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Controllability properties of bilinear systems in dimension 2

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Abstract

This paper discusses the controllability of bilinear control systems by considering the spectrum of the system and controllability of the projection onto the projective space. Necessary and sufficient conditions are presented for two dimensional systems with bounded and unbounded control range.

Keywords: Bilinear control systems, controllability, spectrum, projected linear system.
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1. Introduction

Bilinear control systems provide a large class of systems with a wide range of applications, including nonlinear systems linearized at a fixed point with respect to the state variable. Controllability is a key property of any control system, describing the sets within which it is possible to steer the system between any two points using an appropriate control function. Controllability of bilinear systems has received a substantial amount of interest over the last 30 years or so, but general characterizations of complete controllability for this class of systems are still not available. Most of the approaches to this problem are algebraic in nature, compare, e.g. [3].

In [2] the authors presented an analytic approach to the study of control systems that can be applied to bilinear systems. In this paper we use some of their setup to obtain new characterizations of controllability for bilinear systems in dimension 2 with bounded and unbounded control range.

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The contents is as follows: in Section 2, we introduce bilinear control systems together with their projection onto the projective space and their spectrum. We also review a key result from [2] about controllability of bilinear systems with bounded control range. In Section 3 we characterize controllability of the projected system on the projective space $\mathbb{P}^1$ for bounded and unbounded one-dimensional control range. In Section 4 we obtain a second characterization of controllability for these systems in terms of the associated Lie brackets. The case of multidimensional control ranges is discussed in Section 5, and Section 6 is dedicated to a discussion of spectral properties of bilinear control systems in dimension 2. Together, the results from these sections deliver insights regarding the controllability of bilinear systems in $\mathbb{R}^2$.

2. Bilinear control systems

For the following definitions and results we refer to [2, Chapters 7 and 12]. We consider a bilinear control system in $\mathbb{R}^d$ given by a family of differential equations

$$\Sigma: \qquad \dot{x}(t) = \left(A + \sum_{i=1}^{m} u_i(t) B_i\right) x(t) = A(u)x(t), \qquad t \in \mathbb{R}, \qquad x(t) \in \mathbb{R}^d, \tag{2.1}$$

where $A, B_1, \ldots, B_m \in gl(d, \mathbb{R})$ are real $d \times d$ matrices and

$$u \in \mathcal{U} = \{u : \mathbb{R} \to U \subset \mathbb{R}^m,\ u \text{ is locally integrable}\}$$

is the set of the admissible controls. The solutions of (2.1) are denoted by $\varphi(t, x, u)$ for the initial value $\varphi(0, x, u) = x \in \mathbb{R}^d$.

The system Σ has the following associated systems:

a) The *angle system* $\mathbb{P}\Sigma$ is defined by the projection of Σ onto the projective space $\mathbb{P}^{d-1}$,

$$\mathbb{P}\Sigma: \qquad \dot{s}(t) = h(A, s(t)) + \sum_{i=1}^{m} u_i(t) h(B_i, s(t)), \qquad s \in \mathbb{P}^{d-1}, \tag{2.2}$$

where $h(A, s) = (A - s^T A s I)s$, with I the identity matrix and $u \in \mathcal{U}$.

b) The *radial system* is defined on $\mathbb{R}^+$ by

$$r(t) = ||\varphi(t, x, u)||,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

The solutions of the projected system (2.2) are denoted by $\mathbb{P}\varphi(t, s, u)$ for the initial value $\mathbb{P}\varphi(0, s, u) = s \in \mathbb{P}^{d-1}$.

For an arbitrary control system on the state space M we recall the definition of the (positive and negative) orbit of a point: $\mathcal{O}^+(x) = \{y \in M$, there exists $u \in \mathcal{U}$ and $t > 0$ with $\varphi(t, x, u) = y\}$ and $\mathcal{O}^-(x) = \{y \in M$, there exists $u \in \mathcal{U}$ and $t > 0$ with $\varphi(t, y, u) = x\}$. A control system is called (completely) controllable on M, if for every point $x \in M$ we have $\mathcal{O}^+(x) = \mathcal{O}^-(x) = M$. A control system is said to be accessible from $x \in M$ if $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ have nonvoid interior in M. It is locally accessible from $x \in M$ if the orbits up to time T have nonvoid interior for all $T > 0$. If these properties hold for all $x \in M$, the system is called accessible, and locally accessible, respectively.

The Lie algebra of (2.1) is given by $\mathcal{LA}\{A+\sum_{i=1}^{m} u_i B_i,\, u \in U\} \subset gl(d,\mathbb{R})$. In general, for each $x \in \mathbb{R}^d$, we have

$$\mathcal{LA}\{A+\sum_{i=1}^{m} u_i B_i,\, u \in U\}(x) \subset T_x\mathbb{R}^d,$$

the tangent space of $\mathbb{R}^d$ at $x \in \mathbb{R}^d$. We recall the following definition.

Definition 2.1. The bilinear control system (2.1) satisfies the Lie algebra rank condition at $x \in \mathbb{R}^d$ if

$$\dim(\mathcal{LA}\{A+\sum_{i=1}^{m} u_i B_i,\, u \in U\}(x)) = \dim \mathbb{R}^d = d. \tag{2.3}$$

If (2.3) holds for all $x \in \mathbb{R}^d \backslash \{0\}$, we say that the bilinear system satisfies the Lie algebra rank condition (LARC).

Note that bilinear control systems, as well as the induced angle systems on the projective space are real analytic systems. It is well-known that for these systems the Lie algebra rank condition, accessibility, and local accessibility are equivalent, see e.g. [4]. Hence the Lie algebra rank condition is necessary for controllability of a bilinear system. Note that if (2.1) satisfies (2.3) for all $x \in \mathbb{R}^d \setminus \{0\}$, then the projected system (2.2) satisfies a corresponding Lie algebra rank condition for all $s \in \mathbb{P}^{d-1}$. But, of course, the converse does not necessarily hold true: just consider a system with skew symmetric matrices $A, B_{1,\ldots}, B_m$.

Throughout this paper we will assume that the bilinear control system in $\mathbb{R}^d$ satisfies the Lie algebra rank condition (2.3).

For a matrix $A \in gl(d,\mathbb{R})$ we denote its real Jordan form by $\mathcal{J}(A)$. Note for all $A \in gl(d,\mathbb{R})$ there exists an invertible matrix $P \in Gl(d,\mathbb{R})$ such that $\mathcal{J}(A) = P^{-1}AP$. The following lemma shows the invariance of controllability, the Lie algebra rank condition, and the eigenvalues of a bilinear system under the real Jordan transformation.

Lemma 2.2. *Consider the bilinear control system* (2.1) *and let $\mathcal{J}(A) = P^{-1}AP$ be the real Jordan form of A. Let*

$$\dot{y}(t) = \left(\mathcal{J}(A) + \sum_{i=1}^{m} u_i(t) P^{-1} B_i P\right) y(t) \tag{2.4}$$

be the transformed system under $\mathcal{J}$. Then the following facts hold:

1. *If $\varphi(t,x,u(\cdot))$ is a trajectory of* (2.1) *with initial value $\varphi(0,x,u(\cdot)) = x$, then $\psi(t,y,u(\cdot)) = P^{-1}\varphi(t,x,u(\cdot))$ is the trajectory of* (2.4) *for the initial value $y = P^{-1}x$.*
2. *$\mathcal{J}$ preserves controllability of the system in $\mathbb{R}^d \setminus \{0\}$ and of the projected system on $\mathbb{P}^{d-1}$.*
3. *$\mathcal{J}$ preserves the Lie algebra rank condition of the system in $\mathbb{R}^d \setminus \{0\}$.*
4. *$\mathcal{J}$ preserves the eigenvalues of the system for constant controls $u(t) \equiv u \in U$.*

Proof.

1. This follows from linearity of the system $\dot{x} = A(u)x$.
2. Suppose that bilinear control system (2.1) is controllable in $\mathbb{R}^d \backslash \{0\}$. Let $y_1, y_2 \in \mathbb{R}^d \backslash \{0\}$, then there exist $x_1, x_2 \in \mathbb{R}^d \backslash \{0\}$, such that $y_1 = P^{-1}x_1$ and $y_2 = P^{-1}x_2$. If (2.1) is controllable, there exist $t > 0$ and $u \in \mathcal{U}$ such that $\varphi(t,u(t),x_1) = x_2$. Applying the transformation $\mathcal{J}$ and using linearity of the r.h.s. of (2.1) we obtain $y_2 = P^{-1}x_2 = P^{-1}\varphi(t,u(t),x_1) = \varphi(t,u(t),P^{-1}x_1) = \varphi(t,u(t),y_1)$. Therefore (2.4) is controllable. Using the fact that $\mathbb{P}(P^{-1}(\mathbb{P}x)) = \mathbb{P}(P^{-1}x)$ for all $x \in \mathbb{R}^d \setminus \{0\}$, this same argument shows that $\mathcal{J}$ preserves controllability of the projected system on $\mathbb{P}^{d-1}$.

3. Assume that the Lie algebra rank condition (2.3) holds for the system (2.1). Then we have for all $y \in \mathbb{R}^d \setminus \{0\}$

$$\begin{aligned}
&\dim\left(\mathcal{LA}\left\{\mathcal{J}(A)+\sum_{i=1}^{m}u_iP^{-1}B_iP,\ u\in U\right\}(y)\right)\\
&=\dim\left(\mathcal{LA}\left\{P^{-1}AP+\sum_{i=1}^{m}u_iP^{-1}B_iP,\ u\in U\right\}(y)\right)\\
&=\dim\left(P^{-1}\mathcal{LA}\left\{A+\sum_{i=1}^{m}u_iB_i,\ u\in U\right\}P(y)\right)\\
&=\dim\left(P^{-1}\mathcal{LA}\left\{A+\sum_{i=1}^{m}u_iB_i,\ u\in U\right\}(Py)\right)=d.
\end{aligned}$$

4. This part is standard.

□

Different spectral concepts have been defined for bilinear control systems to characterize the behavior of the radial component, and hence stability and stabilizability. These include the Floquet, the Lyapunov, and the Morse spectrum. As it turns out, the spectra also serve for a characterization of controllability of bilinear systems in $\mathbb{R}^d$, in combination with controllability of the angle system. We will need the following spectral concepts.

The spectrum Spec(C) of a constant matrix $C \in gl(d,\mathbb{R})$ is defined as the set of eigenvalues of C. The distinct (complex) eigenvalues of C will be denoted by $\mu_1,\dots,\mu_r$. The real version of the generalized eigenspace is denoted by $E(C,\mu_k) \subset \mathbb{R}^d$ or simply E_k for $k = 1,\dots,r \le d$. We order the distinct real parts of the eigenvalues as $\lambda_1 < \cdots < \lambda_l$, $1 \le l \le r \le d$, and define the Lyapunov space of λ_j as $L(\lambda_j) = \oplus E_k$, where the direct sum is taken over all (generalized) real eigenspaces associated to eigenvalues with real part equal to λ_j. Note that

$$\oplus_{j=1}^{l} L(\lambda_j) = \mathbb{R}^d .$$

Let $\varphi(\cdot,x_0)$ be a solution of the linear differential equation $\dot{x} = Cx$ with $\varphi(0,x_0) = x_0$. Its Lyapunov exponent for $x_0 \neq 0$ is defined as

$$\lambda(x_0) = \limsup_{t\to\infty}\frac{1}{t}\log\|\varphi(t,x_0)\|,$$

where log denotes the natural logarithm and $\|\cdot\|$ is any norm in $\mathbb{R}^d$. The Lyapunov exponents are determined on the Lyapunov space $L(\lambda_j)$, $j = 1,...,l$ of C in the following way: we have that the Lyapunov exponent $\lambda(x)$ of a solution $\varphi(\cdot,x)$ (with $x \neq 0$) satisfies $\lambda(x) = \lim_{t\to\pm\infty}\frac{1}{t}\log\|\varphi(t,x_0)\| = \lambda_j$ if and only if $x \in L(\lambda_j)$. Hence, associated to a matrix $C \in gl(d,\mathbb{R})$ there are exactly l Lyapunov exponents, which correspond to the different real parts of the eigenvalues of C.

The Lyapunov exponents of a linear differential equation $\dot{x} = Cx$ can also be recovered from the projected system on $\mathbb{P}^{d-1}$, which is given by $\dot{s} = h(C,s) = (C - q(C,s)\cdot I)s$ with $q(C,s) = s^TCs$, compare (2.2). The Lyapunov exponents satisfy $\lambda(x) = \limsup_{t\to\infty}\frac{1}{t}\int_0^t q(C,\mathbb{P}\varphi(\tau,x))d\tau$.

For bilinear control systems we need the following concepts.

For a solution $\varphi(t,x,u)$ of (2.1) with $x \neq 0$ and $u \in \mathcal{U}$ the Lyapunov exponent is defined as

$$\lambda(u,x) = \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi(t,x,u(\cdot))\| .$$

Note that $\lambda(u,x) = \limsup_{t\to\infty} \frac{1}{t}\int_0^t q(A(u(\tau)), \mathbb{P}\varphi(\tau,x,u(\cdot)))d\tau$. For the bilinear control system (2.1) its Lyapunov spectrum consists of all Lyapunov exponents, i.e.,

$$\Sigma_{Ly} = \{\lambda(u,x),\, (u,x) \in \mathcal{U} \times \mathbb{P}^{d-1}\}.$$

A subset of the Lyapunov spectrum is the Floquet spectrum: let $u \in \mathcal{U}_{pc,T} := \{u : [0,T] \to U,$ piecewise constant$\}$. For a point $x \in \mathbb{P}^{d-1}$ and $u \in \mathcal{U}_{pc,T}$ for some $T \geq 0$ let the solution $\mathbb{P}\varphi(\cdot,x,u)$ of (2.2) be T-periodic. Then the Floquet exponent of $\varphi(\cdot,x,u)$ is the Lyapunov exponent $\lambda(u,x)$. For the bilinear control system (2.1) the Floquet spectrum Σ_{Fl} consists of all its Floquet exponents.

For the following control analysis of bilinear systems in $\mathbb{R}^d$, certain extremal Lyapunov exponents play a crucial role. Extremal Lyapunov exponents are those exponential growth rates that are defined globally as suprema and/or infima over the initial values and the control. We need the following quantities for the bilinear system (2.1):

$$\kappa = \sup_{u\in\mathcal{U}} \sup_{x\neq 0} \lambda(u,x), \qquad \kappa^* = \inf_{u\in\mathcal{U}} \inf_{x\neq 0} \lambda(u,x).$$

Note that in case of a compact control range U we have $-\infty < \kappa^* \leq \kappa < \infty$.

The following result clarifies the relation between the different spectral concepts in case the projected system (2.2) is completely controllable:

Theorem 2.3. *Consider the bilinear control system* (2.1) *and its projected system* (2.2) *satisfying* (2.3). *We assume that the control range $U \subset \mathbb{R}^m$ is compact and that projected system* (2.2) *is completely controllable on $\mathbb{P}^{d-1}$. Then the spectra satisfies*

$$[\kappa^*, \kappa] = \mathrm{Cl}(\Sigma_{Fl}) = \Sigma_{Ly}.$$

This theorem is part of Corollary 7.3.23 in [2].

The following result shows that controllability of the bilinear system (2.1) can be characterized via the entire spectral interval $[\kappa^*, \kappa]$.

Theorem 2.4. *Consider the bilinear control system* (2.1) *and its projected system* (2.2) *satisfying* (2.3). *We assume that the control range $U \subset \mathbb{R}^m$ is compact. Then the following statements are equivalent:*

1. *The bilinear system* (2.1) *is completely controllable in $\mathbb{R}^d \backslash \{0\}$.*
2. (a) *The projected system $\mathbb{P}\Sigma$* (2.2) *is completely controllable on $\mathbb{P}^{d-1}$, and*
   (b) $0 \in int[\kappa^*, \kappa]$*, where $int(A)$ denotes the interior of a set A.*

This theorem is a special case of Corollary 12.2.6 in [2].

In the spirit of Theorem 2.4 we analyze in the following sections the controllability of the projected system (2.2) and the spectrum of a bilinear control system in $\mathbb{R}^2$.

3. Controllability of bilinear systems on the projective space $\mathbb{P}^1$

We consider the bilinear control system (2.1) on $\mathbb{R}^2$, given by:

$$\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \left(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \sum_{i=1}^{m} u_i(t) \begin{pmatrix} b_1^{(i)} & b_2^{(i)} \\ b_3^{(i)} & b_4^{(i)} \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{3.1}$$

and we will use the notation

$$n = a_1 + \sum_{i=1}^{m} u_i(t) b_1^{(i)}, \quad m = a_2 + \sum_{i=1}^{m} u_i(t) b_2^{(i)},$$
$$p = a_3 + \sum_{i=1}^{m} u_i(t) b_3^{(i)}, \quad q = a_4 + \sum_{i=1}^{m} u_i(t) b_4^{(i)}.$$

Thus the bilinear control system is,

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = Q(t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad Q = \begin{pmatrix} n & m \\ p & q \end{pmatrix}.$$

Projecting (3.1) onto the projective space $\mathbb{P}^1$ results in the angle system (written in coordinates of $\mathbb{P}^1 \subset \mathbb{S}^1 \subset \mathbb{R}^2$)

$$\dot{\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}} = \begin{pmatrix} n(1-s_1^2) - (m+p)s_1 s_2 - q s_2^2 & m \\ p & -n s_1^2 - (m+p) s_1 s_2 + q(1-s_2^2) \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}. \tag{3.2}$$

The following discussion provides motivation for our results: assume that $Q(t)$ is constant. Then the projected system is controllable on $\mathbb{P}^1$ iff Q is skew-symmetric, i.e., it is of the form $Q = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for some $b \neq 0$, iff the solutions of (3.1) are rotations. In this section we will show that for a bilinear control system on $\mathbb{P}^1$ with one-dimensional control range satisfying the Lie algebra rank condition (2.3) in $\mathbb{R}^2$, this sufficient condition is also necessary.

We consider the following type of systems having one control input:

$$\Sigma_u^2: \qquad \dot{x} = (A + uB)x, \qquad x \in \mathbb{R}^2,\ A, B \in gl(2, \mathbb{R}),\ u(t) \in U \subset \mathbb{R}. \tag{3.3}$$

Definition 3.1. For the bilinear system (3.3) the associated discriminant polynomial (for constant $u \in U$) is defined as $y_{[A+uB]}(u) = \alpha u^2 + \beta u + \gamma$, $\alpha = (tr(B))^2 - 4\det(B)$, $\beta = 2tr(AB) - tr(A)tr(B)$ and $\gamma = (tr(A))^2 - 4\det(A)$, where $tr(A)$ denotes the trace of a matrix A. We will often use $y(u)$ for the polynomial $y_{[A+uB]}(u)$.

The following lemma establishes that the discriminant polynomial is invariant under a change of basis.

Lemma 3.2. *We consider the set of matrices $\{A + uB,\ u \in U\}$ and the real Jordan transformation of A in the form $\mathcal{J} : gl(d, \mathbb{R}) \to gl(d, \mathbb{R})$, $\mathcal{J}(C) = P^{-1}CP$. Then*

$$y_{[A+uB]}(u) = y_{[\mathcal{J}(A)+P^{-1}BP]}(u).$$

Proof. The discriminant polynomial of $\mathcal{J}(A + uB)$ is

$$y_{[\mathcal{J}(A)+P^{-1}BP]}(u) = \alpha_1 u^2 + \beta_1 u + \gamma_1,$$

where $\alpha_1 = (\mathrm{tr}(P^{-1}BP))^2 - 4\det(P^{-1}BP)$, $\beta_1 = 2\mathrm{tr}(\mathcal{J}(A)P^{-1}BP) - \mathrm{tr}(\mathcal{J}(A))\mathrm{tr}(P^{-1}BP)$, and $\gamma_1 = (\mathrm{tr}(\mathcal{J}(A)))^2 - 4\det(\mathcal{J}(A))$. With $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ and $\det(XY) = \det(X)\det(Y)$, we obtain $\alpha_1 = \alpha$, $\beta_1 = \beta$, and $\gamma_1 = \gamma$, where α, β, γ are the coefficients of $y_{[A+uB]}(u)$ from Definition 3.1. Hence, $y_{[A+uB]}(u) = y_{[\mathcal{J}(A+uB)]}(u)$. □

The following theorem will be established in a series of lemmas.

Theorem 3.3. *Consider the bilinear control system* (3.3) *with* $U = \mathbb{R}$ *and assume that the Lie algebra rank condition* (2.3) *holds for the system* Σ_u^2 *on* $\mathbb{R}^2$. *Then* Σ_u^2 *is controllable on* $\mathbb{P}^1$ *if and only if there exists a constant control* $u \in \mathbb{R}$ *such that the matrix* $A + uB$ *has a complex eigenvalue, i.e., iff* $\inf\{(a_4 + ub_4 - (a_1 + ub_1))^2 + 4(a_2 + ub_2)(a_3 + ub_3),\ u \in U\} < 0$.

Lemma 3.4. *The matrix* $A + uB$ *has a complex eigenvalue iff* $\inf\{(a_4 + ub_4 - (a_1 + ub_1))^2 + 4(a_2 + ub_2)(a_3 + ub_3),\ u \in U\} < 0$.

Proof. The discriminant of the characteristic polynomial of the matrix $A + uB$ is the polynomial $y(u) = \alpha u^2 + \beta u + \gamma$ from Definition 3.1, with critical point $u_0 = -\frac{\beta}{\alpha}$, and $y(u_0) = \gamma - \frac{\beta^2}{\alpha}$. This proves the claim. □

Note that the short discussion above after equation (3.2) establishes that the system Σ_u^2 is controllable on $\mathbb{P}^1$ if there exists a control $u \in U \subset \mathbb{R}$ such that the matrix $A + uB$ has a complex eigenvalue. To see the converse, we now examine the different cases characterizing the coefficients of $y(u) = \alpha u^2 + \beta u + \gamma$ for unbounded control range $U = \mathbb{R}$.

Case 1: $\alpha < 0$

In this case $\inf\{y(u),\ u \in \mathbb{R}\} = -\infty$, B has a pair of complex eigenvalues, and the system Σ_u^2 is controllable on $\mathbb{P}^1$.

Case 2: $\alpha = 0$ and $\beta \neq 0$

In this case $\inf\{y(u),\ u \in \mathbb{R}\} = -\infty$, B has a pair of equal real eigenvalues, and the system Σ_u^2 is controllable on $\mathbb{P}^1$.

Case 3: $\alpha = 0$ and $\beta = 0$

In this case $y(u) \equiv \gamma$ with three subcases:

Case 3a: $\alpha = 0$, $\beta = 0$, and $\gamma < 0$

In this case $\inf\{y(u),\ u \in \mathbb{R}\} = \gamma < 0$, B has a pair of equal real eigenvalues and A has a complex pair of eigenvalues, and the system Σ_u^2 is controllable on $\mathbb{P}^1$.

Case 3b: $\alpha = 0$, $\beta = 0$ and $\gamma = 0$

In this case the system Σ_u^2 does not satisfy the Lie Algebra rank condition (2.3) in $\mathbb{R}^2$: note that $\gamma = 0$ implies that A has a pair of equal real eigenvalues and according to Lemma 2.2 we may assume that A is in real Jordan form, i.e., $a_3 = 0$. Assume first that $a_2 \neq 0$, then the condition $\beta = 0$ forces $b_3 = 0$, so $\alpha = 0$ implies $b_1 = b_4$.

That is:

$$A = \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix}.$$

Observe that the Lie bracket satisfies $[A, B] = 0$. Thus $\mathcal{LA}\{A + uB,\ u \in \mathbb{R}\} = span\{A, B\}$ and

$$(A, B)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1x_1 + a_2x_2 & b_1x_1 + b_2x_2 \\ a_1x_2 & b_1x_2 \end{pmatrix},$$

hence $\dim(\mathcal{LA}\{A + uB,\ u \in \mathbb{R}\} \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = 1$ and the control system does not satisfy (2.3).

If $a_2 = 0$ and $b_1 = b_4$ then $b_2 = 0$ or $b_3 = 0$, which together with A diagonal implies that A and B are simultaneously triangular. Hence the system does not satisfy the Lie algebra rank condition (2.3). If $a_2 = 0$ and $b_1 \neq b_4$ then $b_2 b_3 < 0$; let us suppose w.l.o.g. that $b_2 < 0$ and $b_3 > 0$. We have $[A, B] = 0$, and

$$(A, B)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 x_1 & b_1 x_1 + b_2 x_2 \\ a_1 x_2 & b_3 x_1 + b_4 x_2 \end{pmatrix}.$$

Note that $\det((A, B)x) = a_1(b_3 x_1^2 + (b_1 - b_4)x_1 x_2 - b_2 x_2^2)$, and with $(b_1 - b_4)^2 = 4(-b_2)b_3$ we have $\det((A, B)x) = (\sqrt{b_3}x_1 + \sqrt{-b_2}x_2)^2 a_1$. For $x = (-\sqrt{-b_2}/\sqrt{b_3}, 1) \neq 0$, this implies $\det((A, B)x) = 0$, so the control system does not satisfy (2.3).

Case 3c: $\alpha = 0$, $\beta = 0$, and $\gamma > 0$

In this case $y(u) > 0$ for all $u \in \mathbb{R}$, and we show that the system Σ_u^2 does not satisfy the Lie Algebra rank condition (2.3) in $\mathbb{R}^2$: the assumption $\gamma > 0$ means that A has real and distinct eigenvalues, hence its real Jordan form is diagonal. Now $\beta = 0$ implies $b_1 = b_4$, hence we have shown that $b_2 = 0$ or $b_3 = 0$ holds.

If $b_2 = 0$ and $b_3 = 0$, then $B = b_1 I$ where I is the identity matrix, hence $[A, B] = 0$ and

$$(A, B)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 x_1 & b_1 x_1 \\ a_4 x_2 & b_1 x_2 \end{pmatrix},$$

hence $\dim(\mathcal{LA}\{A + uB,\, u \in \mathbb{R}\}\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = 1$ and the control system does not satisfy (2.3).

If $b_2 \neq 0$ and $b_3 = 0$, then the matrices are of the form

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix}.$$

The Lie brackets are computed as

$$[A, B] = \begin{pmatrix} 0 & b_2(a_1 - a_4) \\ 0 & 0 \end{pmatrix}, \qquad [A, [A, B]] = (a_1 - a_4)[A, B], \qquad [B, [A, B]] = 0.$$

Therefore $\mathcal{LA}\{A + uB,\, u \in \mathbb{R}\} = span\{A, B, [A, B]\}$ and

$$(A, B, [A, B])\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 x_1 & b_1 x_1 + b_2 x_2 & b_2(a_1 - a_4)x_2 \\ a_4 x_2 & b_1 x_2 & 0 \end{pmatrix},$$

hence $\dim(\mathcal{LA}\{A + uB,\, u \in \mathbb{R}\}\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = 1$ and the control system does not satisfy (2.3).

A similar computation establishes the case $b_2 = 0$ and $b_3 \neq 0$.

Case 4: $\alpha > 0$

In this case we consider three subcases depending on $\gamma - \frac{\beta^2}{\alpha}$:

Case 4a: $\alpha > 0$ and $\gamma - \frac{\beta^2}{\alpha} < 0$

In this case we have $y(u_0) < 0$ for $u_0 = -\frac{\beta}{\alpha}$ and the system is controllable on $\mathbb{P}^1$.

Case 4b: $\alpha > 0$ and $\gamma - \frac{\beta^2}{\alpha} > 0$

In this case we have $y(u) > 0$ for all $u \in \mathbb{R}$ and $A + uB$ has distinct real eigenvalues. This implies that the control sets of the projected system (3.2) on $\mathbb{P}^1$ are determined by the eigendirections for constant controls $u \in \mathbb{R}$, compare [2, Chapter 8.2]. We will show that the ranges of the eigendirections are disjoint, hence the projected system has two disjoint control sets and therefore is not controllable.

The condition $\alpha\gamma > \beta^2$ implies $\gamma > 0$. We may assume that A is in Jordan canonical form, i.e., $a_2 = a_3 = 0$, and

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

The eigenvalues of the system for constant controls are:

$$\lambda_1(u) = \frac{(a_1 + ub_1 + a_4 + ub_4) + \sqrt{y(u)}}{2},$$

$$\lambda_2(u) = \frac{(a_1 + ub_1 + a_4 + ub_4) - \sqrt{y(u)}}{2},$$

where $y(u) = \alpha u^2 + 2\beta u + \gamma > 0$.

If $b_2 = 0$ and $b_3 = 0$, then B is diagonal matrix and the control system does not satisfy the Lie algebra rank condition (2.3).

If $b_2 \neq 0$ and $b_3 = 0$, or $b_2 = 0$ and $b_3 \neq 0$, then the analysis is similar to Case 3.c and the control system does not satisfy (2.3).

If $b_2 \neq 0$ and $b_3 \neq 0$ we need to analyze the eigendirections of the system for constant controls. W.l.o.g. we may normalize the x_1-component of the eigenvectors to 1, the case with $x_1 = 0$ and x_2 normalized to 1 is similar.

The eigendirections for $u \neq 0$ are:

$$x_2 = \frac{\lambda_1(u) - (a_1 + ub_1)}{b_2 u} = \frac{(a_4 + ub_4) + \sqrt{y(u)} - (a_1 + ub_1)}{2b_2 u},$$

$$x_2 = \frac{\lambda_2(u) - (a_1 + ub_1)}{b_2 u} = \frac{(a_4 + ub_4) - \sqrt{y(u)} - (a_1 + ub_1)}{2b_2 u}.$$

In polar coordinates with $\theta = \arctan\left(\frac{x_2}{x_1}\right)$ we obtain

$$\theta_1(u) = \arctan\left(\frac{(a_4 + ub_4) + \sqrt{y(u)} - (a_1 + ub_1)}{2b_2 u}\right),$$

$$\theta_2(u) = \arctan\left(\frac{(a_4 + ub_4) - \sqrt{y(u)} - (a_1 + ub_1)}{2b_2 u}\right).$$

Claim: The ranges of the eigendirections have empty intersection, i.e., for all u_1, u_2 in $\mathbb{R} \setminus \{0\}$ with $u_1 \neq u_2$, we have $\theta_1(u_1) \neq \theta_2(u_2)$.

Proof. Suppose that there exist u_1, u_2 in $\mathbb{R} \setminus \{0\}$ with $u_1 \neq u_2$ such that $\theta_1(u_1) = \theta_2(u_2)$. Then

$$\arctan\left(\frac{(a_4 + u_1 b_4) + \sqrt{y(u_1)} - (a_1 + u_1 b_1)}{2b_2 u_1}\right) = \arctan\left(\frac{(a_4 + u_2 b_4) - \sqrt{y(u_2)} - (a_1 + u_2 b_1)}{2b_2 u_2}\right),$$

which yields by applying the tan-function

$$\frac{(a_4+u_1b_4)+\sqrt{y(u_1)}-(a_1+u_1b_1)}{2b_2u_1}=\frac{(a_4+u_2b_4)-\sqrt{y(u_2)}-(a_1+u_2b_1)}{2b_2u_2},$$
$$u_2(a_4-a_1)+u_2\sqrt{y(u_1)}=u_1(a_4-a_1)-u_1\sqrt{y(u_2)},$$
$$u_2\sqrt{y(u_1)}+u_1\sqrt{y(u_2)}=(u_1-u_2)(a_4-a_1),$$
$$u_2^2y(u_1)+2u_2u_1\sqrt{y(u_1)}\sqrt{y(u_2)}+u_1^2y(u_2)=(u_1-u_2)^2(a_4-a_1)^2.$$

Since A is diagonal matrix, $\gamma=(a_4-a_1)^2$ and we obtain in terms of $y(u)$

$$\alpha u_1^2u_2^2+2\beta u_1u_2^2+\gamma u_2^2+\alpha u_2^2u_1^2+2\beta u_2u_1^2+\gamma u_1^2+2u_2u_1\sqrt{y(u_1)}\sqrt{y(u_2)}=\gamma u_1^2+\gamma u_2^2-2\gamma u_1u_2,$$
$$2u_1u_2\left(\alpha u_1u_2+\beta(u_1+u_2)+\sqrt{y(u_1)}\sqrt{y(u_2)}\right)=-2\gamma u_1u_2,$$
$$\alpha u_1u_2+\beta(u_1+u_2)+\sqrt{y(u_1)}\sqrt{y(u_2)}=-\gamma,$$
$$\alpha u_1u_2+\beta(u_1+u_2)+\gamma=-\sqrt{y(u_1)}\sqrt{y(u_2)},$$
$$(\alpha u_1u_2)^2+2\alpha\beta u_1u_2(u_1+u_2)+\beta^2(u_1+u_2)^2+2\alpha\gamma u_1u_2+2\beta\gamma(u_1+u_2)+\gamma^2=y(u_1)y(u_2).$$

Comparing both sides we have

$$\left\{\begin{array}{l}\alpha^2u_1^2u_2^2+2\alpha\beta u_1^2u_2+2\alpha\beta u_1u_2^2\\+\beta^2u_1^2+2\beta^2u_1u_2+\beta^2u_2^2\\+2\alpha\gamma u_1u_2+2\beta\gamma u_1+2\beta\gamma u_2+\gamma^2\end{array}\right\}=\left\{\begin{array}{l}\alpha^2u_1^2u_2^2+2\alpha\beta u_1^2u_2+\alpha\gamma u_1^2\\+2\alpha\beta u_1u_2^2+4\beta^2u_1u_2+2\beta\gamma u_1\\+\alpha\gamma u_2^2+2\beta\gamma u_2+\gamma^2\end{array}\right\},$$

and therefore

$$\beta^2u_1^2+\beta^2u_2^2+2\alpha\gamma u_1u_2=\alpha\gamma u_1^2+2\beta^2u_1u_2+\alpha\gamma u_2^2,$$
$$\beta^2(u_1^2-2u_1u_2+u_2^2)=\alpha\gamma(u_1^2-2u_1u_2+u_2^2),$$
$$\beta^2(u_1-u_2)^2=\alpha\gamma(u_1-u_2)^2.$$

For $u_1\neq u_2$, we have, $\beta^2=\alpha\gamma$, which contradicts the assumption $\alpha\gamma>\beta^2$. □

Case 4c: $\alpha>0$ and $\gamma-\frac{\beta^2}{\alpha}=0$

In this case we have $y(u_0)=0$ for $u_0=-\frac{\beta}{\alpha}$ and $y(u)>0$ for all $u\in\mathbb{R}\setminus\{u_0\}$.

Case 4.c.i: $\beta=0$. This implies $\gamma=0$ and hence $y(u)=\alpha u^2\geq 0$. Therefore, the A matrix has real and equal eigenvalues. Without loss of generality, we may assume that A is in Jordan canonical form, i.e., $a_3=0$ and $a_1=a_4$. Consider first the case $a_2\neq 0$. Then the condition $\beta=0$ forces $b_3=0$, and the control system does not satisfy (2.3), compare the proof of Case 3.b.

On the other hand, if $a_2=0$, then A is a diagonal matrix. If the coefficients of B satisfy $b_2=0$ or $b_3=0$, then the control system does not satisfy (2.3); the proof is similar to Case 3.c.

If $b_2\neq 0$ and $b_3\neq 0$, then the matrices A and B are given as

$$A=\begin{pmatrix}a_1&0\\0&a_1\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix}b_1&b_2\\b_3&b_4\end{pmatrix}.$$

Therefore, the eigenvalues of the control system are:

$$\lambda_1(u)=\frac{(2a_1+ub_1+ub_4)+\sqrt{y(u)}}{2},\qquad\lambda_2(u)=\frac{(2a_1+ub_1+ub_4)-\sqrt{y(u)}}{2},$$

where $y(u) = \alpha u^2 \geq 0$. For the eigenvectors $x_i = \begin{pmatrix} x_1^i \\ x_2^i \end{pmatrix}$ corresponding to these eigenvalues we may assume, without loss of generality, that $x_1^i = 1$. For $u \neq 0$ we have

$$x_2^1 = \frac{\lambda_1(u) - (a_1 + ub_1)}{b_2 u} = \frac{u(b_4 - b_1) + \sqrt{\alpha u^2}}{2b_2 u},$$

$$x_2^2 = \frac{\lambda_2(u) - (a_1 + ub_1)}{b_2 u} = \frac{u(b_4 - b_1) - \sqrt{\alpha u^2}}{2b_2 u},$$

or in polar coordinates

$$\theta_1^1(u) = \arctan\left(\frac{(b_4 - b_1) + \sqrt{\alpha}}{2b_2}\right),$$

$$\theta_2^2(u) = \arctan\left(\frac{(b_4 - b_1) - \sqrt{\alpha}}{2b_2}\right).$$

These are different constants and hence the ranges of the eigendirections have empty intersection and the control system (3.2) on $\mathbb{P}^1$ is not controllable.

Case 4.c.ii: $\beta \neq 0$. In this case we have $\gamma > 0$ and the A matrix has real distinct eigenvalues. Without loss of generality, we may assume that A is in Jordan canonical form with $a_2 = a_3 = 0$ and $a_1 \neq a_4$. The condition $\alpha\gamma = \beta^2$ implies $b_2 b_3 = 0$. If $b_2 = b_3 = 0$, then A, B are simultaneously diagonal, and the control system obviously does not satisfy (2.3). If $b_2 \neq 0$ and $b_3 = 0$, or $b_2 = 0$ and $b_3 \neq 0$, then the system does not satisfy (2.3) since the matrices A and B are simultaneously triangularizable, compare the proof of Case 3.b.

This completes the discussion of the discriminant $y(u)$ for unbounded control range $U = \mathbb{R}$. We summarize our findings in the proof of Theorem 3.3:

Proof of Theorem 3.3. The discussion above after equation (3.2) establishes that the system Σ_u^2 is controllable on $\mathbb{P}^1$ if there exists a control $u \in U \subset \mathbb{R}$ such that the matrix $A + uB$ has a complex eigenvalue. To see the converse, note that the cases discussed above constitute a complete list of possible cases. The Cases 1, 2, 3.a or 4.a are valid if and only if there exists $u \in \mathbb{R}$ such that the $A + uB$ has complex eigenvalues. These are also the only cases for which the system (3.2) is controllable. □

The following corollary establishes conditions for the controllability when the range of the control is a subset $U \subset \mathbb{R}$. We consider the following bilinear control system

$$\dot{x} = (A + uB)x, \quad x \in \mathbb{R}^2, \quad u : \mathbb{R} \to U = [\underline{u}, \overline{u}] \subset \mathbb{R}, \tag{3.4}$$

which we denote by Σ_{ub}^2. Here $\underline{u} = -\infty$ and $\overline{u} = \infty$ are allowed. We obtain the following corollary.

Corollary 3.5. *Consider the bilinear control system* (3.4) *with* $U = [\underline{u}, \overline{u}] \subset \mathbb{R}$ *and assume that the Lie algebra rank condition* (2.3) *holds for the system* Σ_{up}^2 *in* $\mathbb{R}^2$*. Then* Σ_{up}^2 *is controllable on* $\mathbb{P}^1$ *if and only if there exists a constant control* $u \in [\underline{u}, \overline{u}]$ *such that the matrix* $A + uB$ *has a complex eigenvalue.*

Proof. Obviously, the projection of system Σ_{ub}^2 is not controllable on $\mathbb{P}^1$ if the system Σ_u^2 with control range $U = \mathbb{R}$ is not controllable on $\mathbb{P}^1$. This restricts the proof to the Cases 1, 2, 3.a and 4.a as discussed above. But these arguments go through when the discriminant polynomial $y(u)$ is restricted to $U = [\underline{u}, \overline{u}] \subset \mathbb{R}$. □

We next discuss a few examples to illustrate Theorem 3.3 and Corollary 3.5.

Example 3.6. Let us consider the bilinear control system,

$$\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + u \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2(1+u) & 2u \\ 2u & 1+u \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The discriminant polynomial reads $y(u) = 17u^2 + 2u + 1 = (1+u)^2 + 16u^2 > 0$ with $\alpha\gamma > \beta^2$. This is the Case 4.b, and we have to analyze the eigendirections: the eigenvalues are

$$\lambda_1(u) = \frac{3(1+u) + \sqrt{(1+u)^2 + 16u^2}}{2}, \quad \lambda_2(u) = \frac{3(1+u) - \sqrt{(1+u)^2 + 16u^2}}{2},$$

and for $u \neq 0$, we obtain as eigendirections (with $x_1^i = 1$) for $\lambda_1(u)$

$$x_2^1 = \frac{\frac{3(1+u)+\sqrt{(1+u)^2+16u^2}}{2} - 2(1+u)}{2u} = \frac{\sqrt{(1+u)^2 + 16u^2} - (1+u)}{4u},$$

and for $\lambda_2(u)$

$$x_2^2 = \frac{\frac{3(1+u)-\sqrt{(1+u)^2+16u^2}}{2} - 2(1+u)}{2u} = \frac{-\sqrt{(1+u)^2 + 16u^2} - (1+u)}{4u}.$$

In polar coordinates,

$$\theta_1(u) = \arctan\left(\frac{\sqrt{(1+u)^2 + 16u^2} - (1+u)}{4u} \right) \qquad \text{for } \lambda_1(u) \text{ and } x_1 = 1,$$

$$\theta_2(u) = \arctan\left(\frac{-\sqrt{(1+u)^2 + 16u^2} - (1+u)}{4u} \right) \qquad \text{for } \lambda_2(u) \text{ and } x_1 = 1.$$

The graph of the eigendirections, parametrized by the angle $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2})$, is shown in the following figure. Since the eigendirections do not overlap, we see that the bilinear control system is not controllable on $\mathbb{P}^1$.

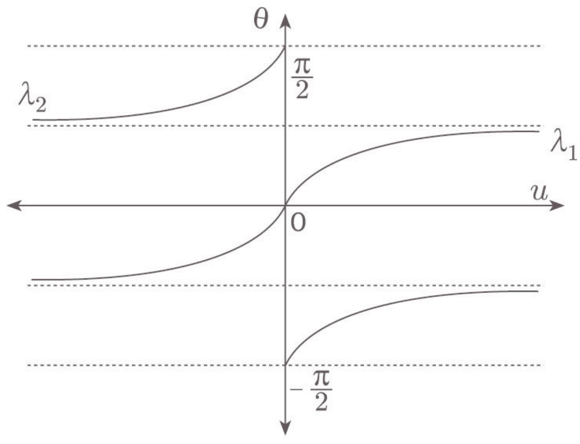

Figure 1

Example 3.7. We consider the bilinear control system

$$\begin{pmatrix} \dot{x_1} \\ x_2 \end{pmatrix} = \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + u \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2(1+u) & 2u \\ 0 & 1+u \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The discriminant polynomial reads $y(u) = u^2 + 2u + 1 = (1+u)^2 > 0$ for all $u \in \mathbb{R}$, with $\alpha\gamma = \beta^2$. This is the Case 4.c.ii. This control system does not satisfy the Lie algebra rank condition (2.3), since

$$[A, B] = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad [A, [A, B]] = [A, B], \; [B, [A, B]] = [A, B],$$

and hence

$$(A, B, [A, B]) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 & 2x_1 + 2x_2 & 2x_2 \\ x_2 & x_2 & 0 \end{pmatrix}$$

has rank 1 for $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Example 3.8. Consider the bilinear control system,

$$\begin{pmatrix} \dot{x_1} \\ x_2 \end{pmatrix} = \left(\begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix} + u \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5+3u & -1-u \\ 1+u & 3+u \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The discriminant polynomial reads $y(u) \equiv 0$ and hence we are in Case 4.c.ii. The eigenvalues are $\lambda(u) = 2(2+u)$. We analyze the Lie algebra rank condition for this system using the real Jordan form of A. The similarity matrix is

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{with} \qquad P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

With $z = P^{-1}x$ we have the bilinear control system $P^{-1}\dot{x} = (P^{-1}AP + uP^{-1}BP)P^{-1}x$, i.e.,

$$\begin{pmatrix} \dot{z_1} \\ z_2 \end{pmatrix} = \left(\begin{pmatrix} 4 & 0 \\ 1 & 4 \end{pmatrix} + u \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \right) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 4+2u & 0 \\ 1+u & 4+2u \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Computing the Lie bracket we obtain $[P^{-1}AP, P^{-1}BP] = 0$, and therefore

$$(P^{-1}AP, P^{-1}BP, [P^{-1}AP, P^{-1}BP]) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 4z_1 & 2z_1 & 0 \\ z_1 + 4z_2 & z_1 + 2z_2 & 0 \end{pmatrix},$$

which has rank 1 for $z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

4. The Lie bracket of the bilinear system

In this section we give a characterization of the controllability of the angular system (2.2) in terms of the Lie bracket of the bilinear system.

We denote by $\mathcal{LA}\{A, B\}$ the Lie algebra generated by the two matrices A, $B \in gl(2, \mathbb{R})$. The set $\mathcal{LA}\{A, B\}(x)$ is a subspace of the tangent space T_x at $x \in \mathbb{R}^2 \setminus \{0\}$, the basis of which can be found among the vectors Ax, Bx, $[A, B]x$, $[A, [A, B]]x$, $[B, [A, B]]x$, $\cdots$. Note that if the matrices A and B have a common eigenvector, then the condition LARC (2.3) cannot be satisfied for the bilinear system: if $v \in \mathbb{R}^2 \setminus \{0\}$ is an eigenvector for both matrices, then $[A, B]v = 0$ and the sequence of vectors is reduced to the two linear dependent ones Av and Bv.

Recall that the characteristic polynomial of $A + uB$ is $\lambda^2 + \mathrm{tr}(A + uB)\lambda + \det(A + uB)$ with eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left(\mathrm{tr}(A + uB) \pm \sqrt{[\mathrm{tr}(A + uB)]^2 - 4\det(A + uB)}\right)$$

and discriminant $y_{[A+uB]}(u) = [\mathrm{tr}(A + uB)]^2 - 4\det(A + uB)$. Using

$$\det(A + uB) = \det(B)u^2 + [\mathrm{tr}(AB) - \mathrm{tr}(A)\mathrm{tr}(B)]u + \det(A)$$

we can write

$$y_{[A+uB]} = y_{[B]}u^2 + 2[2\mathrm{tr}(AB) - \mathrm{tr}(A)\mathrm{tr}(B)]u + y_{[A]}, \tag{4.1}$$

where $y_{[A]}$, $y_{[B]}$ are the discriminants of the characteristic polynomials of A and B, respectively.

To simplify matters we will work with trace zero matrices, i.e., we consider $\xi : gl(2, \mathbb{R}) \to sl(2, \mathbb{R})$ defined by

$$\xi(X) = X - \frac{1}{2}\mathrm{tr}(X)I.$$

Since the characteristic polynomial of a matrix X is given by $p(\lambda) = \det(X - \lambda I) = \lambda^2 - \mathrm{tr}(X)\lambda + \det(X)$, we obtain

$$\det \xi(X) = p\left(\frac{1}{2}\mathrm{tr}(X)\right) = -\frac{1}{4}(\mathrm{tr}^2(X) - 4\det(X)) = -\frac{1}{4}y_{[X]}.$$

Proposition 4.1. *Let $A, B \in sl(2, \mathbb{R})$. Then it holds:*

a) $AB + BA = \mathrm{tr}(AB)I$;
b) $\det[A, B] = 4\det(AB) - \mathrm{tr}^2(AB) = -y_{[AB]}$;
c) *The discriminant of* $\det(A + uB)$ *is* $-\det[A, B]$.

Proof.

a) This follows directly from the definition of the trace.

b) From *a*) we have $[A, B] = \mathrm{tr}(BA)I - 2BA$, hence

$$\begin{aligned}\det[A, B] &= \det(2BA - \mathrm{tr}(BA)I)\\ &= 4\det\left(BA - \frac{1}{2}\mathrm{tr}(BA)I\right)\\ &= 4\det\xi(BA)\\ &= -\mathrm{tr}^2(BA) + 4\det(BA).\end{aligned}$$

c) It follows directly from (4.1) that the discriminant of $\det(A + uB)$ is $\mathrm{tr}^2(AB) - 4\det(AB)$.

□

Part *b*) of Proposition 4.1 can be generalized to arbitrary matrices:

Proposition 4.2. *Let $A, B \in gl(2, \mathbb{R})$. Then it holds:*

a) $\det[A,B] = \frac{1}{4}y_{[A]}y_{[B]} - \frac{1}{4}(2\mathrm{tr}(AB) - \mathrm{tr}(A)\mathrm{tr}(B))^2$.
b) *The discriminant of* $y_{[A+uB]}$ *is* $-16\det[A,B]$.

Proof.

a) Proposition 4.1 yields

$$\det[\xi(A),\xi(B)] = 4\det(\xi(A)\xi(B)) - \mathrm{tr}^2(\xi(A)\xi(B)).$$

We observe that

$$[\xi(A),\xi(B)] = \left[A - \frac{1}{2}\mathrm{tr}(A)I, B - \frac{1}{2}\mathrm{tr}(B)I\right] = [A,B]$$

and therefore

$$\begin{aligned} 4\det(\xi(A)\xi(B)) - \mathrm{tr}^2(\xi(A)\xi(B)) &= 4\left(-\frac{1}{4}y_{[A]}\right)\left(-\frac{1}{4}y_{[B]}\right) - \left(\mathrm{tr}(AB) - \frac{1}{2}\mathrm{tr}(A)\mathrm{tr}(B)\right)^2 \\ &= \frac{1}{4}y_{[A]}y_{[B]} - \frac{1}{4}(2\mathrm{tr}(AB) - \mathrm{tr}(A)\mathrm{tr}(B))^2. \end{aligned}$$

b) This follows directly from *a*) and (4.1).

□

The formulas obtained above allow us to characterize the cases in which the matrix $A + uB$ has complex eigenvalues, i.e., to characterize the situation in which the control system (3.3) $\dot{x} = (A+uB)x$ is controllable on the projective space $\mathbb{P}^1$. We note first of all that the sign of $y_{[A+uB]}$ is determined by the sign of $\det[A,B]$. Hence the only cases in which $A + uB$ has complex eigenvalues are

I. $\det[A,B] < 0$,
II. $\det[A,B] > 0$ and $y_{[B]} < 0$,
III. $\det[A,B] = 0$ and $y_{[A]} < 0$ or $y_{[B]} < 0$.

In these cases controllability on $\mathbb{P}^1$ follows immediately since we have a $u \in U$ such that the solutions of $\dot{x} = (A + uB)x$ are rotations. Next we discuss the converse.

Consider the case $\det[A,B] > 0$: by Proposition 4.2, the matrices A and B have either complex eigenvalues, or the eigenvalues are real. If $y_{[B]} > 0$, then we also have $y_{[A]} > 0$. It suffices to look at

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_4 \end{pmatrix}.$$

From Proposition 4.2 we see that $\det[A,B] = a_2a_3(b_1 - b_4)^2$ and hence $a_2a_3 > 0$. On the other hand, the eigenvectors of A are

$$(x_{1,2}, 1) = \left(\frac{(a_1 - a_4) \pm \sqrt{y_{[A]}}}{2a_2}, 1\right),$$

and therefore

$$x_1x_2 = \frac{1}{4a_2^2}[(a_1 - a_4)^2 - y_{[A]}] = -\frac{a_3}{a_2}.$$

We obtain that $x_1x_2 > 0$ iff $a_2a_3 < 0$, yielding a contradiction.

Next consider $\det[A,B] > 0$ and $y_{[B]} = 0$, so we have to look at

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix}.$$

This means that $\det[A,B] = -(a_3 b_2)^2$. Hence we do not have controllability on $\mathbb{P}^1$ if $\det[A,B] > 0$ and $y_{[B]} \geq 0$.

Note that the cases $y_{[A]} = 0$ or $y_{[B]} = 0$ immediately imply that the system $\dot{x} = (A + uB)x$ does not satisfy the Lie algebra rank condition (2.3). Hence we only need to consider $y_{[A]} > 0$ and $y_{[B]} > 0$ with

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

In this case $\det[A,B] = b_2 b_3 (a_1 - a_2)^2$ and hence $b_2 b_3 = 0$. Therefore A and B have common eigenvector and hence the system cannot satisfy the Lie algebra rank condition (2.3).

From this discussion we obtain the following result, which is parallel to Theorem 3.3.

Theorem 4.3. *Consider the bilinear system $\dot{x} = (A + uB)x$ and assume that it satisfies the Lie algebra rank condition* (2.3)*. Then the following statements are equivalent:*

a) *The angular system on $\mathbb{P}^1$ is controllable.*
b) *Either* $\det[A,B] < 0$*, or* $\det[A,B] \geq 0$ *and A or B have complex eigenvalues.*

5. Controllability with multiple inputs

In this section we consider the bilinear control system

$$\dot{x}(t) = \left(A + \sum_{i=1}^{m} u_i(t) B_i \right) x(t) = A(u)x(t), \qquad t \in \mathbb{R}, \qquad x(t) \in \mathbb{R}^d$$

with multiple inputs $u \in \mathcal{U} = \{u : \mathbb{R} \to U \subset \mathbb{R}^m,\ u \text{ is locally integrable}\}$, where $U \subset \mathbb{R}^m$ is compact, and convex with $0 \in intU$. We continue to assume the Lie algebra rank condition (2.3), and are again interested in controllability of the projected system (2.2) on the projective space $\mathbb{P}^1$. The following example shows that we cannot expect the characterization via complex eigenvalues to hold in this case.

Example 5.1. Consider

$$\begin{aligned} \dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} &= \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + u(t) \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + v(t) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 2 + v(t)) & u(t) - v(t) \\ u(t) - v(t) & 1 + 2u(t) + v(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned} \tag{5.1}$$

Let us look at two subsystems, Σ_1 given by $v = 0$, i.e.,

$$\begin{aligned} \dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} &= \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + u(t) \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & u(t) \\ u(t) & 1 + 2u(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

and Σ_2 given by $u = 0$

$$\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + v(t) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= \begin{pmatrix} 2 + v(t)) & -v(t) \\ -v(t) & 1 + +v(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

One easily checks that these two subsystems satisfy the Lie algebra rank condition for U as given above. The eigenvalues are computed as

$$\Sigma_1 \;:\; \lambda_{1,2}(u) = \frac{3}{2} + u \pm \frac{1}{2}\sqrt{(1-2u)^2 + 4u^2},$$
$$\Sigma_2 \;:\; \lambda_{1,2}(v) = \frac{3}{2} + v \pm \frac{1}{2}\sqrt{1 + 4v^2}$$

and hence by Section 2 neither subsystem is controllable on the projective space $\mathbb{P}^1$.

Computing the eigenvalues of the combined system (5.1) we obtain

$$\lambda(u, v) = \frac{3}{2} + u + v \pm \frac{1}{2}\sqrt{(1-2u)^2 + 4(u-v)^2}.$$

These values are all real and the combined system has no rotation.

We take as control range the set, $U \times V = [-2, 2] \times [-6, 6]$. For $v = 0$ we obtain eigenvalues of Σ_1 as follows: for $u = 0$ we have $\lambda_1(0, 0) = 1$ and $\lambda_2(0, 0) = 2$, and for $u = 2$ we obtain $\lambda_1(2, 0) = 1$ and $\lambda_2(2, 0) = 6$. Computing the corresponding eigenvectors shows that the set $D_1 = (\pi/2, 7\pi/8) \subset \mathbb{P}^1$ is contained in the smaller control set of Σ_1, while $D_2 = [0, \pi/3]$ is contained in the greater control set of Σ_1. (Here "smaller" and "greater" refer to the order of control sets, compare [2, Chapter 7]).

Now consider the constant control $u = 0$ and $v = 5$. The eigenvalues for this control are $\lambda_{1,2}(0, 5) = \frac{3}{2} + 5 \pm \frac{1}{2}\sqrt{101}$, and the eigenspaces (in radians of the projective space) are $\mathbb{P}E(\lambda_1(0, 5)) = 0.835$ and $\mathbb{P}E(\lambda_2(0, 5)) = 2.399$. Note that $\mathbb{P}E(\lambda_1(0, 5)) \subset D_2$ and $\mathbb{P}E(\lambda_2(0, 5)) \subset D_1$, hence D_1 can be reached from D_2 using the constant control $(u, v) = (0, 5)$ and the system (5.1) is controllable on the projective space.

Our characterization of controllability of the system (2.2) on the projective space is in terms of the location of eigenspaces on $\mathbb{P}^1$. We will use the notation $\mathbb{P}\varphi(t, \cdot, u)$ for the solutions at time $t \in \mathbb{R}$ of (2.2) for constant controls $u \in U$. The maps $\{\mathbb{P}\varphi(t, \cdot, u), t \in [0, T]\}$ are called a directed family for $x \in \mathbb{P}^1$ if $\mathbb{P}\varphi(\cdot, x, u) : [0, T] \to \mathbb{P}^1$ is injective. This means, intuitively, that $\{\mathbb{P}\varphi(\cdot, x, u), t \in [0, T]\}$ wraps around (a part of) $\mathbb{P}^1$ in one direction. Note that if there is a control $u(\cdot) \in \mathcal{U}$ and a time $t \geq 0$ with $y = \mathbb{P}\varphi(t, x, u)$ then there exists a piecewise constant control $v(\cdot) \in \mathcal{U}_{pc}$ and $s \geq 0$ with $y = \mathbb{P}\varphi(s, x, v)$. This motivates the following definition.

Definition 5.2. The family of maps $\{\mathbb{P}\varphi(t, \cdot, u), t \in [0, T]\}$ for some $u(\cdot) \in \mathcal{U}_{pc}$ is called a controlled rotation if there exists $x \in \mathbb{P}^1$ such that $\{\mathbb{P}\varphi(t, \cdot, u), t \in [0, T]\}$ is a directed family for x, and $\{\mathbb{P}\varphi(t, x, u), t \in [0, T]\} = \mathbb{P}^1$.

Proposition 5.3. *The system* (2.2) *is controllable on* $\mathbb{P}^1$ *iff there exists a controlled rotation of* (2.2).

Proof. If there exists a controlled rotation, then the system is obviously controllable. To see the converse, let $x \in \mathbb{P}^1$ and denote by $\mathcal{A}^+(x)$ the points of $\mathbb{P}^1$ that can be reached from x via counterclockwise angular motion, and by $\mathcal{A}^-(x)$ the points of $\mathbb{P}^1$ that can be reached from x via clockwise motion. Since the system is assumed to be controllable, we have $\mathcal{A}^+(x) \cap \mathcal{A}^-(x) \neq \emptyset$. Let $y \in \mathcal{A}^+(x) \cap \mathcal{A}^-(x)$, then $x \in \mathcal{A}^+(y)$ or $x \in \mathcal{A}^-(y)$. In any case we obtain a controlled rotation anchored at x. □

It remains to describe controlled rotations in simple ways. We will use the following notation: let $\mathcal{M} := \{A(u) := A + \sum_{i=1}^m u_i B_i, u \in U\}$ denote the set of system matrices, and let $A(u) \in \mathcal{M}$ with two distinct real eigenvalues $\lambda_1 < \lambda_2$ and associated eigenspaces E_1 and E_2. Note that the flow $\{\mathbb{P}\varphi(t, \cdot, u), t \geq 0\}$ on $\mathbb{P}^1$ is always from $\mathbb{P}E_1$ to $\mathbb{P}E_2$. For two points $x, y \in \mathbb{P}^1$ the set $[x, y]$ denotes the interval of $\mathbb{P}^1$ in the "counterclockwise" order.

1. The set of matrices $\mathcal{M}$ contains a matrix with a complex pair of eigenvalues, iff the system admits an actual rotation. In this case we have 'controlled' rotation with one constant $u \in U$.
2. Assume now that $\mathcal{M}$ contains only matrices with real eigenvalues, and that there is a matrix $A(u^0)$ with double real eigenvalue and only one eigendirection E^0, such that (w.l.o.g.) $\{\mathbb{P}\varphi(t, \cdot, u^0), t \geq 0\}$ follows counterclockwise motion on $\mathbb{P}^1$. Then part of $\{\mathbb{P}\varphi(t, \cdot, u^0), t \geq 0\}$ belongs to a controlled rotation iff there exists $u^1 \in U$ such that $\{\mathbb{P}\varphi(t, \cdot, u^1), t \geq 0\}$ moves counterclockwise in a neighborhood of $\mathbb{P}E^0$. This occurs iff (i) $A(u^1)$ has a double real eigenvalue and only one eigendirection $E^1 \neq E^0$, such that $\{\mathbb{P}\varphi(t, \cdot, u^1), t \geq 0\}$ moves counterclockwise on $\mathbb{P}^1$, or (ii) $A(u^1)$ has two distinct real eigenvalues with $\mathbb{P}E^0 \in [\mathbb{P}E_1^1, \mathbb{P}E_2^1]$. In both cases we obtain a controlled rotation with two control values and two switches of the control value. Note that matrices $A(u) \in \mathcal{M}$ with one double real eigenvalue and two linearly independent eigenvectors cannot be part of a controlled rotation.
3. Assume finally that $\mathcal{M}$ contains only matrices with two distinct real eigenvalues. In this case there obviously exists a controlled counterclockwise rotation if there are $u^1, ..., u^n, u^{n+1} = u^1 \in U$ such that $\mathbb{P}E_1(u^2) < \mathbb{P}E_2(u^1) < \mathbb{P}E_1(u^3) < \mathbb{P}E_2(u^2) < \cdots < \mathbb{P}E_2(u^n) < \mathbb{P}E_1(u^2) < \mathbb{P}E_2(u^1) = \mathbb{P}E_2(u^{n+1})$, and analogously for clockwise controlled rotations.

With these observations we obtain the following result.

Theorem 5.4. *For the system* (2.2) *under the Lie algebra rank condition* (2.3) *the following statements are equivalent:*

1. *The system is controllable.*
2. *The system admits a controlled rotation.*
3. *The matrix set $\mathcal{M} := \{A + \sum_{i=1}^m u_i B_i, u \in U\}$ satisfies at least one of the following conditions:*
   (a) *$\mathcal{M}$ contains a matrix with a pair of complex eigenvalues, or*
   (b) *$\mathcal{M}$ contains a matrix $A(u^0)$ with double real eigenvalue and only one eigendirection E^0, and a second matrix $A(u^1)$ whose flow moves in the same direction as $\{\mathbb{P}\varphi(t, \cdot, u^0), t \geq 0\}$ in a neighborhood of $\mathbb{P}E^0$, or*
   (c) *$\mathcal{M}$ contains a finite sequence $\{A(u^i), i = 1, ..., n\}$ of matrices with distinct real eigenvalues whose eigenspaces satisfy either $\mathbb{P}E_1(u^2) < \mathbb{P}E_2(u^1) < \mathbb{P}E_1(u^3) < \mathbb{P}E_2(u^2) < \cdots < \mathbb{P}E_2(u^n) < \mathbb{P}E_1(u^2) < \mathbb{P}E_2(u^1)$ in case of a counterclockwise controlled rotation, or $\mathbb{P}E_1(u^2) > \mathbb{P}E_2(u^1) > \mathbb{P}E_1(u^3) > \mathbb{P}E_2(u^2) > \cdots > \mathbb{P}E_2(u^n) > \mathbb{P}E_1(u^2) > \mathbb{P}E_2(u^1)$ in case of a clockwise controlled rotation.*

Proof. All we need to show is that the existence of a controlled rotation implies one of the three situations in 3. If the set $\mathcal{M}$ contains a matrix with a pair of complex eigenvalues, we are done. Hence we now assume that all matrices $A(u) \in \mathcal{M}$ have real eigenvalues. Furthermore, we assume (w.l.o.g.) that the given controlled rotation flows counterclockwise on $\mathbb{P}^1$.

Let $x \in \mathbb{P}^1$ and denote $U^+(x) := \{u \in U$, the flow of $A(u)$ on $\mathbb{P}^1$ at x is counterclockwise$\}$. This means that (i) $x \in (\mathbb{P}E_1(u), \mathbb{P}E_2(u))$ for all $u \in U^+(x)$ for which $A(u)$ has two (distinct) eigenvalues, and (ii) $x \in (\mathbb{P}E_0(u), \mathbb{P}E_0(u))$ for all $u \in U^+(x)$ for which $A(u)$ has a double real eigenvalue and only one eigendirection E_0.

We first consider the case that there is a $u \in U^+(x)$ such that (ii) holds. By assumption of the existence of a counterclockwise controlled rotation, $U^+(\mathbb{P}E_0) \neq \emptyset$, which together with smoothness of the vector fields on $\mathbb{P}^1$ implies the existence of a controlled rotation as in 3 (b).

We now assume that for all $u \in U^+(x)$ the matrix $A(u)$ has two distinct eigenvalues. Denote $y^1 := \sup\{\mathbb{P}E_2(u),\ u \in U^+(x)\}$, with the supremum taken in the counterclockwise order on $\mathbb{P}^1$. Note that the supremum is well-defined since $x \in (\mathbb{P}E_1(u), \mathbb{P}E_2(u))$ for all $u \in U^+(x)$. Define $y^2 := \sup\{\mathbb{P}E_2(u),\ u \in U^+(y^1)\}$, and so on until the first $n \in \mathbb{N}$ with $x \in (y^n, y^{n+1})$. We need to show that such an n exists: assume to the contrary that $y := \lim_{k\to\infty} y^k \leq x$. Then by assumption on the existence of a controlled rotation there exists $v \in U^+(x)$ and $\mathbb{P}\varphi(\cdot, y, v)$ is counterclockwise in a neighborhood of y, which is a contradiction. Note that by construction $y^{n+1} = y^1$. Using again the smoothness of the vector fields of (2.2) for constant $u \in U$, we arrive at a finite sequence of control values $u^1, ..., u^n, u^{n+1} = u^1$ and switch points $z^1, ..., z^n$ on $\mathbb{P}^1$ with $x \in (z^n, z^1)$ such that the flow defined by the vector fields $\mathbb{P}A(u^{i+1})$ on the intervals $[z^i, z^{i+1})$ is a controlled, counterclockwise rotation.

Note that by construction we have the following order of the eigenspaces of the $A(u^i) \in \mathcal{M}$ on $\mathbb{P}^1$: $\mathbb{P}E_1(u^2) < \mathbb{P}E_2(u^1) < \mathbb{P}E_1(u^3) < \mathbb{P}E_2(u^2) < \cdots < \mathbb{P}E_2(u^n) < \mathbb{P}E_1(u^2) < \mathbb{P}E_2(u^1) = \mathbb{P}E_2(u^{n+1})$.

Analogously, all arguments for clockwise controlled rotations can be described. □

We close this section by giving an example for a bilinear system with two controls for which the subsystems do not satisfy the Lie algebra rank condition (2.3), but the complete system does and, moreover, it is controllable.

Example 5.5. Consider the bilinear control system with one control

$$\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + u(t) \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2(1+u(t)) & 2u(t) \\ 2u(t) & 1+u(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The projected angular system on $\mathbb{S}^1$ satisfies the differential equation

$$\dot{\theta} = (1 + u(t)) \sin(\theta) \cos(\theta) - 2u(t) \sin^2(\theta),$$

which for all $u \in \mathbb{R}$ has a common fixed point $\theta = 0$, and hence the system does not satisfy the Lie algebra rank condition (2.3). In particular, the system is not controllable on $\mathbb{P}^1$. The same holds true for the following system

$$\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + v(t) \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2+v(t) & 0 \\ 2v(t) & 1+v(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Now we combine the two systems as

$$\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + u(t) \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} + v(t) \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2+2u(t)+v(t) & 2u(t) \\ 2v(t) & 1+u(t)+v(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This system has the projected angular equation on $\mathbb{S}^1$

$$f((u(t), v(t)), \alpha) = -(1 + u(t) \sin(\alpha) + 2(u(t) + v(t)) \cos(\alpha) + 2(u(t) - v(t))$$

and one can check easily that (2.3) is satisfied. Computing the eigenvalues for constant (u, v) we obtain

$$\lambda(u,v) = \frac{3}{2} + \frac{3}{2}u + v \pm \frac{1}{2}\sqrt{(1+u)^2 + 16uv}.$$

For $u = 1$ and $v = -1$ the eigenvalue is a complex pair and hence the system is controllable on $\mathbb{P}^1$ if, e.g., $(1, -1) \in U$.

6. The spectrum of bilinear systems

In this section we consider the Lyapunov spectrum Σ_{Ly} of the bilinear control system (3.3) $\dot{x} = (A + uB)x$, $x \in \mathbb{R}^2$, $A, B \in gl(2, \mathbb{R})$, $u(t) \in \mathbb{R}$, i.e., we consider the set of Lyapunov exponents

$$\Sigma_{Ly} = \{\, \lambda(u,x),\ (u,x) \in \mathcal{U} \times (\mathbb{R}^2 \setminus \{0\})\, \},$$
$$\lambda(u,x) = \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi(t,x,u)\|\,,$$

where $\varphi(t, x, u)$ is a solution of (3.3).

We continue to assume the Lie algebra rank condition (2.3), and we work under the condition that the projected system (2.2) on $\mathbb{P}^1$ is controllable. According to Theorem 2.4 we need to characterize the situation $0 \in Int(\Sigma_{Ly})$.

The eigenvalues, whose real parts are the Lyapunov exponents for constant $u \in U$, are given by

$$\lambda_{1,2}(u) = \frac{1}{2}\left(\mathrm{tr}(A+uB) \pm \sqrt{[\mathrm{tr}(A+uB)]^2 - 4\det(A+uB)}\right).$$

For $u \in U$ we define the discriminant

$$y(u) = [\mathrm{tr}(A+uB)]^2 - 4\det(A+uB),$$

and obtain $y(u) = \alpha u^2 + 2\beta u + \gamma$, with

$$\begin{aligned}
\alpha &= \mathrm{tr}^2(B) - 4\det(B),\\
\beta &= 2\mathrm{tr}(AB) - \mathrm{tr}(A)\mathrm{tr}(B),\\
\gamma &= \mathrm{tr}^2(A) - 4\det(A)\,.
\end{aligned}$$

We will use the notation

$$\Sigma_{\mathrm{Re}} := \{\mathrm{Re}(\lambda_{1,2}(u)),\ u \in U\}$$

for the real parts of the eigenvalues of the system matrices, and

$$\mathcal{C}_- := y^{-1}[(-\infty,0)],\ \mathcal{C}_0 := y^{-1}[\{0\}],\ \mathcal{C}_+ := y^{-1}[(0,+\infty)]$$

for $u \in U$ with negative, zero, and positive discriminant, respectively. Note that $\Sigma_{\mathrm{Re}} \subset \Sigma_{Ly}$.

Lemma 6.1. *If the system* (2.2) *is controllable on* $\mathbb{P}^1$ *and if* $\mathrm{tr}(B) \neq 0$*, then* $0 \in int(\Sigma_{Ly})$*.*

Proof. The real parts of the eigenvalues of $A + uB$ are

$$Re(\lambda_{1,2}(u)) = \frac{1}{2}(\mathrm{tr}(A) + u\mathrm{tr}(B) \pm \chi_{\mathcal{C}_+}(u)\sqrt{y(u)})\,,$$

where χ_S denotes the characteristic function of a set S. Note that these values cover all of $\mathbb{R}$. □

Lemma 6.2. *Assume that the system* (2.2) *is controllable on* $\mathbb{P}^1$ *and that* $\mathrm{tr}(B) = 0$. *Then* $0 \in int(\Sigma_{\mathrm{Re}})$ *iff* $0 < \mathrm{tr}^2(AB) - 4\det(AB)$, *i.e., the matrix AB has distinct real eigenvalues.*

Proof. We distinguish three cases according to the sign of $\det(B)$.

1. Let $\det(B) > 0$. The maximum of $y(u)$ is $y\left(-\frac{\beta}{\alpha}\right) = \gamma - \frac{\beta^2}{\alpha}$. Note that there exist matrices in $\{A + uB, u \in \mathbb{R}\}$ with different sign of the real parts of eigenvalues iff

$$\mathrm{tr}(A) - \sqrt{y\left(-\frac{\beta}{\alpha}\right)} < 0 < \mathrm{tr}(A) + \sqrt{y\left(-\frac{\beta}{\alpha}\right)},$$

i.e.,

$$|\mathrm{tr}(A)| < \sqrt{y\left(-\frac{\beta}{\alpha}\right)},$$

which is equivalent to $4\det(AB) < \mathrm{tr}^2(AB)$.

2. Let $\det(B) = 0$. In this case we have $y(u) = 4\mathrm{tr}(AB)u + (\mathrm{tr}^2(A) - 4\det(A))$ and hence $0 < \mathrm{tr}^2(AB)$ implies $0 \in int(\Sigma_{\mathrm{Re}})$. Note that if $\mathrm{tr}(AB) = 0$, then all eigenvalues have the same real part, i.e., $int(\Sigma_{\mathrm{Re}}) = \emptyset$.
3. Let $\det(B) < 0$. One shows that $0 < \mathrm{tr}^2(AB) - 4\det(AB)$ implies $0 \in int(\Sigma_{\mathrm{Re}})$ following the proof of Lemma 9 in [1]. To see the converse consider

$$\begin{aligned}
\lambda_1(u)\lambda_2(u) &= \frac{1}{4}\begin{cases} \mathrm{tr}^2(A+uB) - \left(\sqrt{y(u)}\right)^2, & u \in \mathcal{C} \cup \mathcal{C}_+, \\ |\lambda_1(u)|^2, & u \in \mathcal{C}_-, \end{cases} \\
&= \frac{1}{4}\begin{cases} 4\det(A+uB), & u \in \mathcal{C} \cup \mathcal{C}_+, \\ \mathrm{tr}^2(A+uB) + \left(\sqrt{-y(u)}\right)^2, & u \in \mathcal{C}_-, \end{cases} \\
&= \det(A+uB).
\end{aligned}$$

This is a quadratic polynomial with different real roots and $\det(B) < 0$ implies that $\mathcal{C}_- \neq \emptyset$. Hence we have $\det(A + u_0B) > 0$ for some $u_0 \in \mathcal{C}_-$. Again with $\det(B) < 0$, we arrive with

$$\det(A + uB) = \det(B)u^2 + [\mathrm{tr}(AB) - \mathrm{tr}(A)\mathrm{tr}(B)]u + \det(A),$$

at the conclusion

$$4\det(AB) < \mathrm{tr}^2(AB).$$

□

We summarize these observations in the following theorem.

Theorem 6.3. *Assume that the bilinear system* (3.3) *satisfies the Lie algebra rank condition in* $\mathbb{R}^2$ *and that the projected system* (2.2) *is controllable on* $\mathbb{P}^1$. *Then*

(1) $0 \in int(\Sigma_{\mathrm{Re}})$ *iff* $\mathrm{tr}(B) \neq 0$ *or* $0 < \mathrm{tr}^2(AB) - 4\det(AB)$.

(2) *In particular, the system* (3.3) *is controllable in* $\mathbb{R}^2\backslash\{0\}$ *if* $\mathrm{tr}(B) \neq 0$, *or if* $\mathrm{tr}(B) = 0$ *and* $0 < \mathrm{tr}^2(AB) - 4\det(AB)$.

References

[1] V. Ayala, L. San Martin, *Controllability of two-dimensional bilinear systems: restricted controls and discrete time,* Proyecciones Journal of Mathematics **18** (1999), 207–223. 3

[2] F. Colonius, W. Kliemann, *The Dynamics of Control,* Birkhäuser, Boston, (2000). 1, 2, 2, 2, 3, 5.1

[3] D. L. Elliott, *Bilinear Control Systems: Matrices in Action*, Springer, Dordrecht, (2009). 1

[4] H. J. Sussmann, *Orbits of families of vector fields and integrability of distributions,* Trans. Amer. Math. Soc., **180** (1973), 171–188. 2