Journal of Dynamical and Control Systems, Vol. 18, No. 3, July 2012, 323–338 (©2012)
DOI: 10.1007/s10883-012-9146-3

SOLUTIONS OF SINGULAR CONTROL SYSTEMS ON LIE GROUPS

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Abstract. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. A singular control system $\mathcal{S}_G$ on G is defined by a pair (E, D) of $\mathfrak{g}$-derivations. Through a fiber bundle decomposition of TG in [1], the authors decompose $\mathcal{S}_G$ in two subsystems $\mathcal{S}_{G/V}$ and $\mathcal{S}_V$, as in the linear case on Euclidean spaces, see for instance [9]. Here, $V \subset G$ is the Lie subgroup with Lie algebra $\mathfrak{v}$, the generalized 0-eigenspace of E. On the other hand, D defines the drift vector field of the system. We assume that the subspace $\mathfrak{v}$ is invariant under D. With this hypothesis we show a process to determine the solution of $\mathcal{S}_G$ through every state $x = yv$, where v is any admissible initial condition on V. From this information, we are able to build the global solution. Finally, in order to illustrate our processes we develop some examples on nilpotent simply connected Lie groups.

1. Introduction

The class of singular control systems involves algebraic relations between the derivatives of their states. The study of such systems on $\mathbb{R}^n$ began at 1970's with the work of the authors in [8], and the subject has been well developed on Euclidean spaces, see [5]. In [1] the authors extend this notion to Lie groups.

In the sequel we denote by G an arbitrary connected finite-dimensional Lie group with Lie algebra $\mathfrak{g}$ spanned by the right invariant vector fields on G. We also denote by $\partial\mathfrak{g}$ the set of all $\mathfrak{g}$–derivations which is the Lie algebra of the linear transformations $E : \mathfrak{g} \to \mathfrak{g}$ such that $\forall\ X, Y \in \mathfrak{g}$, $E[X,\ Y] = [EX,\ Y] + [X,\ EY]$.

2000 *Mathematics Subject Classification.* 93B03, 93B11, 93B27.

Key words and phrases. Lie algebra derivation, Jordan decomposition, singular control system, homogeneous space, algebraic differential equations.

This research was partially supported by Proyectos FONDECYT No. 1100375, Programa AMAZONAS SENIOR Processo Número: 507/2007, FAPEAM; Proyecto FONDECYT No. 3100137, and INCTMat–Avanço Global e Integrado da Matemática Brasileira e Contribuçoes a Região, CAPES, UFAM, Brasil.

We recall that $\mathfrak{g}$ is isomorphic to the tangent space T_eG of G at the identity element e. Thus, a right invariant vector field Y on G is determined by its value at e. In particular, $Y(x) = (dr_x)_e Y(e)$ and its flow is given by $Y_t(x) = r_x(\exp(tY(e)))$, where as usual r_x denotes the right translation by $x \in G$ and (dr_x) its derivative. Here, $\exp : \mathfrak{g} \to G$ is the exponential map.

By definition, a singular control system $\mathcal{S}_G$ on G is determined by the family of differential equations:

$$\mathcal{S}_G : \; E_{x(t)}\left(\dot{x}(t)\right) = X\left(x(t)\right) + \sum_{j=1}^{m} u_j(t) Y^j\left(x(t)\right), \quad x(t) \in G, \tag{1}$$

parametrized by $u \in \mathcal{U} = \{u : \mathbb{R} \longrightarrow \mathbb{R}^m, \; u \text{ piecewise constant function}\}$, the set of piecewise constant admissible controls with values on $\mathbb{R}^m$. E is a noninvertible derivation of $\mathfrak{g}$, this is, an element of $\partial\mathfrak{g}$. The drift vector field X belongs to the normalizer of $\mathfrak{g}$ inside of the Lie algebra $\mathcal{X}(G)$ of all smooth vector fields on G, i.e.,

$$\mathrm{norm}_{\mathcal{X}(G)}(\mathfrak{g}) = \{X \in \mathcal{X}(G) | \; [X, Y] \in \mathfrak{g} \text{ for all } Y \in \mathfrak{g}\}.$$

Here $[X, Y]$ is the Lie bracket of X and Y, and the control vectors Y^j, $j = 1, 2, \cdots, m$, are elements in $\mathfrak{g}$. The operator $E_x : T_xG \longrightarrow T_xG$ is defined by $(l_x)_* \circ E \circ (l_{x^{-1}})_*$, where $E : T_eG \longrightarrow T_eG$, and $l_x : G \longrightarrow G$ denote the left translations by $x \in G$ with $(l_x)_*$ its derivative whose inverse is $(l_{x^{-1}})_* : T_xG \longrightarrow T_eG$.

In the sequel we collect all the hypotheses that we need. Throughout this paper the subgroup V of G will be assumed to be closed, because in this case the quotient set G/V is a homogeneous space, see for instance [9]. In order to decompose G as the product $G/V \times V$, we assume that E is an integrable derivation which means that $\mathfrak{g} = \mathfrak{v} + \mathfrak{h}$, where $\mathfrak{h}$ is a subalgebra and $\mathfrak{v}$ is the corresponding Lie algebra of V. Furthermore, the assumption that the drift vector field X is projectable on the homogeneous space G/V leads to a decomposition of (1) in two systems, one on G/V and the one other on V.

Let $\mathfrak{v}$ be the generalized eigenspace corresponding to $0 \in \mathrm{Spec}(E)$. From the Jordan decomposition we know that $\mathfrak{v}$ is a Lie subalgebra of $\mathfrak{g}$. Through a fiber bundle decomposition of TG induced by $E \in \partial\mathfrak{g}$ and under the assumption that the drift vector field X is projectable on the homogeneous space G/V, the authors in [1] have showed that a singular control system $\mathcal{S}_G$ on the Lie group G, can be decomposed in two subsystems:

The *projection on the homogeneous space*

$$\mathcal{S}_{G/V} : \; \dot{y}(t) = \left(E_{y(t)}\right)^{-1} \circ X^h\left(y(t)\right) + \left(E_{y(t)}\right)^{-1} \circ \left(Y^u\right)^h\left(y(t)\right), \tag{2}$$

where $y(t) \in G/V$, and the *algebraic–differential control system*

$$\mathcal{S}_V : E_{x(t)}\left(y(t)\dot{v}(t)\right) = \left(r_{v(t)}\right)_* X^v\left(y(t)\right) + \left(r_{v(t)}\right)_* \left(Y^u\right)^v\left(y(t)\right) + \left(l_{y(t)}\right)_* X^v\left(v(t)\right), \quad (3)$$

where $Y^u = \sum_{j=1}^m u_j Y^j$ is the invariant vector field associated with the control $u \in \mathcal{U}$. Also $(\cdot)^h$ and $(\cdot)^v$ are the horizontal and vertical components of $(\cdot)$. The notation $y(t)\dot{v}(t)$ means $(l_{y(t)})_{*_v}(\dot{v})$.

The algebraic–differential subsystem plays a crucial role in the understanding of the trajectories for singular control systems on Lie groups. Actually, the solvability of (1) depends just on when we are able to solve (3). In fact, since we prove that the subsystem on G/V can be lifted to a linear control system on G, its solution can be computed through the projection of the corresponding solutions of the lifted system, see [2]. In this paper under the above regularity assumptions and through an iterative process, we establish a closed formula for the solution of the algebraic–differential control subsystem (3) and hence the solution of (1). In particular, when the Lie group is an Euclidean space we recover the classical solution, see for instance [10]. Our work is related to earlier work of the authors in [1].

The solvability problem for the class of singular control systems on $\mathbb{R}^n$ has been studied by many authors, see for instance, [3–6, 10].

The paper is organized as follows. In Section 2 we review some results that appear in [1]. In particular, the fiber bundle decomposition induced by the singular derivation E on TG. In Section 3 we show how to get the global solution for any admissible initial condition by analyzing both subsystems separately. Section 4 contains a number of examples.

2. The fibre bundle E-decomposition of TG

Let G be a Lie group with Lie algebra $\mathfrak{g}$. For a derivation $E \in \partial\mathfrak{g}$, denote by $\mathrm{Spec}(E)$ the set of eigenvalues of E and consider its Jordan decomposition $\sum_{i=1}^r \mathfrak{g}_{\lambda_i}$ where $\mathfrak{g}_{\lambda_i}$ is the generalized eigenspace of E corresponding to $\lambda_i \in \mathrm{Spec}(E)$, $i = 1, 2, \cdots, r$, given by

$$\mathfrak{g}_{\lambda_i} = \{X \in \mathfrak{g} : \ (E - \lambda_i I)^{n_i} X = 0, \ n_i \text{ the algebraic multiplicity of } \lambda_i \ \}.$$

Let $\mathfrak{v}$ be the generalized 0–eigenspace of E and $\mathfrak{h}$ the direct sum of all subspaces $\mathfrak{g}_{\lambda_i}$, with $\lambda_i \neq 0$, i.e.,

$$\mathfrak{v} = \mathrm{Ker} E^k = \mathfrak{g}_0, \quad \mathfrak{h} = \bigoplus_{\lambda_i \neq 0} \mathfrak{g}_{\lambda_i}. \quad (4)$$

Furthermore, $[\mathfrak{g}_{\lambda_i}, \mathfrak{g}_{\lambda_j}] \subset \mathfrak{g}_{\lambda_i+\lambda_j}$, and $\mathfrak{g}_{\lambda_i+\lambda_j} = 0$ when $\lambda_i+\lambda_j \notin \mathrm{Spec}(E)$. We notice that $\dim(\mathfrak{g}) = \dim(\mathfrak{v}) + \dim(\mathfrak{h})$. In fact, $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{h}$. On the other hand, we recall that $\mathfrak{v}$ has a structure of a Lie subalgebra of $\mathfrak{g}$, see [7]. In particular, there exists a connected Lie subgroup V of G with Lie algebra $\mathfrak{v}$.

We recall Proposition 3.2 in [1] where the authors have introduced an invariant connection to create the horizontal and vertical components of the singular control system.

Proposition 1. *Let $E \in \partial\mathfrak{g}$ be a derivation and $V \subset G$ the connected closed Lie subgroup with Lie algebra $\mathfrak{v}$. If $\pi : G \longrightarrow G/V$ is the canonical projection of G onto the homogeneous space G/V, then $(G, \pi, G/V, V)$ is a principal fibre bundle. Furthermore, the vectorial space $\mathfrak{h}$ in (4) induces a left invariant connection $\Gamma_{\mathfrak{h}}$ on $(G, \pi, G/V, V)$ given by $\Gamma_{\mathfrak{h}}(x) = \{P_x \in T_xG : \ \omega_x(P_x) = 0\}$. Here, for each $x \in G$, $\omega_x : T_xG \to \mathfrak{v}$ satisfies $\omega_x(P_x) = Z \in \mathfrak{v} \Leftrightarrow (Z^*)_x = P_x^v$, where Z^* denotes the invariant vector field on G induced by $Z \in \mathfrak{v}$. Also, P_x^v is the vertical component of $P_x \in T_xG$.*

From Proposition 1, for each $x \in G$ we have the following decomposition

$$T_xG = \Gamma_{\mathfrak{h}}(x) + \mathfrak{v}_x. \tag{5}$$

Here, $\mathfrak{v}_x$ is the vertical component of the tangent space of G at a state x which is constituted by all vectors tangent to the fibre through x, and $\Gamma_{\mathfrak{h}}(x)$ is called the horizontal component of T_xG. Therefore, for each $t \in \mathbb{R}$ holds $\dot{x}(t) = \dot{x}_h(t) + \dot{x}_v(t)$, where $\dot{x}_h(t)$ and $\dot{x}_v(t)$ are horizontal and vertical components of the vector $\dot{x}(t) \in T_{x(t)}G$. Hence, given a vector field $P \in \mathcal{X}(G)$, the connection $\Gamma_{\mathfrak{h}}$ induces two well–defined smooth vector fields on G: the horizontal component P^h of P, and the vertical component P^v of P.

3. The solution for any admissible initial condition

As was discussed in [1] any solution $x(t)$ of the singular control system $\mathcal{S}_G$ has the form $x(t) = y(t)v(t), \ t \in \mathbb{R}$ where $y(t)$ is an integral curve of the projected linear control system (2) on the homogeneous space G/V and $v(t)$ is a one parameter group of the closed subgroup V which together with $y(t)$ satisfies the algebraic–differential control subsystem (3). For that it is necessary to decompose G as the product $G/V \times V$. On simply connected Lie groups this condition is equivalent to the integrability of E, i.e., E decomposes $\mathfrak{g} = \mathfrak{v} + \mathfrak{h}$ and $\mathfrak{h}$ is a subalgebra. From now, we assume that E is an integrable derivation.

As in the Euclidean case, not any state is an initial condition of the system. Therefore, we need to introduce the following notion.

Definition 2. A point x_0 is said to be an admissible initial condition for $\mathcal{S}_G$ if there exists a control u such that $x(x_0, u, t)$ is an integral curve of the system.

We denote by I the set of the admissible initial conditions. In the next two subsections we describe this set. In order to build the solutions of (1), we analyze both subsystems separately (2) and (3).

3.1. **The solution of the projected subsystem on G/V.** We denote by $\mathfrak{aut}(G)$ the Lie algebra of $\mathrm{Aut}(G)$, the Lie group of G–automorphisms. In the sequel, we consider an element $D \in \mathfrak{aut}(G)$ which induces the drift vector field X of the system. As a matter of fact, X is an infinitesimal automorphism, i.e., its flow is a one parameter group of automorphims. Recall that $\mathfrak{aut}(G) \subset \partial\mathfrak{g}$ and when G is simply connected we have $\mathfrak{aut}(G) = \partial\mathfrak{g}$. Furthermore, if G is semisimple any derivation is inner. Hence the flow of the vector field X induced by $D \in \mathfrak{g}$ is determined by conjugation. So, there exists $Z \in \mathfrak{g}$ such that

$$X_t(x) = \exp(tZ)x\exp(-tZ)\ \forall t \in \mathbb{R},\ x \in G.$$

In general the homogeneous space G/V is not a Lie group. This fact depends on the structure of the subalgebra $\mathfrak{v}$. The vector space $\mathfrak{h} = \mathfrak{g}/\mathfrak{v}$ is isomorphic to a subalgebra of $\mathfrak{g}$ if and only if $\mathfrak{v}$ is an ideal. We start with the following

Proposition 3. *Let $E \in \partial\mathfrak{g}$ be a noninvertible derivation. Then, $\mathfrak{v} \subset \mathfrak{g}$ is an ideal if and only if E is nilpotent on $\mathfrak{g}$. In this case $\mathfrak{v} = \mathfrak{g}$.*

Proof. From the Jordan decomposition, we know that for any λ in the spectrum of E, $[\mathfrak{g}_0, \mathfrak{g}_\lambda] \subset \mathfrak{g}_\lambda$. Therefore, if $\mathfrak{g}_0$ is an ideal it turns out that $\lambda = 0$, and the proof follows. □

Therefore, if $E \in \partial\mathfrak{g}$ is not nilpotent the quotient set G/V is not a Lie group. However, since we are assuming that the corresponding Lie group V with Lie algebra $\mathfrak{v}$ is closed, we can build the homogeneous space. In the next Proposition, we charaterize when the drift vector field X is projectable on the manifold G/V.

Proposition 4. *Let $\mathcal{S}_G$ be a singular control systems on G as* (1) *with $E \in \partial\mathfrak{g}$ and X is induced by $D \in \mathfrak{aut}(G)$. Let V be a closed subgroup of G, G/V the homogeneous space of left cosets of V. Then, X is projectable on G/V if and only if $D(\mathfrak{v}) \subset \mathfrak{v}$.*

Proof. Through the canonical projection $\pi :\ G \longrightarrow G/V$, X is projectable on G/V if and only if $\pi_*(X) = \pi_*(X^h)$. Equivalently, the corresponding flows on G are related by

$$\pi(X_t(x)) = \pi(X_t(xy)),\ t \in \mathbb{R},\ x \in G,\ y \in V. \tag{6}$$

Let $v(t)$ be the one parameter group in V given by $(X^v)_t$. Since $X_t(xy) = X_t(x)X_t(y)$, $t \in \mathbb{R}$, $x \in G$, $y \in V$, from (6) we obtain

$$\pi(X_t(xy)) = \pi((X^h)_t(x)X_t(y)) = (X^h)_t(x)X_t(y)V, \tag{7}$$

replacing (7) by (6) we get that X is projectable if and only if $X_t(y) \in V$, for each $t \in \mathbb{R}$ and $y \in V$. It is known from [2] that $X_t(\exp Y) = \exp\left(e^{tD}Y\right) \forall Y \in \mathfrak{g}, \forall t \in \mathbb{R}$. It follows that $\exp\left(e^{tD}Y\right) \in V \quad \forall\, Y \in \mathfrak{v}, \forall t \in \mathbb{R}$. So, $D(\mathfrak{v}) \subset \mathfrak{v}$, and the proof is concluded. □

On the other hand, for any constant control u the projection of the invariant vector field $Y^u = \sum_{j=1}^{m} Y^j$ is well defined on G/V. Hence, we get the projected subsystem given by

$$\mathcal{S}_{G/V} \ : \ \dot{y}(t) = \widehat{X}^h\left(y(t)\right) + \sum_{j=1}^{m} u_j(t)\left(\widehat{Y}^j\right)^h\left(y(t)\right), \qquad y(t) \in G/V, \tag{8}$$

where $\widehat{X}^h\left(y(t)\right) = \left(E_{y(t)}\right)^{-1} \circ X^h\left(y(t)\right)$ and $\left(\widehat{Y}^j\right)^h\left(y(t)\right) = \left(E_{y(t)}\right)^{-1} \circ \left(Y^j\right)^h\left(y(t)\right), j = 1, \cdots, m.$

Any solution of this subsystem can be obtained by the projection of a trajectory of a linear control system on G, for which we already know the solution as was discussed in [2]. In fact, we have

Proposition 5. *There exists a linear control system on G*

$$\mathcal{L}_G : \dot{x}(t) = \widehat{X}^{lift}\left(x(t)\right) + \sum_{j=1}^{m} u_j(t)\left(\widehat{Y}^j\right)^{lift}\left(x(t)\right), \qquad x(t) \in G$$

which projects down onto $\mathcal{S}_{G/V}$.

Proof. Since E and D are both $\mathfrak{g}$-derivations and $\mathfrak{v}$-invariant, it follows that $\widehat{D} = E_{\mathfrak{h}}^{-1}D$ is a $\mathfrak{h}$-derivation. Let us denote by $\widehat{D}^{lift}$ a $\mathfrak{g}$-derivation such that its restriction to $\mathfrak{h}$ coincide with $\widehat{D}$. Thus, $\widehat{X}^{lift} = \widehat{X}^h$ on G/V. On the other hand, we define $\left(\widehat{Y}^j\right)^{lift}$ as the only invariant vector fields determined by $\widehat{Y}^j(e) \in \mathfrak{h}$.

Given an element $y \in G/V$ and a control $u \in \mathcal{U}$ we take an element x_0 in the fiber of y. Therefore, $\pi(x(x_0, u, t))$ is a $\mathcal{S}_{G/V}$-solution, where $x(x_0, u, t)$ is the solution of the linear control system with initial condition x_0 and control u. Therefore, each $y \in G/V$ is an admissible initial condition for $\mathcal{S}_{G/V}$. ☐

3.2. **The solution of the algebraic–differential subsystem.** In this section we establish a formula for the solution of the algebraic–differential control subsystem (3).

Let $x = yv$ where $y \in G/V$ and $v \in V$. Since X is an infinitesimal automorphism,

$$X_t\left(yv\right) = X_t^h\left(y\right) X_t^v\left(v\right), \qquad y \in G/V, \qquad v \in V.$$

Since $X_t\left(y\right) = X_t^h\left(y\right) \in G/V$, then $X\left(y\right) = \left.\dfrac{d}{dt}\right|_{t=0} X_t\left(y\right) \in T_y(G/V)$. Also $X\left(y\right) = X^h\left(y\right) + X^v\left(y\right) \in T_y(G/V)$ so $X^v\left(y\right) = 0$. On the other hand, $X^v(v) = \dot{v}$. In fact, the flow of the vector field X is related with the flow of its components X^h and X^v through the existence of a one parameter group $v(t)$ in V as follows $X_t(x) = X_t^h(x)v(t),\ t \in \mathbb{R},\ x \in G$. For $v \in V$ holds $X_t(v) = X_t^h(v)v(t) = v(t)$. By differentiating we get the desired conclusion.

Therefore, the equation (3) can be rewritten as

$$\mathcal{S}_V : E_{x(t)}\left(y(t)\dot{v}(t)\right) = \left(r_{v(t)}\right)_* \left(Y^u\right)^v \left(y(t)\right) + \left(l_{y(t)}\right)_* X^v \left(v(t)\right). \quad (9)$$

Since $(l_{v^{-1}})_* = (l_{x^{-1}})_* \circ (l_y)_*$, we have

$$E_{x(t)}\left(y(t)\dot{v}(t)\right) = (l_x)_* \circ E \circ (l_{x^{-1}})_* \circ (l_y)_* \dot{v}(t) = (l_x)_* \circ E \circ (l_{v^{-1}})_* \dot{v}(t).$$

Also, $\left(r_{v(t)}\right)_* \left(Y^u\right)^v \left(y(t)\right) = \sum_{j=1}^{m} u_j(t) \left(Y^j\right)^v \left(x(t)\right)$, because Y^u is a right invariant vector field. Consequently, applying $(l_{x(t)^{-1}})_*$ on both sides of (9), equation (3) takes its final form,

$$E \circ (l_{v(t)^{-1}})_* \dot{v}(t) = (l_{v(t)^{-1}})_* \dot{v}(t) + \sum_{j=1}^{m} u_j(t) (l_{x(t)^{-1}})_* \left(Y^j\right)^v \left(x(t)\right). \quad (10)$$

In our approach $\mathfrak{v} = \operatorname{Ker} E^k$, hence by the Jordan decomposition itself the derivation E is an invariant operator on $\mathfrak{v}$. Certainly, $E_{\mathfrak{v}}$ is nilpotent. Through the main result in Theorem 6 we construct the solution of the algebraic–differential control subsystem (10) by an iterative process taking in account the nilpotence indice of E on $\mathfrak{v}$. When E is not nilpotent, i.e., when $\mathfrak{v} \subset \mathfrak{g}$, $\mathfrak{v} \neq \mathfrak{g}$, the decomposition of the singular control system $\mathcal{S}_G$ in two subsystems takes place. Therefore, we first solve the subsystem on the homogeneous space. With the solution $y(t)$ at hands we apply the processes on $\mathfrak{v}$ where the derivation E is nilpotent.

Theorem 6. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Consider a singular control system on G as* (1)*, where the vector field X is induced by a derivation $D \in \mathfrak{aut}(G)$. Assume that the connected Lie subgroup V of G with Lie algebra $\mathfrak{v}$ is closed and X projects on the homogeneous space G/V. Then, if there exists a curve $v(t) \in V$ with $v(0) = e$, where e denotes the identity of G, which satisfies the algebraic–differential control subsystem* (10)*, we have*

(i) *If $E|_{\mathfrak{v}} = 0$, then*

$$(l_{v(t)^{-1}})_* \dot{v}(t) = -\sum_{j=1}^{m} u_j(t) \circ (l_{x(t)^{-1}})_* \left(Y^j\right)^v \left(x(t)\right). \quad (11)$$

(ii) *If $E|_{\mathfrak{v}}$ is nilpotent of index $l > 1$, then*

$$(l_{v(t)^{-1}})_* \dot{v}(t) = -\sum_{i=0}^{l-1} \sum_{j=1}^{m} u_j(t) \left(E|_{\mathfrak{v}}\right)^i \circ (l_{x(t)^{-1}})_* \left(Y^j\right)^v \left(x(t)\right). \quad (12)$$

Equivalently, the last equation reads

$$\dot{v}(t) = -\sum_{i=0}^{l-1} \sum_{j=1}^{m} u_j(t) (l_{v(t)})_* \circ \left(E|_{\mathfrak{v}}\right)^i \circ (l_{x(t)^{-1}})_* \left(Y^j\right)^v \left(x(t)\right).$$

Proof. The proof is a direct consequence of our construction.
If the nilpotence index of $E|_{\mathfrak{v}}$ is one then $E \circ (l_{v^{-1}})_* \dot{v} = 0$. From (10) it follows that

$$(l_{v(t)^{-1}})_* \dot{v}(t) = -\sum_{j=1}^{m} u_j(t) (l_{x(t)^{-1}})_* \left(Y^j\right)^v (x(t))$$

and then we arrive to the desired formula.

We proceed to prove (12) by induction on l. The case $l = 2$ is given as follows: if the nilpotence index of $E|_{\mathfrak{v}}$ is two, then $\dot{v} \notin \mathrm{Ker} E$ but $\dot{v} \in \mathrm{Ker} E^2$. Since

$$E^2 \circ (l_{v^{-1}})_* \dot{v} = 0, \tag{13}$$

by applying E to both sides of (10) we obtain

$$E^2 \circ (l_{v^{-1}})_* \dot{v} = E \circ (l_{v^{-1}})_* \dot{v} + \sum_{j=1}^{m} u_j E \circ (l_{x^{-1}})_* \left(Y^j\right)^v (x) .$$

Therefore,

$$E \circ (l_{v^{-1}})_* \dot{v} = -\sum_{j=1}^{m} u_j E \circ (l_{x^{-1}})_* \left(Y^j\right)^v (x) . \tag{14}$$

Replacing (14) by (10) we get

$$(l_{v^{-1}})_* \dot{v} = -\sum_{j=1}^{m} u_j E \circ (l_{x^{-1}})_* \left(Y^j\right)^v (x) - \sum_{j=1}^{m} u_j (l_{x^{-1}})_* \left(Y^j\right)^v (x) .$$

Now, suppose that the induction hypothesis holds true for a given $l > 2$, that is

$$(l_{v(t)^{-1}})_* \dot{v}(t) = -\sum_{i=0}^{l-2} \sum_{j=1}^{m} u_j(t) \left(E|_{\mathfrak{v}}\right)^i \circ (l_{x(t)^{-1}})_* \left(Y^j\right)^v (x(t)) \tag{15}$$

is true. By applying E to both sides of (15) we obtain

$$E(l_{v(t)^{-1}})_* \dot{v}(t) = -\sum_{i=0}^{l-2} \sum_{j=1}^{m} u_j(t) \left(E|_{\mathfrak{v}}\right)^{i+1} \circ (l_{x(t)^{-1}})_* \left(Y^j\right)^v (x(t)) .$$

By reordering the indices

$$E(l_{v(t)^{-1}})_* \dot{v}(t) = -\sum_{i=1}^{l-1} \sum_{j=1}^{m} u_j(t) \left(E|_{\mathfrak{v}}\right)^i \circ (l_{x(t)^{-1}})_* \left(Y^j\right)^v (x(t)) . \tag{16}$$

Replacing (16) by (10) we get

$$(l_{v(t)^{-1}})_*\dot{v}(t) = -\sum_{j=1}^{m} u_j(t)(l_{x(t)^{-1}})_* \left(Y^j\right)^v (x(t)) - \sum_{i=1}^{l-1}\sum_{j=1}^{m} u_j(t) \left(E|_{\mathfrak{v}}\right)^i \circ (l_{x(t)^{-1}})_* \left(Y^j\right)^v (x(t)),$$

which proves our claim. □

Finally, we have

Theorem 7. *Let $\mathcal{S}_G$ be a singular control system on the Lie group G, and we assume the same hypothesis of the previous theorem. The set of the admissible initial conditions is given by*

$$I = G/V \times I_V$$

where I_V is the set of the initial conditions of the subsystem $\mathcal{S}_V$.

Proof. Since we are assuming that E is integrable it turns out that G is diffeomorphic to the product $G/V \times V$. On the other hand, any state of G/V is an admissible initial condition. Thus, from the shape of a general solution $x(t) = y(t)v(t)$ the proof follows. □

4. Examples

In order to illustrate our construction we develop three examples. In the first one the derivation E is nilpotent, in particular there is not a horizontal component. In the second example the algebra $\mathfrak{v}$ has dimension two. Finally, we show the third example in which the derivation $D \in \partial\mathfrak{g}$ does not leaves invariant $\mathfrak{v}$, the Lie algebra of the closed subgroup V of G. In this situation, according to the Proposition 4 the drift vector field is not projectable on the homogeneous space G/V.

Example 1. Let us consider the simply connected three dimensional nilpotent Heisenberg Lie group

$$G = \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} : \quad x_1, x_2, x_3 \in \mathbb{R} \right\},$$

with Lie algebra $\mathfrak{g} = \mathbb{R}Y^1 + \mathbb{R}Y^2 + \mathbb{R}Y^3$, with right invariant generators

$$Y^1 = \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3}, \quad Y^2 = \frac{\partial}{\partial x_2}, \quad \text{and} \quad Y^3 = \frac{\partial}{\partial x_3},$$

The only non–null Lie bracket is $\left[Y^2, Y^1\right] = Y^3$. The multiplication and inverse map are given by

$$\begin{aligned}(x_1, x_2, x_3)(z_1, z_2, z_3) &=(x_1 + z_1, x_2 + z_2, x_3 + z_3 + x_1z_2),\\ (x_1, x_2, x_3)^{-1} &=(-x_1, -x_2, x_1x_2 - x_3).\end{aligned}$$

Also the exponential and logarithm mappings read as

$$\exp(a_1Y^1 + a_2Y^2 + a_3Y^3) = (a_1, a_2, a_3 + \frac{1}{2}a_1a_2),$$
$$\log(x_1, x_2, x_3) = x_1Y^1 + x_2Y^2 + (x_3 - \frac{1}{2}x_1x_2)Y^3.$$

A straightforward computation shows that

$$\partial\mathfrak{g} = \left\{ \begin{pmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ d_{31} & d_{32} & d_{11}+d_{22} \end{pmatrix} : \quad d_{ij} \in \mathbb{R},\ i = 1,2,3;\ j = 1,2 \right\}.$$

We consider $E \in \partial\mathfrak{g}$ determined by $d_{31} = 1$ and zero otherwise. Since E is nilpotent, Proposition 3 implies that $\mathfrak{v} = \mathfrak{g}$. So, $G = V$. Let $x = (x_1, x_2, x_3) \in G$, we have

$$(dl_x)_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x_1 & 1 \end{pmatrix}.$$

Hence, the vertical space reads

$$\mathfrak{v}_x = (dl_x)_0\, \mathfrak{v} = \operatorname{span}\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_3} \right\}$$

and the horizontal space $\Gamma_{\mathfrak{h}}(x) = \operatorname{Ker}(\omega_x)$ is trivial, because $\mathfrak{v} = \mathfrak{g}$. In this situation, $T_xG = \mathfrak{v}_x$.

Next we consider the singular control system on the Lie group G as follows

$$\mathcal{S}_G:\ E_{x(t)}\left(\dot{x}(t)\right) = X\left(x(t)\right) + u_1(t)Y^1\left(x(t)\right) + u_2(t)Y^3\left(x(t)\right), \quad x(t) \in G,$$

where the drift vector field X is induced by the derivation $D = \operatorname{diag}(-1,\ -1,\ -2) \in \partial\mathfrak{g}$. By definition

$$X(x) = \left.\frac{d}{dt}\right|_{t=0} X_t(x).$$

Since G is simply connected and nilpotent, the exponential map is a global difeomorphism. Hence, the flow of the infinitesimal automorphism X is given by $X_t(x) = \exp\left(e^{tD}\log(x)\right)$, see [1] for a general proof. A simple calculus shows that

$$X_t(x) = \left(x_1e^{-t},\ x_2e^{-t},\ (x_3 - \frac{1}{2}x_1x_2)e^{-2t} + \frac{1}{2}x_1x_2e^{-2t} \right) \in \operatorname{Aut}(G) \quad \forall t \in \mathbb{R}.$$

Hence, $X(x) = -x_1\frac{\partial}{\partial x_1} - x_2\left(\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}\right) - 2x_3\frac{\partial}{\partial x_3}$. Since, $X^v(x) = X(x)$ then $X^h(x) = 0$.

On the other hand, for each constant control u the projections of the invariant vector fields are well defined on G/V and are given by

$$\left(Y^1\right)^v(x) = Y^1(x),\ \left(Y^3\right)^v(x) = Y^3(x) \text{ and } \left(Y^1\right)^h(x) = 0 = \left(Y^3\right)^h(x).$$

Now, we assume there exists a solution $x(t)$ of the singular control system $\mathcal{S}_G$ with control u and initial condition x_0. It turns out that there exists a one parameter group $v(t) = (v_1(t), v_2(t), v_3(t))$ of $V = G$ such that $v(0) = e$, where $e = (0, 0, 0)$ denotes the identity of G. Since $\dot{y}(t) = 0$ then $y(t) = e$. In order to find the solutions of the algebraic–differential subsystem control, notice that $E^2_{\mathfrak{v}} = E^2 = 0$. From (11) we obtain

$$\begin{aligned}&(l_{v(t)^{-1}})_*\dot{v}(t)\\ &\quad= -\sum_{i=0}^{1}\left(E|_{\mathfrak{v}}\right)^i \circ (l_{x(t)^{-1}})_*\left[u_1(t)\left(Y^1\right)^v(x(t)) + u_2(t)\left(Y^3\right)^v(x(t))\right],\end{aligned}$$

which is equivalent to the system

$$\begin{cases}\dot{v}_1 = -u_1\\ \dot{v}_2 = 0\\ -v_1\dot{v}_2 + \dot{v}_3 = -u_1(x_2+1) - u_2.\end{cases} \tag{17}$$

Hence, for each constant control u the associated solution starting from the identity element reads as

$$v(t) = (-u_1 t,\ 0,\ -u_1 t - u_2 t).$$

Since we are considering the class of unrestricted constant control, it follows that the solution coming from any $u \in \mathcal{U}$ is a concatenation of solutions of constant controls. Then the solutions generate a two dimensional Lie subgroup of G,

$$I = I_V = \exp\left\{Y^1 + Y^3, Y^3\right\}.$$

It turns out that the systems has solution just for those element of I that we call the admissible initial condition of the system.

Example 2. Let us consider the five dimensional nilpotent simply connected Lie group whose Lie algebra $\mathfrak{g} = \mathbb{R}Y^1 + \mathbb{R}Y^2 + \mathbb{R}Y^3 + \mathbb{R}Y^4 + \mathbb{R}Y^5$ is generated by the following right invariant vector fields

$$\begin{aligned}&Y^1 = \frac{\partial}{\partial x_1}, \quad Y^2 = \frac{\partial}{\partial x_2}, \quad Y^3 = x_2\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\\ &Y^4 = \frac{\partial}{\partial x_4}, \quad \text{and} \quad Y^5 = x_4\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}.\end{aligned}$$

The only non–null Lie brackets are $\left[Y^2, Y^3\right] = Y^1$, $\left[Y^4, Y^5\right] = Y^1$. The multiplication and inverse in G are given by

$$\begin{aligned} z * x =& (z_1 + x_1 + z_3x_2 + z_5x_4,\ z_2 + x_2,\ z_3 + x_3,\ z_4 + x_4,\ z_5 + x_5), \\ x^{-1} =& (-x_1 + x_2x_3 + x_4x_5, -x_2, -x_3, -x_4, -x_5). \end{aligned}$$

The exponential and logarithm mappings are

$$\exp(a_1Y^1 + a_2Y^2 + a_3Y^3 + a_4Y^4 + a_5Y^5) = \left(a_1 + \frac{1}{2}a_2a_3 + \frac{1}{2}a_4a_5, a_2, a_3, a_4, a_5\right),$$

$$\log(x_1, x_2, x_3, x_4, x_5) = \left(x_1 - \frac{1}{2}x_2x_3 - \frac{1}{2}x_4x_5\right) Y^1 + x_2Y^2 + x_3Y^3 + x_4Y^4 + x_5Y^5.$$

A straightforward computation shows that

$$\partial\mathfrak{g} = \left\{ \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} \\ 0 & d_{22} & d_{23} & d_{24} & d_{25} \\ 0 & d_{32} & d_{11} - d_{22} & d_{34} & d_{35} \\ 0 & -d_{35} & d_{25} & d_{44} & d_{45} \\ 0 & d_{34} & -d_{24} & d_{54} & d_{11} - d_{44} \end{pmatrix} : \quad d_{ij} \in \mathbb{R} \right\}.$$

Next, we consider $E \in \partial\mathfrak{g}$ determined by $d_{11} = d_{12} = d_{22} = d_{35} = d_{44} = 1$ and zero otherwise. Since $\mathrm{Spec}(E) = \{0, 1\}$ the Jordan decomposition (4) induced by E on $\mathfrak{g}$ is given by: $\mathfrak{v} = \mathbb{R}\{Y^3, Y^5\}$ and $\mathfrak{h} = \mathbb{R}\{Y^1, Y^2, Y^4\}$. We notice that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. In general this is not true. For instance, the derivation $E = \mathrm{diag}\,(0,\ 1,\ -1,\ 1,\ -1)$ leads to $\mathfrak{v} = \mathbb{R}Y^1$. Therefore, the subspace $\mathfrak{h}$ is not a subalgebra. In our case E is integrable, in particular $G = G/V \times V$ and our decomposition takes place.For any $x = (x_1, x_2, x_3, x_4, x_5) \in G$, we have

$$\begin{aligned} l_x(a, b, c, p, q) &= (x_1 + a + x_3b + x_5p, x_2 + b, x_3 + c, x_4 + p, x_5 + q), \\ \mathfrak{v}_x &= (dl_x)_0\, \mathfrak{v} = \mathrm{span}\left\{ \left.\frac{\partial}{\partial x_3}\right|_x, \left.\frac{\partial}{\partial x_5}\right|_x \right\}, \\ \Gamma_{\mathfrak{h}}(x) &= \mathrm{span}\left\{ \left.\frac{\partial}{\partial x_1}\right|_x, \left.\left(x_3\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\right|_x, \left.\left(x_5\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}\right)\right|_x \right\}. \end{aligned}$$

Thus, we get the E-decomposition: $T_xG = \Gamma_{\mathfrak{h}}(x) \oplus \mathfrak{v}_x$.

We consider the singular control system

$$\mathcal{S}_G :\ E_{x(t)}\left(\dot{x}(t)\right) = X\left(x(t)\right) + u(t)Y\left(x(t)\right), \qquad x(t) \in G,$$

where the drift vector field X is induced by the derivation $D = \operatorname{diag}(1,\ -1,\ 2,\ -1,\ 2)$. The Lie group G is simply connected and nilpotent, thus

$$X(x) = \left.\frac{d}{dt}\right|_{t=0} \exp\left(e^{tD}\log(x)\right) = (x_1,\ -x_2,\ 2x_3,\ -x_4,\ 2x_5).$$

Since $D(\mathfrak{v}) \subset \mathfrak{v}$ holds, from Proposition 4, X is projectable on the homogeneous space G/V. Hence, the vertical and horizontal projections of X are

$$X^v(x) = 2x_3 \left.\frac{\partial}{\partial x_3}\right|_x + 2x_5 \left.\frac{\partial}{\partial x_5}\right|_x \qquad \text{and}$$
$$X^h(x) = x_1 \left.\frac{\partial}{\partial x_1}\right|_x - x_2 \left.\left(x_3 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\right|_x - x_4 \left.\left(x_5 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x4}\right)\right|_x .$$

On the other hand, for each constant control u the projection of the invariant vector field $Y = Y^5$ is well defined on G/V and their vertical and horizontal projections are given by

$$Y^v(x) = \left.\frac{\partial}{\partial x_5}\right|_x \quad \text{and} \quad Y^h(x) = x_4 \left.\frac{\partial}{\partial x_1}\right|_x .$$

If $x(t)$ is a solution of the singular control system $\mathcal{S}_G$ with control u and initial condition x_0, then the one parameter group $v(t)$ together with $y(u, x_0, t)$ satisfies

$$x(t) = y(t)v(t) = (y_1(t),\ y_2(t),\ y_3(t),\ y_4(t),\ y_5(t))\,(0,\ 0,\ \alpha(t),\ 0,\ \beta(t)),$$

where $y(0) = y_0$ and $\alpha(0) = \beta(0) = 0$.

A simple computation shows that $\left(E_{y(t)}\right)^{-1}$ is given by

$$\begin{pmatrix} 1 & y_5 - 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

with $y \in G/V$. Therefore, the projected linear control system (2) induced by $\mathcal{S}_G$ leads to

$$\mathcal{S}_{G/V} : \qquad \begin{cases} \dot{y}_1 = (y_1 + y_2 - y_2 y_3 - y_4 y_5 - y_2 y_5) + u y_4 \\ \dot{y}_2 = -y_2 \\ \dot{y}_3 = 0 \\ \dot{y}_4 = -y_2 - y_4 \\ \dot{y}_5 = 0, \end{cases} \tag{18}$$

where $u \in \mathbb{R}$.

The solution for (18), for each constant control u with initial condition y_0 is given by

$$y(t) = \Big(y_{10}e^t + y_{20}\,(1 - y_{30} - y_{50})\sinh t + (u - y_{50})\left[\left(y_{40} - \frac{1}{2}y_{20}\right)\sinh t + \frac{1}{2}y_{20}te^{-t}\right], \quad y_{20}e^{-t},\ y_{30},\ (y_{40} - y_{20}t)\,e^{-t}, y_{50}\Big).$$

In order to find the solutions of the algebraic–differential subsystem control, we notice that $\left(E|_{\mathfrak{v}}\right)^2 = 0$ and $\dot{v}(t) \in \mathrm{Ker}E^2 = \mathrm{span}\{Y^3, Y^5\}$. Hence, from the formula (12) we get

$$\begin{cases} \dot{v}_1 - v_3\dot{v}_2 - v_5\dot{v}_4 = 0 \\ \dot{v}_2 = 0 \\ \dot{v}_3 = -u \\ \dot{v}_4 = 0 \\ \dot{v}_5 = -u, \end{cases}$$

it follows that $v(t) = (0,\ 0,\ -ut,\ 0,\ -ut)$.

As before, the solution coming from any $u \in \mathcal{U}$ is a concatenation of solutions of constant controls. In particular, the algebraic–differential system generate a 1-dimensional Lie subgroup.

From the previous analysis we get that

$$I = G/V \times I_V = \exp\left\{Y^1, Y^2, Y^4, Y^3 + Y^5\right\}$$

is a submanifold of dimension 4.

More explicitly, the solutions $x(t) = y(t)v(t)$ for the singular control system $\mathcal{S}_G$ are:

- If $u = 0$, then $v(t) = 0$. So, starting at $y(0)$

$$x(t) = \Big(y_{10}e^t + y_{20}\,(1 - y_{30} - y_{50})\sinh t + (u - y_{50})\left[\left(y_{40} - \frac{1}{2}y_{20}\right)\sinh t + \frac{1}{2}y_{20}te^{-t}\right], \quad y_{20}e^{-t},\ y_{30},\ (y_{40} - y_{20}t)\,e^{-t}, y_{50}\Big).$$

- If $u \neq 0$, starting at x_0 the solution reads

$$x(t) = \Big(y_{10}e^t + y_{20}\left(1 - y_{30} - y_{50}\right)\sinh t + (u - y_{50})\left[\left(y_{40} - \frac{1}{2}y_{20}\right)\sinh t + \frac{1}{2}y_{20}te^{-t}\right], \quad y_{20}e^{-t},\ y_{30} - ut,\ \left(y_{40} - y_{20}t\right)e^{-t}, y_{50} - ut \Big).$$

Example 3. We consider the Heisenberg group of the Example 1. Next, we consider the non nilpotent derivation $E \in \partial\mathfrak{g}$ determined by $d_{12} = d_{22} = d_{32} = 1$ and zero otherwise. Since $\mathrm{Spec}(E) = \{0, 1\}$ the Jordan decomposition (4) induced by E on $\mathfrak{g}$ is given by: $\mathfrak{v} = \mathbb{R}\{Y^1\}$ and $\mathfrak{h}$ is the subalgebra $\mathbb{R}\{Y^2, Y^3\}$. We notice that E is integrable, in particular $G = G/V \times V$ and our decomposition takes place. It is easily seen that E left $\mathfrak{v}$ invariant.

Now, we consider the derivation D determined by $d_{21} = d_{31} = 1$ and zero otherwise. It is easy to verify that D does not leave $\mathfrak{v}$ invariant. Hence the drift vector field X is not projectable on the homogeneous space G/V. In fact, for any $(x, y, z) \in G$ its flow is given by

$$X_t(x, y, z) = \left(x,\ tx + y,\ tx + z + \frac{1}{2}x^2t\right) \quad \forall t \in \mathbb{R},$$

therefore there are $w \in V$ and $t_0 \in \mathbb{R}$ such that $X_{t_0}(w) \notin V$. For instance, for $w = (x_0, 0, 0) \in V$ and $t_0 = 1$.

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(Received September 02 2010, received in revised form August 04 2011)

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