Controllability Properties of a Class of Control Systems on Lie Groups

Victor Ayala[1]* and Luiz A. B. San Martin[2]**

[1] Departamento de Matemática Universidad Católica del Norte
Casilla 1280
Antofagasta, Chile
vayala@socompa.ucn.cl

[2] Instituto de Matemática
Universidade Estadual de Campinas
Cx. Postal 6065
13081-970 Campinas SP, Brasil

Abstract. Linear control systems on Lie groups were introduced by Markus [3] and also studied by Ayala and Tirao in [1]. For this class of control systems we establish controllability results in the compact case and also in the semi-simple non-compact case. Wealso show that the forward orbit from the identity is not in general a semigroup.

1 Introduction

In this paper we study controllability properties of linear control systems on Lie groups. This class of control systems was introduced by Markus [3] and also studied by Ayala and Tirao [1]. A linear control system on a connected Lie group G is determined by the family of differential equations on G, parametrized by the class U of the piecewise constant addmisible control:

$$\dot{x}(t) = X\left(x(t)\right) + \sum_{i=1}^{m} u_i(t), Y_i\left(x(t)\right) \tag{1}$$

Here $x(t) \in G$, and X stand for an infinitesimal automorphism of G, i.e., the flow $(X_t)_{t\in\mathbb{R}}$ is a 1-parameter group of $\mathrm{Aut}(G)$, the group of autmorphism of G. The control vectors Y_i are elements of the Lie algebra $\mathfrak{g}$ of G, considered as right invariant vector fields on G. First of all, we discuss the group of diffeomorphism generated by the system. We denote by $\mathrm{Af}\,(G)$ the *affine group* of G i.,e., the semi-direct product $\mathrm{Af}\,(G) = \mathrm{Aut}\,(G) \times_s G$. If G is simple connected then it is well known that the Lie algebra $aut\,(G)$ of $\mathrm{Aut}\,(G)$ is $\mathrm{Der}\,(\mathfrak{g})$. On the other hand, we show that for a connected Lie group G, $aut\,(G)$ is a

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subalgebra of Der $(\mathfrak{g})$. We denote by $\mathfrak{g}_\Sigma$ the subalgebra of $\mathfrak{a}f\ (G)$ generated by the vector fields of the system, and by G_Σ the connected subgroup of Af (G) whose Lie algebra is $\mathfrak{g}_\Sigma$. Of course G_Σ is exactly the group of diffeomorphisms generated by the control system, that is the systems group. The vector fields of the system are then right invariant vector fields in G_Σ. The original system lifts to a right invariant control system on G_Σ, whose forward orbit from the identity is a semigroup which we denote by S_Σ. And the (1) system itself is induced on G by the invariant system on G_Σ. We assume that the system satisfies the Lie algebra rank condition. This amounts to suppose that G has codimension one in the systems group G_Σ. By construction G identifies with a homogeneous space of G_Σ, and the system (1) is controllable if and only if S_Σ is transitive in this homogenous space. On the other hand, If G is a semi-simple non-compact Lie group denote by $\pi : G_\Sigma \to G$ the canonical homomorphism onto G_Σ/Z, induced by the decomposition of the Lie algebra $\mathfrak{g}_\Sigma = \mathfrak{z} \oplus \mathfrak{g}$, where $\mathfrak{z}$ is the one-dimensional center of $\mathfrak{g}_\Sigma$ and $Z = \exp(\mathfrak{z})$. Then, the invariant system on G_Σ defines through π an induced invariant control system on G. In [1] the authors extend the well known Kalman rank condition for a linear control systems on $\mathbb{R}^n$. In fact, they proved that the Lie algebra rank condition characterize controllability for abelian Lie groups. For a connected Lie group G they also prove local controllability from the identity element of G if the dimension of the subspace

$$V = \text{Span}\left\{Y_i, ad^{(j)}(X)(Y_i) : i = 1, 2, ..., m, j \geq 1\right\} \subset \mathfrak{g}$$

is the dimension of G. We consider the global controllability property for a linear control system on a connected Lie group. According to the following cases, we prove:

1. In a compact and connected Lie group G, the linear system (1) is controllable if and only if it satisfies the Lie algebra rank condition.
2. The induced invariant system on G is controllable if the linear system (1) is controllable from the identity.

To get the last result we use the notion of left reversible semigroup and the following theorem proved in [5]: If L is semi-simple noncompact group and T is a semigroup with nonvoid interior then T is not left reversible unless $T = L$. This paper is organized as follows: In Section 2 we study the group of diffeomorphisms generated by the linear control system 1. Section 3 contains the main results about controllability: the compact case and the semisimple non-compact case. Section 4 contains one example showing a system which is locally controllable from the identity but not controllable from the same point on Sl $(2, \mathbb{R})$. In particular, this example shows that the forward orbit of (1) from the identity is not in general a semigroup.