



Isochronous sets of invariant control systems

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ABSTRACT

Let G be a connected Lie group with Lie algebra \mathfrak{g} and $\Sigma = (G, D)$ a controllable invariant control system. A subset $A \subset G$ is said to be *isochronous* if there exists a uniform time $T_A > 0$ such that any two arbitrary elements in A can be connected by a positive orbit of Σ at exact time T_A . In this paper, we search for classes of Lie groups G such that any Σ has the following property: there exists an increasing sequence of open neighborhoods $(V_n)_{n \geq 0}$ of the identity in G such that the group can be decomposed in isochronous rings $W_n = V_{n+1} - V_n$. We characterize this property in algebraic terms and we show that three classes of Lie groups satisfy this property: completely solvable simply connected Lie groups, semisimple Lie groups and reductive Lie groups.

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1. Introduction

Let G be a connected Lie group with Lie algebra \mathfrak{g} . As in [Definition 5](#), consider a right invariant control system Σ on G . A subset A of G with non empty interior is said to be isochronous if there exists a positive uniform time T_A such that any two points in A can be connected by a positive trajectory of Σ at exactly T_A units of time. We also say that T_A is an isochronous time for Σ .

In this paper we analyze the existence of an isochronous decomposition for a large class of Lie groups. This notion appears naturally from the fact that for a general non compact Lie group, a global uniform time for Σ may not longer exist.

According to [Theorem 4.5](#) in [\[1\]](#), the reachable set $S_\Sigma(e, t)$ of Σ from the identity element e of G at exact time t satisfies

$$S_\Sigma(e, t) \subset \exp(tX)G_0$$

where X is the drift vector field of Σ and G_0 is the associated normal Lie subgroup of G with Lie algebra \mathfrak{L}_0 , the ideal generated by the control vectors.

$$\mathfrak{L}_0 = \text{Span} \{[P, Q] : P, Q \in D\} + \text{Span} \{P - Q : P, Q \in D\}$$

D is as in [Definition 5](#).

Furthermore, with respect to the G_0 -topology

$$cl(\text{int}_{G_0}(S_\Sigma(e, t))) = cl(S_\Sigma(e, t)).$$

As usual $\text{int}(P)$ means the interior of a set P and cl its closure. Thus, assuming the Lie algebra rank condition, it turns out that if the co-dimension of \mathfrak{L}_0 is 1 we cannot expect controllability at exact time. Therefore, we will not have controllability at uniform time. Of course, controllability itself is also a necessary condition for our study.

From the Lie algebra structure theory, see [\[2,3\]](#), specifically from the Levi decomposition, we know that any arbitrary Lie algebra \mathfrak{g} can be decomposed as $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$. Here, $\mathfrak{r} = \mathfrak{r}(\mathfrak{g})$ is the solvable radical of \mathfrak{g} and \mathfrak{s} its semisimple part. As appointed by an anonymous referee, it should be noted that \mathfrak{g} is just a direct sum of subspaces. However, \mathfrak{g} is a semidirect product of Lie algebras through the adjoint representation of \mathfrak{s} on \mathfrak{r} . In particular, there exists a connected Lie group $G = R \times_{\mathfrak{s}} S$ with Lie algebra \mathfrak{g} . In this context, R and S are Lie groups with Lie algebras \mathfrak{r} and \mathfrak{s} respectively. The group R is connected, normal, and solvable; S is connected and semisimple; and G is the semidirect product between R and S . The semidirect product in the group is defined through conjugacies.

On the other hand, if R is a non simply connected Abelian Lie group, [Example 11](#) shows the existence of a controllable invariant control system such that any reachable set has empty interior. In other words, this kind of space state is not suitable for us. We concentrate the analysis on the following classes of Lie groups:

$$\mathbb{G}_{CS} = \{R : R \text{ is cs simply connected}\}$$

$$\mathbb{G}_{SS} = \{S : S \text{ is semisimple}\}$$

$$\mathbb{G}_r = \{R \times S : R \in \mathbb{G}_{CS} \text{ and } R \text{ is abelian, } S \in \mathbb{G}_{SS}\}.$$

Recall that a solvable Lie group is said to be completely solvable, (cs), if for any element Z in its Lie algebra, the linear transformation $ad(Z)$ has just real eigenvalues.

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If the control range K is bounded, Lemma 4.4 in [4] shows that for any $t > 0$ the reachable sets $S_{\Sigma}(e, \leq t)$ and $S_{\Sigma}(e, t)$ are compact sets. So, if G is not compact, under the bounded admissible control class we cannot expect controllability at uniform time for the entire manifold.

Through the paper we look for topological and algebraic conditions to impose on G and Σ in order to decompose the space state in sets where any two states can be reached by a positive Σ -trajectory in a universal time. We get,

Theorem 1. *Let Σ an invariant control system on G . Suppose Σ is controllable, then $\mathcal{L}_0 = \mathfrak{g}$ if and only if G can be covered by an increasing sequence $(V_n)_{n \geq 0}$ of open isochronous neighborhoods of the identity.*

By analogy with vector spaces we call the set $W_n = V_{n+1} - V_n$ an *isochronous ring*. In particular, under the hypothesis of Theorem 1 we decompose the space state as $G = \bigcup_{n \geq 0} W_n$, where $V_0 = \emptyset$.

We first analyze the situation on \mathbb{G}_{cs} . Denote by $D_{\mathcal{U}}$ the Lie algebra of \mathfrak{g} generated by the control vectors of Σ , as in Definition 5,

$$D_{\mathcal{U}} = \text{Span}_{\mathcal{L}\mathcal{A}}\{Y^1, \dots, Y^m\}.$$

According to [5], see also [6], when $K = \mathbb{R}^m$ the hypothesis $D_{\mathcal{U}} = \mathfrak{g}$ is a necessary and sufficient condition for the controllability of Σ , when $G = R \in \mathbb{G}_{cs}$ is nilpotent, see also [7,8] for related topics. Furthermore, in [9] the author extends this result for a simply connected completely solvable Lie group, see also [10].

By working with the class of unrestricted control and assuming that the Lie algebra generated by the control vectors $D_{\mathcal{U}}$ is \mathfrak{g} in [11], the author makes an important contribution to the study of exact controllability at arbitrary time.

Theorem 2. *Let $\Sigma = (G, D)$ be an invariant control system on a connected Lie group G . Then $D_{\mathcal{U}} = \mathfrak{g}$ implies $S_{\Sigma}(e, T) = G$ for all $T > 0$.*

On the other hand, in the context of invariant systems on semisimple Lie groups, for any class of admissible controls the authors prove in [4]:

Theorem 3. *Let G be a connected and compact semisimple Lie group and Σ an invariant control system on G . If Σ satisfies the Lie algebra rank condition, then Σ is controllable at uniform time.*

In Section 3 we generalize this theorem as a particular case of Theorem 1, when $G \in \mathbb{G}_{ss}$. Actually, Theorem 3 says that the whole group G is an isochronous set for Σ . Of course, the compactness property of G is essential to find a global isochronous time for the space state. The proof of Theorem 3 is based on the following topological results, see [1].

Theorem 4. *Let M be a manifold whose universal cover space is compact. Then, every system having the accessibility property has the strong accessibility property.*

In [12], the authors give a more elementary proof of Theorem 3. Since the Lie algebra \mathfrak{g} of G is semisimple, in particular it does not contain ideals of codimension 1. It turns out that for any positive time t , the accessibility set $S_{\Sigma}(e, t)$ does not have empty interior in the G -topology. Therefore, the system has the strong accessibility property and the proof follows.

We apply the main results of the paper to the class of bilinear control systems in \mathbb{R}^n . As usual, we consider the Lie algebra \mathfrak{g} generated by the bilinear control system Σ_{Bil} such that the induced invariant control system Σ on $G \in \mathbb{G}$ satisfies the hypothesis of Theorem 1. Pick a point $x_0 \in \mathbb{R}^n$, then there exists a decomposition of $G(x_0)$ in isochronous rings.

This article is organized as follows. Section 2 contains some preliminaries; in particular, the definition of invariant control system and the isochronous set notion. In Section 3, we prove Theorem 1 in which we characterize the isochronous property in

algebraic terms and then we show that it is satisfied by the classes \mathbb{G}_{cs} , \mathbb{G}_{ss} and \mathbb{G}_r . In order to show the role of the unrestricted class of control and the simply connected hypothesis, in Section 3 we also comment on some results of controllability on nilpotent, solvable and completely solvable Lie groups, which appeared in [5,10,9]. Section 4 contains a number of examples.

2. Preliminaries

From a geometric point of view, a control system is determined by a manifold: the space of states and a family of differential equations. In our special case, the system reads

Definition 5. An invariant control system $\Sigma = (G, D)$ is determined by a Lie group G with Lie algebra \mathfrak{g} and the dynamics D given by

$$D = \left\{ X^u = X + \sum_{j=1}^m u_j Y^j : u \in \mathcal{U} \right\}.$$

Here, the drift vector field X and the control vectors $Y^j, j = 1, 2, \dots, m$, are elements of \mathfrak{g} considered as right invariant vector fields. Any admissible control u is an element of the class

$$\mathcal{U}_K = \{u : \mathbb{R} \rightarrow K \subset \mathbb{R}^m \mid u(t) \text{ is a piecewise constant function}\}.$$

The possibilities for the control range are: the unrestricted controls $K = \mathbb{R}^m$; the bounded controls $K = [-1, 1]^m$, and the bang–bang controls $K = \partial[-1, 1]^m$. Except if explicitly mentioned the results are valid for each of the three mentioned classes of controls.

For each $Z \in D$, the associated flow $(Z_t)_{t \in \mathbb{R}}$ is a 1-parameter subset of $\text{Diffeom}(G)$, the set of diffeomorphisms of G . As usual, we consider the sets

$$G_{\Sigma} = \{Z_{t_1}^1 \circ Z_{t_2}^2 \circ \dots \circ Z_{t_k}^k : Z^i \in D, t_i \in \mathbb{R} \text{ and } r \in \mathbb{N}\}$$

and

$$S_{\Sigma} = \{Z_{t_1}^1 \circ Z_{t_2}^2 \circ \dots \circ Z_{t_k}^k : Z^i \in D, t_i \geq 0 \text{ and } r \in \mathbb{N}\}$$

which are a group and a semigroup of $\text{Diffeom}(G)$, respectively.

Definition 6. Let $\Sigma = (G, D)$ be an invariant control system

- (i) Σ is said to be *Transitive* if G_{Σ} acts transitively on G .
- (ii) Σ is said to be *Controllable* if S_{Σ} acts transitively on G .
- (iii) A subset A of G with non empty interior is said to be *Isochronous* if there exists a time $T_A > 0$ such that for any arbitrary two elements $x, y \in A$ there exists $\varphi = Z_{t_1}^1 \circ Z_{t_2}^2 \circ \dots \circ Z_{t_r}^r \in S_{\Sigma}$ with $\sum_{i=1}^r t_i = T_A$ and $\varphi(x) = y$. In this case, T_A is said to be an isochronous time to A .

Let M be an analytical manifold and D a family of analytic vector fields on M . As explained in [1], the zero time orbit of the system $\Sigma = (M, D)$ through the initial condition $x \in M$ is the set:

$$\mathcal{O}_0^M(x) = \{Z_{t_1}^1 \circ Z_{t_2}^2 \circ \dots \circ Z_{t_k}^k(x) : Z^i \in D, t_i \in \mathbb{R}, r \in \mathbb{N} \text{ and } \sum_{i=1}^r t_i = 0\}.$$

In our specific case $\mathcal{O}_0(e) = G_0$ is a normal subgroup of G with Lie algebra

$$\mathcal{L}_0 = \text{Span}\{[P, Q] : P, Q \in D\} + \text{Span}\{P - Q : P, Q \in D\}.$$

Just observe that by definition, for every couple $P, Q \in D$ the Lie brackets

$$[P, Q] = \left(\frac{d}{dt} \right)_{t=0+} Q_{-\sqrt{t}} \circ P_{-\sqrt{t}} \circ Q_{\sqrt{t}} \circ P_{\sqrt{t}}(e) \in \mathcal{L}_0.$$

In this paper we assume that Σ satisfies the Lie algebra rank condition, i.e., $\text{Span}_{\mathcal{L}\mathcal{A}}(D) = \mathfrak{g}$. In particular we get:

Lemma 7. \mathcal{L}_0 is an ideal of codimension 0 or 1.

As a consequence, the Lie subgroup G_0 is of codimension 0 or 1. Let t be a positive time and denote by $S_\Sigma(e, t)$ the reachable set of Σ at exact time t :

$$S_\Sigma(e, t) = \{g \in G : \exists \varphi \in S_\Sigma \text{ with } \varphi(e, t) = g\}.$$

In the sequel we recall a fundamental result for this kind of sets, [1].

Proposition 8. *Let $\Sigma = (G, D)$ be a right invariant control system on a connected Lie group G . Then, for any $t > 0$,*

- (i) $S_\Sigma(e, t) \subset \exp(tX)G_0$
- (ii) $S_\Sigma(e, t)$ has nonvoid interior in the topology of G_0 . Furthermore, $cl(\text{int}_{G_0}(S_\Sigma(e, t))) = cl(S_\Sigma(e, t))$.

It is very well known that the orbit $G_\Sigma(e)$ is a connected Lie subgroup of G with Lie algebra $\text{Span}_{\mathcal{L}, \mathcal{A}}(D)$. Since we consider transitive systems we suppose that Σ satisfies the Lie Algebra Rank Condition: (LARC), i. e., $\text{Span}_{\mathcal{L}, \mathcal{A}}(D) = \mathfrak{g}$. In other words, $G = G_\Sigma(e)$.

3. Isochronous ring decomposition

Let G be a connected Lie group with Lie algebra \mathfrak{g} and Σ an invariant control system on G . We have,

Proposition 9. *If Σ is controllable and $\mathcal{L}_0 = \mathfrak{g}$, there exists a positive time t_+ such that*

$$G = \bigcup_{n \geq 1} S_\Sigma(e, nt_+).$$

Proof. Since G is connected, it follows that $G_0 = G$. Therefore, for any positive time t the set $S_\Sigma(e, t)$ has nonvoid interior and $\text{int } S_\Sigma(e, t)$ is dense in $cl(S_\Sigma(e, t))$, in the topology of the manifold G .

Consider the controllable right invariant system $-\Sigma = (G, -D)$. We have,

$$\bigcup_{t \geq 0} S_\Sigma(e, t) = G = \bigcup_{s \geq 0} S_{-\Sigma}(e, s)$$

where $S_{-\Sigma}(e, s)$ denotes the accessible set of $-\Sigma$ at time $s \geq 0$.

We claim: there are positive times t_0 and s_0 such that

$$\text{int } S_\Sigma(e, t_0) \cap \text{int } S_{-\Sigma}(e, s_0) \neq \emptyset.$$

If not, for each $t > 0$

$$\text{int } S_\Sigma(e, t) \cap \text{int } S_{-\Sigma}(e, s) = \emptyset, \quad \text{for all } s > 0.$$

Thus,

$$\text{int } S_\Sigma(e, t) \cap cl(\text{int } S_{-\Sigma}(e, s)) = \emptyset, \quad \text{for all } s > 0.$$

So, there exists a positive time t such that $\text{int } S_\Sigma(e, t) \cap G = \emptyset$, which is a contradiction. Therefore, we find $x \in G$ with

$$x \in \text{int } S_\Sigma(e, t_0) \cap \text{int } S_{-\Sigma}(e, s_0).$$

As usual, we denote by \exp the exponential map from the Lie algebra \mathfrak{g} to the Lie group G . It follows that there are points $Z_i \in -D$, $s_i \geq 0$ and $\sum_{i=1}^r s_i = s_0$ such that

$$x = \exp(s_r Z_r) \cdots \exp(s_1 Z_1) e.$$

Then,

$$e = \exp(s_1 W_1) \cdots \exp(s_r W_r) x, \quad \text{with } W_i = -Z_i \in D.$$

Since x is also an element of $\text{int } S_\Sigma(e, t_0)$ and the left translations are G -diffeomorphisms, we obtain:

$$e \in \exp(s_1 W_1) \cdots \exp(s_r W_r) \cdot \text{int } S_\Sigma(e, t_0) \subset \text{int } S_\Sigma(t_0 + s_0).$$

Let $t_+ = t_0 + s_0$. Since $\text{int } S_\Sigma(e, t_+)$ is a neighborhood of the identity and G is a connected topological group,

$$G = \bigcup_{n \geq 1} \text{int } S_\Sigma(e, t_+)^n.$$

Since S_Σ is a semigroup, for any natural number n we have

$$S_\Sigma(e, t_+)^n \subset S_\Sigma(e, nt_+).$$

Finally, we get

$$G = \bigcup_{n \geq 1} S_\Sigma(e, nt_+). \quad \square$$

Now, we are able to prove

Theorem 10. *Suppose that Σ is an invariant controllable system on G . Then, $\mathcal{L}_0 = \mathfrak{g}$ if and only if there exists an increasing sequence of isochronous open neighborhoods V_n of the identity that decomposes G in isochronous rings.*

Proof. For the above proposition, there exists a time t_+ such that $e \in \text{int } S_\Sigma(e, t_+)$. Since $-\Sigma$ is also controllable, there exist a time t_- such that $e \in \text{int } S_{-\Sigma}(e, t_-)$. Let $V_1 = \text{int } S_\Sigma(e, t_+) \cap \text{int } S_{-\Sigma}(e, t_-)$. We claim: V_1 is an isochronous set. Let x, y be in V_1 , then there exists a trajectory of Σ steering e to y in t_+ units of time. Also, there exists a trajectory of $-\Sigma$ steering e to x in t_- units of time. Thus, there exists an admissible control of Σ that steers x to e in t_- units of time. By concatenation, there exists a trajectory of Σ steering x to y in $t_- + t_+$ units of time. Then, V_1 is an isochronous set. For $n \geq 1$ define $V_n = V_1^n$ and $V_0 = \emptyset$. By the semigroup property of $S_\Sigma(e)$, it follows that V_n is an isochronous neighborhood of e . Therefore, $G = \bigcup_{n \geq 0} V_n$, where $W_n = V_{n+1} - V_n$ and $V_0 = \emptyset$ is the desired decomposition. For the converse, note that there exists a time t_1 such that $S_\Sigma(e, t_1)$ has non empty interior. We know that $S_\Sigma(e, t_1) \subset \exp(t_1 X)G_0$, thus G_0 has non empty interior. Since G is connected, we obtain $G_0 = G$ and then $\mathcal{L}_0 = \mathfrak{g}$. \square

Example 11. From the Lie structure theory we know that any connected Abelian Lie group G has the form $G = T^k \times \mathbb{R}^n$. Here, T^k is the k -dimensional torus $S^1 \times \cdots \times S^1$ and \mathbb{R}^n is the usual n -dimensional Euclidean space, which is the adjacent topological space of the simply connected part of G . Next, we show a controllable invariant control system Σ on G such that for any positive time t the accessible set $S_\Sigma(e, t)$ has empty interior with respect to the G -topology. In fact, assume $k = 1$ and consider the transitive invariant control system $\Sigma = (S^1 \times \mathbb{R}^n, D)$ where

$$D = \left\{ X^u = X + \sum_{j=1}^m u_j Y^j : u \in \mathcal{U} \right\}.$$

Here, the drift vector field $X = \frac{\partial}{\partial s}$ is the infinitesimal generator of the circle S^1 , the control vectors $Y^j, j = 1, 2, \dots, n$ generates \mathbb{R}^n and $\mathcal{U}_{\mathbb{R}^m}$ is the class of unrestricted admissible controls. For any positive time t ,

$$S_\Sigma(e, t) = \left(\exp t \frac{\partial}{\partial s} (1, 0), S_{\Sigma_{\mathbb{R}^n}}(0, t) \right) = \exp t \frac{\partial}{\partial s} (1, 0) \times \mathbb{R}^n$$

where $S_{\Sigma_{\mathbb{R}^n}}$ is the semigroup associated with the system Σ restricted to \mathbb{R}^n . So, we cannot expect Theorem 1 for the class of Abelian Lie groups. In fact,

$$\text{int}_G S_\Sigma(e, t) = \emptyset.$$

In the next two subsections we are looking for topological and algebraic conditions to meet the hypothesis of Theorem 1.

3.1. Solvable systems

In order to show the role played by the unrestricted class of control and the additional simply connected hypothesis imposed on a completely solvable Lie group G , we comment some controllability results appearing in [5,9]. Furthermore, the notion Lie saturate and strong Lie saturate of a system is also related to our study.

By definition, a Lie algebra \mathfrak{g} is solvable if there exists a natural number $k \geq 1$ such that its derivative series stabilizes at the origin, i.e.,

$$0 = \text{ad}^k(\mathfrak{g}) = [\text{ad}^{k-1}(\mathfrak{g}), \text{ad}^{k-1}(\mathfrak{g})] \subset \cdots \subset \text{ad}^1(\mathfrak{g}) \subset \text{ad}^0(\mathfrak{g}) \\ = \mathfrak{g}, \quad \text{ad}^0 = \text{Id}.$$

A Lie group is said to be solvable if its Lie algebra is solvable. For the class of invariant control systems on a solvable Lie group G , the controllability property can be characterized as follows, see [10].

Theorem 12. Σ is controllable if and only if

- (i) $\text{Span}_{\mathcal{L}\mathcal{A}}(D) = \mathfrak{g}$, and
- (ii) D is not contained in a half space of \mathfrak{g} bounded by a subalgebra.

In order to have a more direct way to check controllability for connected and simply connected solvable Lie groups, in [9] the author introduces the following notion: a solvable Lie algebra is said to be completely solvable if

$$\text{Spec}(\text{ad}(Z)) \subset \mathbb{R}, \quad \text{for every } Z \in \mathfrak{g}$$

to obtain:

Theorem 13. Let Σ be an invariant control system on a Lie group $G \in \mathbb{G}_{\text{cs}}$. Then, Σ is controllable if and only if

$$D_{\mathcal{U}} = \text{Span}_{\mathcal{L}\mathcal{A}} \{Y^1, \dots, Y^m\} = \mathfrak{g}.$$

If \mathfrak{g} is a nilpotent Lie algebra, for any $Z \in \mathfrak{g}$ the set $\text{Spec}(\text{ad}(Z))$ reduces to zero. So, \mathfrak{g} is a completely solvable Lie algebra. In particular, Theorem 13 is a perfect generalization of the results obtained in [5], see also [6], when $G \in \mathbb{G}_{\text{cs}}$ is nilpotent.

To study controllability on semisimple Lie groups, the authors introduced in [13] the notion of Lie saturate $LS(\Gamma)$ of a set $\Gamma \subset \mathfrak{g}$ of invariant vector fields. In our case, it is possible to characterize this notion as follows, [14]

$$LS(D) = \{A \in \text{Span}_{\mathcal{L}\mathcal{A}}(D) : \exp^{\mathbb{R}^+} A \subset \text{cl } S_{\Sigma}(e)\}.$$

It turns out that for any $j = 1, 2, \dots, m$,

$$\lim_{u \rightarrow \pm\infty} \frac{X + u_j Y^j}{|u_j|} = \pm Y^j.$$

In particular, $D_{\mathcal{U}}$ is contained in the Lie saturate $LS(D)$. Controllability follows from the *Lie saturate test* given by: Let G be a connected Lie group. A system (G, D) is controllable if and only if $LS(D) = \mathfrak{g}$. See [13] and also Theorem 4.1 in [14]. Reciprocally, if $D_{\mathcal{U}}$ is not \mathfrak{g} there exists a subalgebra L_1 of codimension 1 containing $D_{\mathcal{U}}$. Therefore,

$$D = \left\{ X + \sum_{j=1}^m u_j Y^j : u \in \mathcal{U} \right\} \subset D_{\mathcal{U}} \subset \mathbb{R}^+ X + L_1.$$

1. If $X \notin L_1$ it follows that $\mathbb{R}^+ + L_1$ is a half space bounded by the subalgebra L_1 . Thus, by Theorem 12 the system is not controllable since $\mathbb{R}^+ + L_1$ contains D .
2. If $X \in L_1$ then $\mathbb{R}^+ X + L_1 = L_1$ contains D . So, the system cannot be controllable since it is not even transitive.

The proof of the main result in [5] uses the co-adjoint representation of G and through the notion of a symplectic vector the author builds an increasing differentiable function on any positive trajectory of the system. The simply connected hypothesis plays the following role,

Proposition 14. Assume $D_{\mathcal{U}}$ is a subalgebra of co-dimension 1 in \mathfrak{g} and denote by H the connected Lie subgroup of G with Lie algebra $D_{\mathcal{U}}$. We have

- (i) If H is closed, Σ is controllable if and only if $X \in D_{\mathcal{U}}$ and the homogeneous space G/H is compact
- (ii) If H is not closed, Σ is controllable if and only if $X \notin D_{\mathcal{U}}$.

If Σ is controllable the symplectic vector theory implies that $D_{\mathcal{U}}$ is an ideal of co-dimension 0 or 1. Co-dimension 1 is not allowed since in that case $G/G_0 \cong S^1$ and then G cannot be simply connected.

By assuming that the Lie algebra generated by the control vectors is \mathfrak{g} in [11] the author makes an important contribution to the study of exact controllability at any time.

Theorem 15. Let $\Sigma = (G, D)$ be an invariant control system on a connected Lie group G . Then $D_{\mathcal{U}} = \mathfrak{g}$ implies $S_{\Sigma}(e, T) = G$ for all $T > 0$.

Related to exact controllability we also have the following notion

Definition 16. The strong Lie saturate of D is the largest subset $LSS(D)$ of $\text{Span}_{\mathcal{L}\mathcal{A}}(D)$ with the property that $\text{cl } S_{LSS(D)}(e, \leq T) = \text{cl } S_{\Sigma}(e, \leq T)$, for all $T > 0$.

Theorem 17. Let G be a completely solvable simply connected Lie group and $\Sigma = (G, D)$ an invariant control system. The following statements are equivalent: (1) Σ is controllable, (2) $D_{\mathcal{U}} = \mathfrak{g}$, (3) $LSS(D) = \mathfrak{g}$, (4) $S_{\Sigma}(e, \leq T) = G$ for all $T > 0$, (5) $S_{\Sigma}(e, T) = G$ for all $T > 0$.

Proof. The equivalence between (1) and (2) is given in [9]. First, we prove the equivalence between (3) and (4). Suppose $LSS(D) = \mathfrak{g}$, then

$$\text{cl}(S_{\Sigma}(e, \leq T)) = \text{cl}(S_{LSS(D)}(e, \leq T)) = G, \quad \text{for all } T > 0.$$

Let $0 < \epsilon < T$, therefore

$$G = \text{int } \text{cl } S_{\Sigma}(e, \leq T - \epsilon) \subset S_{\Sigma}(e, \leq T),$$

see [15]. Reciprocally, suppose $S_{\Sigma}(e, \leq T) = G$ for all $T > 0$. Then $LSS(D) = \text{Lie}(D)$. Since the system is controllable, we have $\text{Lie}(D) = \mathfrak{g}$, which proves our claim. Finally, (4) and (5) are equivalent. In fact, just by definition (5) \Rightarrow (4) \Rightarrow (1) and condition (2) implies (5) from Theorem 16. \square

Therefore, we obtain

Theorem 18. Let Σ be an invariant control system on $G \in \mathbb{G}_{\text{cs}}$. If Σ satisfies any of the conditions in Theorem 17, then G is an isochronous set for every positive time.

3.2. Semisimple systems

According to the previous section, the class of controllable invariant control systems on a simply connected completely solvable Lie group with unrestricted controls has an universal isochronous set: the own space state G . Furthermore, G is reached by $S_{\Sigma}(e, t)$ for any positive time t . In this section we show that for controllable invariant systems on semisimple Lie groups this is not the case.

Definition 19. A Lie algebra \mathfrak{g} is said to be simple if it has a dimension bigger than 1 and does not contain non trivial ideals. \mathfrak{g} is said to be semisimple if its solvable radical $r(\mathfrak{g})$ is the Lie subalgebra zero. Here, $r(\mathfrak{g})$ means the direct sum of all solvable Lie subalgebras of \mathfrak{g} .

We first observe that

Lemma 20. Let \mathfrak{g} be a semisimple Lie algebra. Then \mathfrak{g} doesn't contain ideals of codimension 1.

Proof. Let \mathfrak{h} an ideal of \mathfrak{g} of codimension 1. Thus, $\mathfrak{g}/\mathfrak{h}$ is a semisimple Lie algebra, [3], which is a contradiction. \square

Therefore, we get

Theorem 21. Let $\Sigma = (G, D)$ be a right invariant control system with $G \in \mathbb{G}_{ss}$. If Σ is controllable, G can be covered by an increasing sequence of isochronous open neighborhoods $(V_n)_{n \geq 0}$ of the identity. In particular, G can be decomposed in isochronous rings.

Proof. \mathfrak{L}_0 is an ideal of codimension 0 or 1, then by Lemma 20 it has codimension 0. In particular, $\mathfrak{L}_0 = \mathfrak{g}$. Since we assume controllability, the proof follows from Theorem 1. \square

Corollary 22. Let G be a connected and compact semisimple Lie group. Let $\Sigma = (G, D)$ be a right invariant control system. Then G itself is an isochronous set.

Proof. Under the topological assumption on G the proof of Theorem 10 shows the existence of a natural number n such that $V^n = G$. \square

However, the uniform time is not arbitrary, see [12].

3.3. Reductive Lie groups

We recall that the Lie algebra \mathfrak{g} of a reductive Lie group G has the form $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) + \mathfrak{s}$, where $\mathfrak{z}(\mathfrak{g})$ is the center of \mathfrak{g} , \mathfrak{s} is a semisimple Lie algebra which is also an ideal of \mathfrak{g} . Therefore, in this case \mathfrak{g} and G are direct products of Lie algebras and Lie groups respectively.

We consider the space state $G = Z \times S$ where Z and S are connected Lie groups with Lie algebras \mathfrak{z} and \mathfrak{s} respectively. Here, Z is abelian and simply connected and S is semisimple.

We need the following fact:

Lemma 23. Let \mathfrak{g} be a reductive Lie algebra, with $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) + \mathfrak{s}$. Assume that $\mathfrak{s} = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \cdots \oplus \mathfrak{s}_r$ where $\mathfrak{s}_1, \dots, \mathfrak{s}_r$ are simple Lie subalgebras. If \mathfrak{i} is an ideal of \mathfrak{g} then $\mathfrak{i} = \mathfrak{j} \oplus \mathfrak{s}_{\mathfrak{e}_1} \oplus \cdots \oplus \mathfrak{s}_{\mathfrak{e}_r}$. Where \mathfrak{j} is a vector subspace of $\mathfrak{z}(\mathfrak{g})$.

Proof. Consider the canonical projection $\pi : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. In particular $\pi(\mathfrak{i})$ is an ideal of $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \cong \mathfrak{s}$. Since \mathfrak{s} is semisimple, $\pi(\mathfrak{i})$ has the form

$$\pi(\mathfrak{i}) \cong \mathfrak{s}_{\mathfrak{e}_1} \oplus \cdots \oplus \mathfrak{s}_{\mathfrak{e}_r}$$

where $\mathfrak{s}_{\mathfrak{e}_1}, \dots, \mathfrak{s}_{\mathfrak{e}_r}$ are elements of the decomposition of \mathfrak{s} , [3]. Therefore,

$$\mathfrak{i} \subset \pi^{-1}\pi(\mathfrak{i}) = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{s}_{\mathfrak{e}_1} \oplus \cdots \oplus \mathfrak{s}_{\mathfrak{e}_r}.$$

Since $\mathfrak{i} \cap \mathfrak{s}$ is an ideal of \mathfrak{s} , there exists a vector subspace \mathfrak{j} of $\mathfrak{z}(\mathfrak{g})$ such that

$$\mathfrak{i} = \mathfrak{j} \oplus \mathfrak{s}_{\mathfrak{e}_1} \oplus \cdots \oplus \mathfrak{s}_{\mathfrak{e}_r}. \quad \square$$

Each invariant vector field W in \mathfrak{g} can be decomposed as $W = W_{\mathfrak{a}} + W_{\mathfrak{s}}$. It turns out that any invariant control system $\Sigma = (G, D)$ induces two subsystems: the Abelian system $\Sigma_{Ab} = (R, D_{\mathfrak{a}})$ and the semisimple system $\Sigma_{Semis} = (S, D_{\mathfrak{s}})$. With the previous assumptions and by joining the pieces, we obtain

Theorem 24. Let Σ be a controllable system on a reductive Lie group G with $\mathcal{U}_{\mathbb{R}^m}$. Then, G can be covered by an increasing sequence $(V_n)_{n \geq 0}$ of open isochronous neighborhoods of the identity. In particular, G decomposes in isochronous rings.

Proof. From the hypothesis, the induced systems: $\Sigma_{Ab} = (R, D_{\mathfrak{a}})$ and $\Sigma_{Semis} = (S, D_{\mathfrak{s}})$ are also controllable. By Theorem 13 the ideal generated by the control vectors of $\Sigma_{Ab} = (R, D_{\mathfrak{a}})$, is $\mathfrak{z}(\mathfrak{g})$. On the other hand, Lemma 20, shows that the ideal generated by the control vectors of Σ_{Semis} is \mathfrak{s} . Thus, in the original system, the ideal

generated by the control vectors is $\mathfrak{z}(\mathfrak{g}) + \mathfrak{s} = \mathfrak{g}$, by Lemma 23. From Theorem 10 there exists an increasing sequence of isochronous open neighborhoods V_n of the identity such that decomposes G in isochronous rings. \square

In particular, we prove

Corollary 25. Let $\Sigma = (G, D)$ be a controllable right invariant control system with $G \in \mathbb{G}$, where \mathbb{G} stands for $\mathbb{G}_{cs}, \mathbb{G}_{ss}, \mathbb{G}_r$.

- (i) If G is compact, then G is an isochronous set. Additionally
- (ii) Any bounded subset C of G is an isochronous set.

3.4. General systems

Let \mathfrak{g} be a Lie algebra and $\mathfrak{r} = \mathfrak{r}(\mathfrak{g})$ its solvable radical. From the Levi Decomposition, there exists an algebra \mathfrak{s} which is complementary to \mathfrak{g} . Precisely, $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ decompose as a direct sum of subspaces such that $\mathfrak{g}/\mathfrak{r} \cong \mathfrak{s}$ is semisimple. In other words, \mathfrak{s} is not always an ideal. However, $\mathfrak{g} = \mathfrak{r} \oplus_{\mathfrak{s}} \mathfrak{s}$ is a semidirect product of Lie algebras. In this case \mathfrak{s} acts on \mathfrak{r} through the adjoint representation as follows:

$$\mathfrak{g} = \mathfrak{r} \times \circ \oplus_{\mathfrak{s}} \circ \times \mathfrak{s}$$

with the bracket defined on $(R_1, S_1), (R_2, S_2) \in \mathfrak{g}$ by

$$[(R_1, S_1), (R_2, S_2)] = ([R_1, R_2] + ad(S_1)R_2 - ad(S_2)R_1, [S_1, S_2]).$$

Furthermore, the algebra \mathfrak{s} which is complementary to the solvable radical $\mathfrak{r}(\mathfrak{g})$ is not unique. Each of these algebras is called a Levi component of \mathfrak{g} . In this way, \mathfrak{g} decomposes as a direct sum of a subalgebra isomorphic to \mathfrak{s} and an ideal isomorphic to \mathfrak{r} , both closed by the bracket. This is a more general setup where we could try to find a way to generalize the results in the previous sections. However, in the sequel we show an example given by an anonymous referee that for general systems, controllability of the system on the group implies controllability only on the semisimple factor.

Let G be the Special Euclidean group $SE(3)$, this is the semidirect product of the the simple real special orthogonal group $SO(3)$ and the simply connected Abelian group \mathbb{R}^3 .

Consider the left invariant control system:

$$\frac{d}{dt}(R, v) = (R(u_1 B_1 + u_2 B_2 + u_3 B_3), Rv)$$

where $v \in \mathbb{R}^3$ is a constant, non-zero vector, and B_1, B_2, B_3 is a basis of the Lie algebra $\mathfrak{so}(3)$ of skew-symmetric matrices. This system is controllable, however the induced system on \mathbb{R}^3 is not controllable.

3.5. Bilinear control systems

It is well known that any invariant control system $\Sigma = (G, D)$ induces a homogeneous system on any manifold M when the connected Lie group G acts transitively on M . Precisely, the system is given by $p(\Sigma) = (M, p_*(\mathcal{D}))$. Here, p_* denotes the differential of the p -action. It turns out that for every $x \in M$ and any positive time t

$$p(S_{\Sigma}(e, t), x) = S_{p(\Sigma)}(x, t).$$

Therefore, in the conditions of Theorem 23, we obtain

Corollary 26. If G acts transitively on M then, the sequence $(p(W_n))_{n \geq 0}$ of isochronous rings of $p(\Sigma)$ decomposes M .

4. Examples

1. Let G be the simply connected nilpotent Heisenberg Lie group of dimension 3

$$G = \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{g} = \mathbb{R}X + \mathbb{R}Y^1 + \mathbb{R}Y^2$ is obtained through the generators $X = \frac{\partial}{\partial x_1}$, $Y^1 = x_3 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$, $Y^2 = \frac{\partial}{\partial x_3}$. The only non-vanishing Lie bracket is $[Y^2, Y^1] = X$. We consider the system Σ

$$\dot{x} = (X + u_1 Y^1 + u_2 Y^2)x, \quad x \in G, u \in \mathbb{R}^2.$$

Since $D_{\mathfrak{g}} = \mathfrak{g}$, Theorem 17 shows that Σ is controllable. Therefore, Theorem 18 shows that G is an isochronous set with isochronous time t , for all $t > 0$.

2. On the noncompact simple Lie group $G = SL(2, \mathbb{R})$, we consider the system

$$\dot{x} = [X + uY]x, \quad x \in G, u \in \mathbb{R}$$

where $X = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $Y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. According to [13], see also [14], the system is controllable. Thus, from Theorem 23, G can be decomposed by isochronous rings of Σ .

3. Consider the bilinear control system Σ on \mathbb{R}^3 determined by the dynamic

$$\dot{x} = [X + uY]x, \quad x \in \mathbb{R}^3, -1 \leq u \leq 1$$

which is induced by its lifting the invariant control system on the orthogonal group $SO(3)$. Here X and Y generate the Lie algebra $\mathfrak{so}(3)$ of the skew symmetric real matrices of order 3. In [12], the authors show that Σ is controllable at uniform time. But, the uniform time cannot be arbitrary. It must be equal to or bigger than π . Actually, through the homogeneous system $p_*(\Sigma)$ induced on the sphere S^2 they prove that to connect any point of the Equator to the north pole, $\frac{\pi}{2}$ units of time are necessary. Thus, in general the isochronous time is not arbitrary.

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References

- [1] H.J. Sussmann, V. Jurdjevic, Controllability of Nonlinear Systems, *Journal of Differential Equations* 12 (1972) 95–116.
- [2] S. Helgason, *Differential geometry*, in: *Lie groups and Symmetric Spaces*, Academic Press, New York, 1978.
- [3] Luiz San Martin, *Algebras de Lie*, Editorial UNICAMP, Brasil.
- [4] V. Jurdjevic, H.J. Sussmann, Control systems on Lie groups, *Journal Differential Equations* 12 (1972) 313–329.
- [5] V. Ayala, *Controllability of Nilpotent Systems*, vol. 32, Banach Center Publications, 1995, 35–46.
- [6] V. Ayala, Vergara, Co-adjoint representation and controllability, *Proyecciones* 11 (1992) 37–48.
- [7] V. Ayala, L. San Martin, Controllability properties of a class of control systems on Lie groups, *Lectures Notes in Control and Information Science* 1 (258) (2001) 83–92.
- [8] V. Ayala, J. Tirao, Linear control systems on Lie groups and controllability, *American Mathematical Society, Series: Symposia in Pure Mathematics* 64 (1999) 47–64.
- [9] Y. Sachkov, Controllability of hypersurface and solvable invariant systems, *Journal of Dynamical and Control Systems* 2 (1) (1996) 55–67.
- [10] J.D. Lawson, Maximal subsemigroups of Lie groups that are total, *Proceedings of Edinburgh Mathematical Society*. 30 (1985) 479–501.
- [11] H. Kunita, Support of diffusion processes and controllability problems, in: K. Ito (ed.), *Proceedings International. Symposium on Stochastic Differential Equations*, Wiley, New York, 1978, pp. 163–185.
- [12] V. Ayala, J. Ayala-Hoffmann, I. Tribuzy, Controllability of invariant control systems at uniform time, *Kybernetika* (2009) (in press).
- [13] V. Jurdjevic, I. Kupka, Control systems on semisimple Lie groups and their homogeneous spaces, *Annals of Institute Fourier, Grenoble* 32 (4) (1981) 151–179.
- [14] Y. Sachkov, *Control theory on Lie groups*, Lecture Notes SISSA (2006).
- [15] V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, 1997.