line $l = [e_1] \in \mathbb{P}^1$ is H. In particular, the homogeneous space G/H is isomorphic with $\mathbb{P}^1$. The Lie algebra $\mathfrak{h}$ of H is given by the matrices of the form

$$\begin{pmatrix} \alpha & * \\ 0 & -\alpha \end{pmatrix}, \ \alpha \in \mathbb{R}.$$

That is,

$$\mathfrak{h} = \operatorname{Im}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

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Related to the Cartan–Killing form, the orthogonal space $\mathfrak{h}^\perp$ is the nilpotent Lie algebra

$$\mathfrak{h}^\perp = \{X \in \mathfrak{g} : \forall Y \in \mathfrak{h},\ \mathcal{K}(X,Y) = 0\} = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\}.$$

According to Proposition 1.2, for any $z \in G$, the cotangent space $T^*_{zH}(G/H)$ is identified with

$$\mathrm{Ad}\,(z)\left(h^\perp\right) = \mathrm{Ad}\,(z)\,(\mathfrak{n})\,.$$

This identification gives to the cotangent space $T^*_{zH}\left(\mathbb{P}^1\right)$ a very simple geometric description. In fact, the nilpotent algebra $\mathfrak{n}$, which is the line generated by the matrix

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

is contained in $\mathcal{C}$. Since the action of SL(2) on $\mathfrak{sl}(2)$ is given by conjugation, we obtain that $\mathrm{Ad}\,(z)\,\mathfrak{n}$, $z \in G$, is also a line contained in $\mathcal{C}$. It follows that the fiber bundle $T^*\left(\mathbb{P}^1\right)$ is identified to the entire cone $\mathcal{C}$. ∎

2.1. Trajectories on the cone. Any $Z \in \mathfrak{sl}(2)$ defines, through the linear application $\mathrm{ad}\,Z$, a linear differential equation in $\mathfrak{sl}(2)$, explicitly given by $\dot{P} = [Z,P]$, $P \in \mathfrak{sl}(2)$, whose trajectories are

$$\exp\left(t\ \ \mathrm{ad}\,Z\right)P = Ad(\exp tZ)(P),\ \text{ with } P \in \mathfrak{sl}(2) \text{ and } t \in \mathbb{R}.$$

The tangent plane $T_P\ \mathcal{C}^+$ of $\mathcal{C}^+$ at $P \in \mathcal{C}^+$ is determined by

$$T_P\ \mathcal{C}^+ = \{W \in \mathfrak{sl}(2) : \langle P, W\rangle = 0\}.$$

In particular, each $Z \in \mathfrak{sl}(2)$ induces a differential equation on $\mathcal{C}^+$. Just observe that $\langle P, [Z,P]\rangle = 0$. We will describe these trajectories in $\mathcal{C}^+$ as the intersections of $\mathcal{C}^+$ with the planes

$$Z^\perp = \{P \in \mathcal{C}^+ : \langle P, Z\rangle = c\}$$

orthogonal to Z with respect to the trace form. As a matter of fact, for any $Z \in \mathfrak{sl}(2)$, the map $\mathrm{Ad}\,(\exp Z) = \exp(\mathrm{ad}\,Z)$ is an isometry; i.e., for every $P, W \in \mathfrak{sl}(2)$, $\langle \exp(\mathrm{ad}\,Z)P, \exp(\mathrm{ad}\,Z)W\rangle = \langle P, W\rangle$.

In practice, with respect to the trace form, the plane orthogonal to any matrix of trace zero can be seen with the aid of the inner product $(\cdot\ ,\cdot)$ in $\mathfrak{sl}(2)$ which is defined by

$$(P, W) = \langle P, W^t\rangle = \mathrm{tr}\left(PW^t\right).$$

Actually, if we denote by $Z^{(\perp)}$ the orthogonal plane to Z with respect to $(\cdot\ ,\cdot)$, then $Z^\perp = \left(Z^{(\perp)}\right)^t$. On the other hand, transposition is obtained by a reflection through the plane $\mathfrak{s}$ of symmetric matrices. Therefore, $Z^\perp$ is the reflection through $\mathfrak{s}$ of the plane orthogonal to Z with respect to $(\cdot\ ,\cdot)$.

In [1] the authors give the following description of the trajectories; see Figure 1 and Figure 2. Let us denote by $\mathcal{C}^+_{int}, \mathcal{C}^+_{ext}$ the regions of $\mathbb{R}^3$ inside and outside the cone. We observe that the map det also characterizes these regions. In fact, $\det(W) = 0$ if $W \in \mathcal{C}^+$.

Proposition 2.2. *According to the location of Z with respect to the cone, the trajectories of $\dot{P} = [Z,P]$ in $\mathcal{C}^+$ are*

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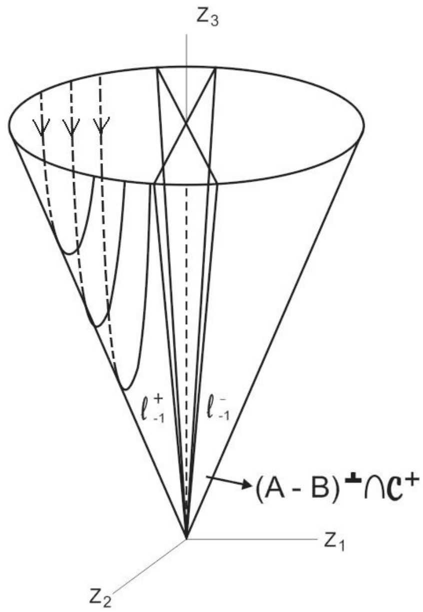

Fig. 1. *Points in the ray of $\mathcal{C}^+$ and parabolas on $\mathcal{C}^+$.*

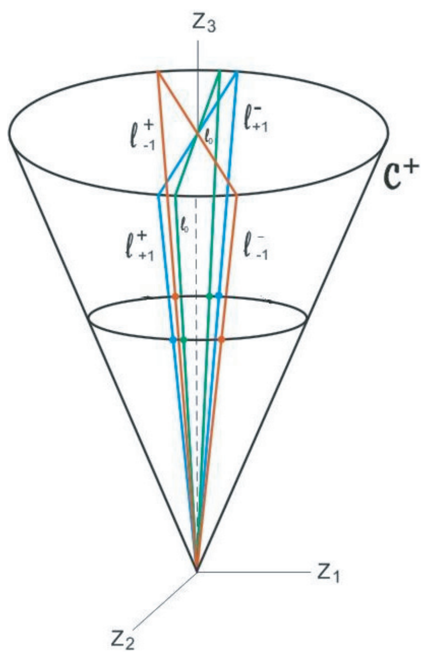

Fig. 2. *Intersection of orthogonal planes with $\mathcal{C}^+$.*

1. *Ellipses around $\mathcal{C}^+$, if $Z \in \mathcal{C}^+_{\text{int}}$.*
2. *Points in the ray of $\mathcal{C}^+$ orthogonal to Z or the parabolas $\{\langle Z, P\rangle = c\} \cap \mathcal{C}^+$, if $Z \in \mathcal{C}^+$.*
3. *The two rays in $Z^\perp \cap \mathcal{C}^+ \setminus \{0\}$ or the hyperbolas $\{\langle Z, P\rangle = c\} \cap \mathcal{C}^+$, if $Z \in \mathcal{C}^+_{\text{ext}}$.*

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3. The time-optimal problem on $\mathbb{P}^1$. In this section, we first establish the PMP for the angle system $\mathbb{P}\Sigma$ on $\mathbb{P}^1$ determined by a bilinear control system Σ in the plane. Next, we formulate the problem through the dynamics on the cone. We analyze the trajectories, in particular the singular trajectories and the time-optimal synthesis.

For our purposes it is enough to use the version of PMP stated and proved in [6]. Recall that the principle gives a first order necessary conditions for continuous-time optimal problems. Its geometric formulation takes place on the cotangent bundle of the state space.

We consider the following bilinear control system Σ in $\mathbb{R}^2$,

$$\dot{x}(t) = (A + uB)x(t), \quad t \in \mathbb{R}, x(t) \in \mathbb{R}^2,$$

where A, $B \in \mathfrak{sl}(2)$, $u \in \mathcal{U}$.

The *angle system* $\mathbb{P}\Sigma$ is defined by the projection of Σ onto the projective line

$$\mathbb{P}\Sigma : \qquad \dot{s}(t) = h(A, s(t)) + u(t)h(B, s(t)), \qquad s \in \mathbb{P}^1,$$

where $h(X, s) = (X - s^T X s I)s$, with I the identity matrix and $u \in \mathcal{U}$.

In this particular case, the principle reads:

If u_* is an optimal control and s_* is the associated trajectory of $\mathbb{P}\Sigma$ on $\mathbb{P}^1$, there exists a constant $\lambda_0 \geq 0$ and an absolutely continuous function $\lambda : [0, T] \longrightarrow T^*\mathbb{P}^1$ such that for almost every $t \in Dom(s_*)$ the adjoint equation,

$$\dot{\lambda} \ = - \ \lambda(A + u_* B),$$

is satisfied. Furthermore, the optimal control u_* minimizes the Hamiltonian

$$\mathcal{H} = \lambda(A + uB)s + \lambda_0$$

on $[-1, 1]$, through the curve $(\lambda(t), s_*(t))$.

The pair (u, s) is called an *extremal pair* if u is an admissible control and if the corresponding trajectory s satisfies the PMP conditions.

In our case the application of the PMP leads to two kinds of controls, the singular and bang-bang one. The difference depends on the fact that the associated solutions belong to the interior or to the boundary of a control set, respectively.

Denote by cl$M$ the closure of a set M and by $\mathbb{S}(x)$ the action of the semigroup of the angle system on the state $x \in \mathbb{P}^1$. We also denote by $\varphi(t, x, u)$ the solution of the system with initial condition x and control u. Of course the projected system may be controllable or not, see [8]. Since we concentrate the analysis in the uncontrollable case, we need the notion of the control set in particular (see, for instance, [5], [10], [11]).

DEFINITION 3.1. *A set $D \subset \mathbb{P}^1$ is called a* control set *of $\mathbb{P}\Sigma$ if*

(i) *for all $x \in D$ we have $D \subset \text{cl}\mathbb{S}(x)$,*

(ii) *for all $x \in D$ there exists a control $u \in \mathcal{U}$ with $\varphi(t, x, u) \in D$ for all $t \geq 0$, and*

(iii) *with respect to the set inclusion, D is maximal with the properties* (i) *and* (ii)*.*

When a control set has nonvoid interior it is called a main control set.

3.1. The problem. Let G be the Lie group SL(2) and consider as before its action on the real projective line $\mathbb{P}^1$. The bilinear control system (1) induces control systems on $\mathbb{P}^1$.

We assume that the bilinear control system (1) satisfies the Lie algebra rank condition; that is, $Span_{\mathcal{LA}} \{A, B\} = \mathfrak{sl}(2)$.

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The cotangent bundle $T^*\mathbb{P}^1$ is identified with the cone $\mathcal{C}$. Therefore, each curve $[0,T] \longrightarrow T^*_{zH}\mathbb{P}^1$ given by the PMP is identified with a curve in $\mathcal{C}^+$.

An admissible piecewise constant control u induces the vector field $X^u = A+uB \in \mathfrak{g}$. Through the previous identifications, the Hamiltonian function for the time optimal control problem takes the form

$$\mathcal{H}_{\widetilde{X}^u} : \mathcal{C}^+ \to \mathbb{R},$$

$$\mathcal{H}_{\widetilde{X}^u}(P) = \langle P, A\rangle + u\langle P, B\rangle + \lambda_0,\ P \in \mathcal{C}^+,$$

where $\widetilde{X}^u$ is the lifted vector field on $\mathcal{C}^+$ induced by X^u on $\mathbb{P}^1$ and $\lambda_0 \in \mathbb{R}$.

The Hamiltonian to be minimized is the map

$$\mathcal{H} : T^*\mathbb{P}^1 \to \mathbb{R},\quad \mathcal{H}(P) = \min_{v\,\in\,[-1,1]} \mathcal{H}(P).$$

By Proposition 2.1 and the form of the Hamiltonian $\mathcal{H}$, it follows that the equation for $\lambda(\cdot)$ can be rewritten as

$$\frac{dP(t)}{dt} = [A + uB,\ P(t)]\,,\ P(t) \in \mathcal{C}^+. \tag{4}$$

In fact, each $P \in \mathfrak{sl}(2)$ is associated with $\lambda \in \mathfrak{sl}(2)^*$ through the Cartan–Killing form $\mathcal{K}$ by $\lambda = \mathcal{K}(\cdot, P)$ and for any arbitrary V

$$\mathcal{K}\left(V, \frac{dP(t)}{dt}\right) = \frac{d\lambda}{dt}(V) = -\lambda(t)\,[V, A + uB] = \mathcal{K}\left([A + uB, P(t)]\,, V\right).$$

3.2. Characterization of the extremal controls. In order to obtain necessary conditions to determine the optimal control, we apply the PMP. An admissible control $u : [a,b] \longrightarrow [-1,1]$ is said to be bang-bang if $u(t) \in \{-1,1\}$ a.e. in $[a,b]$. Moreover, if $u(t) \in \{-1,1\}$ and $u(t)$ is constant for almost every $t \in [a,b]$, then u is called a bang control.

As pointed out by Sussmann, the bang-bang theorem for bilinear control systems does not need to be true as in the linear case. We recall Theorem 8.3.1 in [12].

Theorem 3.2. *Consider a real analytic system*

$$\dot{x}\ = f(x) + ug(x),\ x \in M,\ |u| \le 1$$

on a real analytical manifold M. Suppose that, for each $p \in M$ and each integer $m > 0$, there exists a neighborhood U of p such that, on U, $[g, (\mathrm{ad} f)^m(g)]$ is a linear combination

$$\Sigma_{i=0}^{m+1} \alpha_i (\mathrm{ad} f)^i(g)$$

with analytic coefficients α_i, such that $|\alpha_{m+1}| < 1$, for every $q \in U$. Then, the system has the bang-bang property with a bounded number of switchings (b-condition).

We use these results to narrow the class of possible optimal trajectories. According to Theorem 3.2, to apply the bang-bang theorem in our special situation it is sufficient condition to have

$$\mathrm{ad}_A(B) = Span\,\{B,\ \ [A,B]\,, [A,[A,B]]\} = \mathfrak{sl}(2)$$

and $|\alpha_2| < 1$.

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Let us denote by u_* the optimal control minimizing the Hamiltonian; then

$$\langle P, A\rangle + u_*\langle P, B\rangle + \lambda_0 = \min_{u(\cdot)\,\in\,[-1,1]}(\langle P, A\rangle + u\langle P, B\rangle + \lambda_0).$$

Equivalently,

$$u_*\langle P, B\rangle = \min_{-1\,\leq\,u\,\leq\,1} u\langle P, B\rangle.$$

If we write $\phi(P) = \langle P, B\rangle = \mathrm{tr}(PB)$, we get

$$\phi(P)u_*(t) = \min_{-1\,\leq\,u\,\leq\,1} \phi(P)u.$$

It follows that

$$u_*(t) = \begin{cases} 1 & \text{if} \quad \phi(P) > 0, \\ -1 & \text{if} \quad \phi(P) < 0. \end{cases}$$

Thus, the switching function ϕ determines regions on the cone where the optimal path needs to move.

3.3. Analysis of trajectories and singular controls. We observe that if $\phi(t) = \langle P(t), B\rangle = 0$, the control is not determined by the optimality condition of the PMP. On any interval where ϕ has no zeros, the corresponding control is bang, and if it has finitely many zeros, the control is bang-bang. We recall the following.

DEFINITION 3.3. *A control u is said to be singular on an open interval $I \subset [0, T]$, if it is computed from the solution of the resultant equation $\phi^{(r)} \equiv 0$ for some r, for which the control variable u appears explicitly in the derivative $\phi^{(r)}(t)$.*

If a control u is singular on I, then the corresponding function $\phi(t)$ and all its derivatives must vanish on I. Therefore, to find the corresponding control to a singular trajectory, one should compute the derivatives of $\phi(t) = \langle P(t), B\rangle$.

On the other hand, we need to analyze when the singular controls are optimal. In the following we use the term singular trajectory on an open interval I if the switching function ϕ vanishes.

DEFINITION 3.4. *Consider a single-input control system which is linear in the control. The order of a singular trajectory on an interval $I \subset [0, T]$ is the integer k such that $r = 2k$, $\phi^{(r)}(t) \equiv 0$ on I, and r is the smallest natural number for which the control variable u appears explicitly in $\phi^{(r)}(t)$.*

A k-order necessary condition for optimality of a singular trajectory of order k, is the generalized Legendre–Clebsch condition in [7], given by the following theorem.

THEOREM 3.5. *Assume that u is a singular control and $(\lambda(t), y(t), u(t))$, $t \in [0, T]$ is the associated extremal pair (singular lift). A k-order necessary condition for the optimality of the singular trajectory $y(\cdot)$ is*

$$(-1)^k \left[\frac{\partial}{\partial u}\frac{d^{2k}}{dt^{2k}}\frac{\partial \mathcal{H}}{\partial u}(\lambda(t), y(t), u(t))\right] \geq 0, \tag{5}$$

where $\mathcal{H}$ is the corresponding Hamiltonian.

Suppose that the control u is singular on a nonempty open interval I. Then $\phi(t)$ is constant on I and we get $\dot{\phi}(t) = -\langle P(t), [A, B]\rangle \equiv 0$. A further derivation yields $\ddot{\phi}(t) = \langle P(t), [A + uB, [A, B]]\rangle$.

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Since in $\ddot{\phi}(t) \equiv 0$ the variable of control appears explicitly, it follows that the singular control u is of order $k = 1$ on the interval I. Furthermore, we get

$$\langle P(t), [A, [A, B]]\rangle + u\langle P(t), [B, [A, B]]\rangle \equiv 0.$$

A direct computation implies that $\frac{\partial \mathcal{H}}{\partial u}$ is $\phi(t)$. So, the second-order necessary condition of Legendre–Clebsch in $\mathcal{C}^+$ reads

$$(-1)\frac{\partial}{\partial u}\frac{d^2}{dt^2}\frac{\partial \mathcal{H}}{\partial u} = -\ \langle P(t), [B, [A, B]]\rangle$$

evaluated along the extremal lift.

The singular control is of order 1 on the interval I. Hence, we can solve the control $u(t)$ in feedback form. If the corresponding control value is admissible, i.e., has a value in $(-1, 1)$, then the singular control is well defined. Otherwise the singular control is not admissible.

PROPOSITION 3.6. *The singular trajectories for the time-optimal problem on $\mathcal{C}^+$ is the locus where B and $[A, B]$ are linearly dependent.*

Proof. Consider the adjoint system

$$\dot{P}(t) = [A + u(t)B,\ P(t)]\,, u \in [-1, 1]\,.$$

The singular trajectories need to satisfy $\langle P(t), B\rangle \equiv 0$. Hence,

$$\langle P(t), [A, B]\rangle \equiv 0. \tag{6}$$

On the other hand, from the PMP, we get $\mathcal{H} = 0$; thus $\langle P(t), A\rangle = -1$. By differentiating (6) it follows that $u\ \langle P(t), [B, A]\rangle \equiv 0$. In particular, the singular trajectories are associated with any control $u \in (-1, 1)$.

Therefore, the singular trajectories live in the locus where B and $[A, B]$ are linearly dependent. ∎

Notice that the singular trajectories on the real projective line $\mathbb{P}^1$ are given by the projection of the intersection of $\mathcal{C}^+$ with the orthogonal plane to B with respect to the trace form. We use the Legendre–Clebsch condition to analyze the optimality of singular trajectories. Summarizing, we have the following.

PROPOSITION 3.7. *Let $(y(t), u_*(t))$ be an extremal pair. If the control $u_*(\cdot)$ is singular on an open interval, then it is of order* 1. *Furthermore, if the second-order of Legendre–Clebsch is satisfied, the singular control is given in feedback form as*

$$u_*(t) = -\frac{\langle P(t), [A, [A, B]]\rangle}{\langle P(t), [B, [A, B]]\rangle}.$$

The set of extremal trajectories can be described by a finite-dimensional reduction. More precisely, every extremal trajectory is a finite concatenation of trajectories that are either bang (corresponding to a constant control equal to ± 1) or singular (corresponding to the singular control).

4. Abnormal extremals.

Now we discuss the behavior of abnormal extremals. Recall that an extremal corresponding to $\lambda_0 = 0$ is said to be abnormal extremal; otherwise we call it a *normal extremal*.

The following propositions describe the switching behavior of abnormal extremals.

PROPOSITION 4.1. *An abnormal extremal cannot contain a singular trajectory.*

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Proof. From the maximality condition of the PMP, we get

$$0 = \mathcal{H}_u(P) = \langle P, A\rangle + u\langle P, B\rangle + \lambda_0,\ \ P \in \mathcal{C}^+.$$

Since $\lambda_0 = 0$, we have

$$\langle P, A\rangle = -u\langle P, B\rangle. \tag{7}$$

If the extremal trajectory is singular on an open interval I, then the switching function $\phi(t) = \langle P(t), B\rangle$ vanishes on I. From (7) we obtain $\langle P, A\rangle = 0$ on I. Since $\dot{\phi}(t) = \langle P(t), [A, B]\rangle \equiv 0$ on I and the matrices A, B and $[A, B]$ generate $\mathfrak{sl}(2)$ we get $P(t) \equiv 0$ on I. According to PMP this is a contradiction because the pair $(P(t), \lambda_0)$ should be nonzero. So, there are no singular trajectories which correspond to abnormal extremal trajectories. ∎

PROPOSITION 4.2. *Suppose that the pair $(P(s), s(t))$ is a regular abnormal extremal of the time-optimal problem on $\mathbb{P}^1$. Then, this pair corresponds to the lifting vector field associated with the Hamiltonian $\mathcal{H} = \langle P, A\rangle$, which is given by the differential equation $\dot{P} = [A, P]$.*

Proof. Let $(P(s), s(t))$ be a regular abnormal extremal. From the PMP we know that $\frac{\partial \mathcal{H}}{\partial u} = 0$ along this extremal; i.e., $\langle P(t), B\rangle = 0$. Hence, from the Hamiltonian equation

$$\mathcal{H}_u(P) = \langle P, A\rangle + u\langle P, B\rangle, P \in \mathcal{C}^+,$$

the proof follows. ∎

5. The time-optimal synthesis on $\mathbb{P}^1$. Let q_b and q_e be two arbitrary points on $\mathbb{P}^1$. Our goal is to find a time-optimal trajectory connecting q_b to q_e. We identify $\mathbb{P}^1$ with the clockwise oriented circle $\left\{e^{-i\theta} : -\pi/2 < \theta \leq \pi/2\right\}$. As is well known, if there exists a trajectory connecting q_b with q_e, then there exists a time-optimal trajectory connecting these two points. Our interest here is to show our construction with some examples. In particular, we look at optimal bang-bang paths, that is, optimal trajectories corresponding to bang-bang controls.

In the (z_1, z_2, z_3) coordinates, the equation of $\mathcal{C}^+$ with respect to the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{8}$$

of the Lie algebra $\mathfrak{sl}(2)$ is given by $z_1^2 + z_2^2 = z_3^2$. Since we restrict the analysis to the cone $\mathcal{C}^+$, the condition $z_3 < 0$ remains excluded.

From Corollary 5.2 in [4], we know that the LARC condition of (1) at the SL(2)-level is equivalent to $\det[A, B] \neq 0$, where $[A, B] = AB - BA$.

On the other hand, Theorem 5.2 in [1] gives a characterization of the controllability property of (1); namely it is controllable if and only if the segment

$$A + uB : -1 \leq u \leq 1 \text{ intersects } \mathcal{C}^+_{int}. \tag{9}$$

In the uncontrollable case there are exactly two control sets in $\mathbb{P}^1$, denoted by I^+ and I^- (see [1], [4], [10], and [11] for more general results). They satisfy the following properties:

(i) $I^- \cap I^+ = \emptyset$, I^- is closed, and I^+ is open.

(ii) I^- is invariant; i.e., $\mathbb{S}x \subset I^-$ for all $x \in \acute{I^-}$. On the other hand, I^+ is $\mathbb{S}^{-1}$-invariant.

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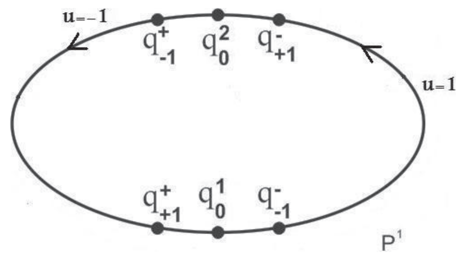

FIG. 3. *Dynamics on* $\mathbb{P}^1$.

(iii) If $g \in \mathbb{S}$ is diagonalizable, then its attractor belongs to I^- and its repellor to the closure of I^+.

Now we take a noncontrollable system with $A+B$, $A-B$ and $B \in \mathcal{C}^+_{ext}$. The trajectories of $A+B$ and $A-B$ are either rays in $Z^\perp \cap \mathcal{C}^+ \setminus \{0\}$ or the hyperbolas $\{\langle Z, P\rangle = c\} \cap \mathcal{C}^+$. We denote by l^+_{+1} and l^-_{+1} the rays associated to $A+B$ and by l^-_{-1} and l^+_{-1} those for $A-B$.

Recall that these rays correspond to eigenvectors of the matrices. We denote the projections of these rays into $\mathbb{P}^1$ by

- q^-_{+1} the attractor and q^+_{+1} the repellor of $A+B$.
- q^-_{-1} the attractor and q^+_{-1} the repellor of $A-B$

The intersection between $\mathcal{C}^+$ and the orthogonal plane to B are the two lines denoted for l^1_0 and l^2_0. We let q^1_0 and q^2_0 be the projections into $\mathbb{P}^1$ of l^1_0 and l^2_0, respectively. See Figure 3. Without loss of generality we assume that $q^-_{+1} < q^-_{-1}$. The other case $q^-_{-1} < q^-_{+1}$ is treated analogously.

In this case the control sets are

$$I^- = \left[q^-_{+1}, q^-_{-1}\right] \;\; I^+ = \left(q^+_{+1}, q^+_{-1}\right).$$

We write

$$R_1 = \left(q^-_{-1}, q^+_{+1}\right), \;\; R_2 = \left(q^+_{-1}, q^-_{+1}\right),$$

for their respective complementary subsets.

We assume that the plane π_B determined by $\langle P, B\rangle = 0$ gives rise to two regions on the cone and also does not intersect a control set. Without loss of generality, assume that I^- is inside of the region determined by $\langle P, B\rangle > 0$ (otherwise we take $-B$ instead of B). In order to construct optimal bang-bang path we apply $u = 1$ inside of $\langle P, B\rangle > 0$ and $u = -1$ when $\langle P, B\rangle < 0$. In terms of the regions I^-, R_1, I^+ and R_2 we have the following synthesis:

(i) Restricted to I^- the system is controllable. According to the dynamics on the circle (and in the cone), there exits an optimal bang-bang path connecting q_b, $q_e \in I^-$ when $q_e < q_b$. However, this is not the case in the other direction. In fact, if $q_b < q_e$, then, starting on q_b under the influence of $u = 1$ the trajectory converges to the attractor q^-_{+1}. So, an optimal curve associated with a bang control from q_b will never reach q_e.

(ii) If $q_b \in R_1$, there exists an optimal bang-bang path connecting q_b with $q_e \in R_1$ when $q_e < q_b$. However, when $q_b < q_e$, starting on $q_b \in R_1$ the trajectory converges to the attractor q^-_{+1}. Hence, when $q_b < q_e$ an optimal curve associated with a bang control from q_b will never reach q_e.

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(iii) Let $q_b \in I^+$. Then the orbit $\mathbb{S}(q)$ is $\mathbb{P}^1 \setminus \{q^+_{-1}\}$. Starting on q_b, there exists an optimal bang-bang path connecting any point in the interval $[q^-_{+1}, q_b)$. However, this is not the case when the ending point belongs to the interval (q_b, q^-_{+1}).

(iv) Let $q_b \in R_2$. Then $\mathbb{S}(q_b) = (q_b, q^-_{-1})$. There exists an optimal bang-bang path connecting q_b with any point $q_e \in (q_b, q^-_{+1})$.

As a concrete example consider Σ with the basis vectors $A = S$ and $B = H$; see (8). The system Σ satisfies LARC but is uncontrollable. The plane $B^\perp$ splits $\mathcal{C}^+$ in two regions

$$\mathcal{C}^+_+ = \mathit{cone} \cap \{z_1 > 0\}\,,\ \mathcal{C}^+_- = \mathit{cone} \cap \{z_1 < 0\}\,.$$

Furthermore, $q_0^1 \in \mathcal{C}^+_+$ and $q_0^2 \in \mathcal{C}^+_-$. In coordinates, the differential equation $\dot{Z} = [A+B, Z]$, $Z \in \mathcal{C}^+$, reads $\dot{z}_1 = 2z_3,\ \dot{z}_2 = -2z_3,\ \dot{z}_3 = 2z_1 - 2z_2$.

So, the direction of the trajectories in $\mathcal{C}^+_+$ goes from $\mathbb{R}^3_{(--+)}$ towards $\mathbb{R}^3_{(-++)}$. (Where the subscript with signals in $\mathbb{R}^3$ refers to the orthants determined by the basis (8).) Analogously, under $u \equiv -1$, the direction of the dynamics on $\mathcal{C}^+_-$ goes from $\mathbb{R}^3_{(--+)}$ to $\mathbb{R}^3_{(-++)}$. Finally, the control sets are given by

$$I^- = \left[\frac{(-1-\sqrt{2},-1)}{\left\|(-1-\sqrt{2},-1)\right\|}, \frac{(1-\sqrt{2},-1)}{\left\|(1-\sqrt{2},-1)\right\|}\right]$$

$$I^+ = \left(\frac{(-1+\sqrt{2},-1)}{\left\|(-1+\sqrt{2},-1)\right\|}, \frac{(1+\sqrt{2},-1)}{\left\|(1+\sqrt{2},-1)\right\|}\right).$$

Despite the fact that $q_b = \frac{1}{2}\frac{(-1-\sqrt{2},-1)}{\left\|(-1-\sqrt{2},-1)\right\|}$ reach, for instance, the middle point $q_e \in \mathit{int}(I^-)$, there is not an optimal bang-bang path connecting q_b with q_e.

The singular extremals satisfy

$$\left\{(z_1, z_2, z_3) \in \mathbb{R}^3 :\ (z_2 - z_3)(z_2 + z_3) = 0\right\}$$

Thus, the only singular trajectories correspond to the lines l_0^1 and l_0^2 on $\mathcal{C}^+$. Since

$$[B,[A,B]] = -4A,\ [A,[A,B]] = -4B,\ \text{and}\ \langle P(t), A\rangle = -1,$$

there exists a singular control u. Since $u = 0$, the singular trajectories on $\mathcal{C}^+$ are the integral curves of $\dot{Z} = [A, Z]$, $Z \in \mathcal{C}^+$ which are projected on $\mathbb{P}^1$ onto the points q_0^1 and q_0^2. However, the admissible singular control u is not optimal. In fact, in this case $\frac{\partial}{\partial u}\frac{d^2}{dt^2}\frac{\partial \mathcal{H}}{\partial u} > 0$, violating the Legendre-Clebsch condition.

Finally we give an exemple with a controllable system.

Example. Let Σ be determined by $A = R$ and $B = H + 2R$, see (8). Since, $A \in \mathit{int}\mathcal{C}^+$ the system is controllable, see (9). The vector $[B,[A,B]]$ belongs to $\mathrm{ad}_A(B)$, but $\alpha_3 = \frac{3}{2} > 1$. So, Σ does not satisfy the conditions of Theorem 3.2. However, $\mathbb{P}\Sigma$ has the bang-bang property. In fact, $< P, B > = 0$ divided the entire cone in $\mathcal{C}^+$ and $\mathcal{C}^-$. According to PMP we must use $u = 1$ on $\mathcal{C}^+$. Since the trajectories of $A + B$ on $\mathcal{C}^+$ are ellipses, the system has the bang-bang property.

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