Semigroup Forum Vol. 69 (2004) 113–140

[illegible]

RESEARCH ARTICLE

Control Sets and Total Positivity

Víctor Ayala,* Wolfgang Kliemann, and Luiz A. B. San Martin†

Communicated by Jimmie D. Lawson

Dedicated to Enrique Lillo

Abstract

The objective of this article is to bring together two different mathematical subjects, namely totally positive matrices and control sets. It describes the control sets of the totally positive matrices and the sign-regular matrices in flag manifolds. In particular, the classical result by Gantmacher and Krein, follows from Theorem 9.7. One expects that with this description some theorems proved by combinatorial techniques have a geometric or dynamic interpretation.

Key words and phrases: Totally positive matrices, semigroups, flag manifolds, control sets.
AMS 2000 subject classification: 20M20, 15A48.

1. Introduction

The objective of this article is to bring together two different mathematical subjects, namely totally positive matrices and control sets for semigroup actions. Recall that a matrix is said to be totally positive if all its minors are non-negative real numbers. Motivated by various applications, these matrices have been extensively studied since the decade of 1930, giving rise to a well established and classical theory which provides a great deal of information about totally positive matrices. We refer to the survey by Ando [1], for an account of this theory as well as its applications, and for further references.

On the other hand we consider the results about semigroups in the Lie theoretic setting of San Martin [8], [9], [10], [11] and San Martin-Tonelli [13]. The main feature of these papers is to put in an algebraic and geometric framework some ideas borrowed from control theory in order to apply them to the study of semigroups in semi-simple Lie groups. Among these ideas a key notion is that of control set, which was studied by Colonius-Kliemann [2] in the context of control systems. In fact, the results of the cited papers are directed towards the understanding of the structural properties of the semigroups by means of the action on the flag manifolds, and hence of the control sets. These results apply in particular to semigroups with interior points in the special linear group $\mathrm{Sl}(d,\mathbb{R})$.

---

*Research partially supported by CONICYT, Proyecto FONDECYT n° [illegible] and Proyecto Incentivo a la Cooperación Internacional n° [illegible].

†Research partially supported by CNPq grant n°. [illegible]

In order to apply this theory we consider here only totally positive matrices having determinant one. It is well known that the set T of totally positive in $\mathrm{Sl}(d,\mathbb{R})$ is also a semigroup with non-empty interior in $\mathrm{Sl}(d,\mathbb{R})$. Thus the results mentioned above apply to T. The main aim of this work is to study properties of totally positive matrices by exploiting its semigroup structure.

We study the action of T on the flag manifolds and describe its control sets. As a consequence we can give geometric interpretations and new proofs of some classical results, like e.g. a result by Gantmacher and Krein about the eigenvectors of a totally positive matrix (see [1], Section 6 and Theorem 9.7 below).

Intimately related to the totally positive matrices are the sign regular matrices (a matrix g is sign regular if for every k its minors of order k have the same sign). We prove that the control sets of T coincide with the control sets of the semigroup S of sign regular matrices in $\mathrm{Sl}(d,\mathbb{R})$, and in fact we work out the properties of T through the action of S. The point here is that S is given as a compression semigroup of its invariant control set in the full flag manifold.

We also consider, for each multi-index $\mathbf{r} = \{r_1,\ldots,r_k\}$, the semigroups $T_{\mathbf{r}}$ and $S_{\mathbf{r}}$ of matrices having positive (respectively fixed sign) minors of orders $r_1,\ldots,r_k$. Their control sets are described and it is proved that $S_{\mathbf{r}}$ is a maximal semigroup in the particular case where $\mathbf{r} = \{r_1\}$ is a singleton whereas, for arbitrary $\mathbf{r}$, the semigroup $S_{\mathbf{r}}$ is maximal of type $\mathbf{r}$, in the sense of Definition 4.1.

Recently, Lusztig [7] introduced the notion of totally positive semigroups in more general simple groups than $\mathrm{Sl}(d,\mathbb{R})$. The analysis of the totally positive matrices through the control sets in flag manifolds must create a bridge between the classical results and those obtained by the mentioned author, which are given in a more abstract setting.

2. Total positivity and sign regularity

A $d \times d$ real matrix g is totally positive provided all its minors of all orders are nonnegative numbers. As usual we denote by $\mathrm{Gl}(d)$ the group of invertible real $d \times d$ matrices and by $\mathrm{Sl}(d)$ the subgroup of determinant one matrices. More generally, if V is a real vector space we write $\mathrm{Gl}(V)$ and $\mathrm{Sl}(V)$ for the corresponding groups of linear mappings.

In this paper we consider only totally positive matrices in $\mathrm{Sl}(d)$. It is tacitly assumed that the whole set of invertible totally positive matrices can be recovered by using the standard trick of writing an arbitrary invertible matrix as $g = sh$ with $s = \sqrt[d]{\det g} \cdot 1$ a scalar matrix (where 1 stands for the identity matrix) and $h \in \mathrm{Sl}(d)$.

For our purposes it is convenient to have the following geometrical description of totally positive matrices. Let $\bigwedge^k$ be the k-fold exterior product of $\mathbb{R}^d$, i.e., $\bigwedge^k = \bigwedge^k \mathbb{R}^d$. Denote by $\beta_1 = \{e_1,\ldots,e_d\}$ the standard basis of $\mathbb{R}^d$ and for each multi-index $\mathbf{r} = \{r_1,\ldots,r_k\}$, with $r_1 < \cdots < r_k$ form the exterior product $e_{\mathbf{r}} = e_{r_1} \wedge \cdots \wedge e_{r_k}$. The set $\beta_k = \{e_{\mathbf{r}}\}$ with $\mathbf{r}$ running through the set of all multi-indices is a basis of $\bigwedge^k$. We endow $\bigwedge^k$ with an inner product $\langle\cdot,\cdot\rangle$, which is defined in such a way that the standard basis β_k is orthonormal. It is well-known that this inner product satisfies

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle)_{i,j} \tag{1}$$

A $d \times d$-matrix g induces a linear map of $\bigwedge^k$ by

$$v_1 \wedge \cdots \wedge v_k \longmapsto gv_1 \wedge \cdots \wedge gv_k$$

where $v_1, \ldots, v_k \in \mathbb{R}^d$. In the sequel we denote this linear map by $\wedge^k g$ or simply by g itself, if no confusion arises. By the very definition, it follows that the k-minors of g are the entries of the matrix of $\wedge^k g$ with respect to the basis β_k.

Now, let $\mathcal{O}_k$ be the first orthant of $\bigwedge^k$ with respect to β_k, i.e., $v \in \mathcal{O}_k$ if and only if its coordinates with respect to β_k are non-negative. Since the cone $\mathcal{O}_k$ is generated by the standard basis, it follows that $g \in T_k$ if and only if $(\wedge^k g) e_{\mathbf{r}} \in \mathcal{O}_k$ for all basic vectors $e_{\mathbf{r}}$. Hence, the entries of $\wedge^k g$ are non-negative if and only if $\wedge^k g(\mathcal{O}_k) \subset \mathcal{O}_k$, implying that g is totally positive if and only $\mathcal{O}_k$ is invariant under $\wedge^k g$ for all k.

For later reference we note that since $\mathcal{O}_k$ is an orthant defined by an orthonormal basis, any $\xi \in \operatorname{int}\mathcal{O}_k$ satisfies $\langle \xi, \eta \rangle > 0$ for any $\eta \in \mathcal{O}_k$.

We denote by T_k the compression semigroup of $\mathcal{O}_k$, that is,

$$T_k = \{g \in \mathrm{Sl}(d) : \wedge^k g(\mathcal{O}_k) \subset \mathcal{O}_k\}. \tag{2}$$

We say that an element of T_k is a *k-positive matrix* while T_k is called the *k-positive semigroup*. The set T_k is actually a subsemigroup of $\mathrm{Sl}(d)$ since for matrices g, h, $\wedge^k(gh) = (\wedge^k g)(\wedge^k h)$. Of course, if T stands for the semigroup of totally positive matrices then $T = T_1 \cap \cdots \cap T_{d-1}$. More generally, given a multi-index $\mathbf{r} = \{r_1, \ldots, r_k\}$, with $r_1 < \cdots < r_k$ we define the semigroup

$$T_{\mathbf{r}} = T_{r_1} \cap \cdots \cap T_{r_k}$$

and say that an element $g \in T_{\mathbf{r}}$ is an $\mathbf{r}$-positive matrix.

We denote by Gr_k the Grassmannian of k-dimensional subspaces of $\mathbb{R}^d$. Also, we let Gr_k^+ be the set of k-dimensional oriented subspaces of $\mathbb{R}^d$. In case $k = 1$, Gr_k and Gr_k^+ are the $(d-1)$-dimensional projective space and sphere, respectively. In these cases we stick to the usual notation and put $\mathrm{Gr}_1 = \mathbb{P}^{d-1}$ and $\mathrm{Gr}_1^+ = \mathbb{S}^{d-1}$, or simply $\mathbb{P}$ or $\mathbb{S}$.

Now, recall that an element of $\bigwedge^k \mathbb{R}^d$ is said to be a decomposable vector if it is of the form

$$v_1 \wedge \cdots \wedge v_k$$

with $v_1, \ldots, v_k \in \mathbb{R}^d$. The oriented Grassmannian Gr_k^+ is in one-to-one correspondence with the set of rays $\mathbb{R}_+ v$ in $\bigwedge^k$ with v a decomposable vector

The correspondence is made by associating to a subspace V, with a prescribed orientation, the ray spanned by $v_1 \wedge \cdots \wedge v_k$ where $\{v_1, \ldots, v_k\}$ is any positively oriented basis of V. In the sequel we denote by $\{v\}$ the ray spanned by v, that is, $\{v\} = \mathbb{R}_+ v$.

The same way, the non-oriented Grassmannian Gr_k identifies with the subset of the projective space $\mathbb{P}\left(\bigwedge^k\right)$ by associating to a k-dimensional subspace the line in $\bigwedge^k$ spanned by $v_1 \wedge \cdots \wedge v_k$ with $\{v_1, \ldots, v_k\}$ a basis of V. In the sequel we denote by $[v]$ the line spanned by v, that is, $[v] = \mathbb{R}v$. We let $\pi : \mathrm{Gr}_k^+ \to \mathrm{Gr}_k$ be the mapping $\pi\{v\} = [v]$, which associates to a ray the line spanned by it. Equivalently, π forgets the orientation of a subspace in Gr_k^+.

Let g be an invertible matrix and $V \in \mathrm{Gr}_k^+(d)$. Then the subspace $gV = \{gx : x \in V\}$ is k-dimensional, which is oriented by the basis $\{gv_1, \ldots, gv_k\}$, where $\{v_1, \ldots, v_k\}$ is a positively oriented basis of V. The mapping $(g, V) \mapsto gV$ defines an action of the linear group $\mathrm{Sl}(d)$ on $\mathrm{Gr}_k^+(d)$. This action is compatible with the action on $\bigwedge^k$. In fact, if $v = v_1 \wedge \cdots \wedge v_k$ is a decomposable vector then $(\wedge^k g)v$ is also decomposable, and if $V \in \mathrm{Gr}_k^+(d)$ corresponds to $\{v\}$ then gV corresponds to $\{(\wedge^k g)v\}$.

In order to relate the k-positive semigroup with the oriented Grassmannian, let C_k^+ be the set rays in Gr_k^+ which are contained in $\mathcal{O}_k$ (*abusing* notation we can write in short: $C_k^+ = \mathrm{Gr}_k^+ \cap \mathcal{O}_k$). Now, it holds

$$T_k = \{g \in \mathrm{Sl}(d) : gC_k^+ \subset C_k^+\}. \tag{3}$$

In fact, if $\mathcal{O}_k$ is invariant under $g \in \mathrm{Sl}(d)$ then C_k^+ is also invariant, since the set of decomposable vectors in $\bigwedge^k$ is $\mathrm{Sl}(d)$-invariant. Conversely, if C_k^+ is invariant then $\wedge^k g$ maps the standard vectors $e_{\mathbf{r}}$ into $\mathcal{O}_k$. Since $\mathcal{O}_k$ is generated by $e_{\mathbf{r}}$ it follows that $\mathcal{O}_k$ is invariant under $\wedge^k g$.

We consider now an analogous construction on the non-oriented Grassmannians. For each $k = 1, \ldots, d$ put $C_k = \pi\left(C_k^+\right)$ and consider the compression semigroup

$$S_k = \{g \in \mathrm{Sl}(d) : gC_k \subset C_k\}. \tag{4}$$

Clearly, $\pi^{-1}(C_k) = C_k^+ \cup C_k^-$ where $C_k^- = -C_k^+$, so that $g \in S_k$ if and only if all its k-minors have the same sign. Thus the elements of the semigroup

$$S = S_1 \cap \cdots \cap S_{d-1} \tag{5}$$

are the sign-regular matrices in $\mathrm{Sl}(d)$ (recall that a matrix is said to be sign-regular if for any k the minors of size k have the same sign). Analogous to the positive case we write

$$S_{\mathbf{r}} = S_{r_1} \cap \cdots \cap S_{r_k}$$

if $\mathbf{r} = \{r_1, \ldots, r_k\}$, with $1 \leq r_1 < \cdots < r_k < d$. In what follows the elements of $S_{\mathbf{r}}$ are called $\mathbf{r}$-sign regular matrices.

Next we check that these semigroups are also compression semigroups. Let $\mathbf{r} = \{r_1, \ldots, r_k\}$, $1 \le r_1 < \cdots < r_k < d$, be a multi-index. A flag of type $\mathbf{r}$ is an increasing sequence of subspaces

$$b = (V_1, \ldots, V_k)$$

of $\mathbb{R}^d$ with $\dim V_i = r_i$ and $V_i \subset V_{i+1}$. The flag manifold $\mathbb{F}(\mathbf{r})$ is the set of all flags of type $\mathbf{r}$. In case of the complete multi-index $\mathbf{r}_c = \{1, 2, \ldots, d-1\}$ we write simply $\mathbb{F}$ for $\mathbb{F}(r_c)$. The elements of $\mathbb{F}$ are the complete flags of subspaces of $\mathbb{R}^d$.

Let $\mathbf{r}_1 \subset \mathbf{r}_2$ be multi-indices. Then there is a natural mapping $\pi^{\mathbf{r}_2}_{\mathbf{r}_1} : \mathbb{F}(\mathbf{r}_2) \to \mathbb{F}(\mathbf{r}_1)$ which associates to a flag $b \in \mathbb{F}(\mathbf{r}_2)$ the flag of type $\mathbf{r}_1$ obtained from b by forgetting the subspaces with dimension $k \in \mathbf{r}_2 \setminus \mathbf{r}_1$. This mapping (projection) is clearly onto. Also, the complete flag manifold $\mathbb{F}$ is the maximal one in the sense that it projects onto every other flag manifold. For these projections we write simply $\pi_{\mathbf{r}} : \mathbb{F} \to \mathbb{F}(\mathbf{r})$. Also, for $\mathbf{r} = \{k\}$, that is, $\mathbb{F}(\mathbf{r}) = \mathrm{Gr}_k$ we write k instead of $\mathbf{r}$. Thus $\pi^{\mathbf{r}}_k$ stands for the projection onto the Grassmannian Gr_k.

Recall that $\mathrm{Sl}(d)$ acts transitively on $\mathbb{F}(\mathbf{r})$ by the natural action

$$g(V_1, \ldots, V_k) = (gV_1, \ldots, gV_k),$$

and the restriction of this action to the orthogonal group $\mathrm{SO}(d)$ is also transitive. Since $\mathrm{SO}(d)$ is a compact Lie group, $\mathbb{F}(\mathbf{r})$ is endowed with a structure of compact manifold. Note that by the very definitization of $\pi^{\mathbf{r}_2}_{\mathbf{r}_1} : \mathbb{F}(\mathbf{r}_2) \to \mathbb{F}(\mathbf{r}_1)$ is equivariant for the actions, in the sense that $g\pi^{\mathbf{r}_2}_{\mathbf{r}_1}(b) = \pi^{\mathbf{r}_2}_{\mathbf{r}_1}(gb)$ for any invertible matrix g and any $b \in \mathbb{F}(\mathbf{r}_2)$. An easy application of this property of the projections yields that the semigroups $S_{\mathbf{r}}$ are compression semigroups.

Proposition 2.1. *Given a multi-index $\mathbf{r} = \{r_1, \ldots, r_k\}$ define the following subset of $\mathbb{F}(\mathbf{r})$:*

$$C_{\mathbf{r}} = (\pi^{\mathbf{r}}_{r_1})^{-1}(C_{r_1}) \cap \cdots \cap (\pi^{\mathbf{r}}_{r_k})^{-1}(C_{r_k}). \tag{6}$$

Then $S_{\mathbf{r}} = \{g \in \mathrm{Sl}(d) : gC_{\mathbf{r}} \subset C_{\mathbf{r}}\}$.

Proof. Suppose that $g \in S_{r_1} \cap \cdots \cap S_{r_k}$. Then by the equivariance of the projections, g leaves invariant each subset $(\pi^{\mathbf{r}}_{r_i})^{-1}(C_{r_i})$, $i = 1, \ldots, k$, and hence their intersection. Conversely, if g leaves invariant $C_{\mathbf{r}}$, then by equivariance again, g leaves invariant each C_{r_i}, and hence $g \in S_{r_i}$, $i = 1, \ldots, k$, showing the claim. ∎

We conclude this section with the following remarks about the semigroups defined so far:

1. Let D be the subgroup of diagonal matrices with positive entries. Then D is contained in T, and hence in $T_{\mathbf{r}}$ and $S_{\mathbf{r}}$ for an arbitrary multi-index $\mathbf{r}$.

2. Since the double covering $\pi : \mathrm{Gr}_k^+ \to \mathrm{Gr}_k$ satisfies $\pi(gV) = g\pi(V)$ for all invertible matrix g and $V \in \mathrm{Gr}_k^+$, it follows that $T_k \subset S_k$ and hence $T \subset S$. Moreover, if $g \in S_k$ then $gC_k^+ = C_k^+$ or $-C_k^-$. Hence, $g^2C_k^+ = C_k^+$, so that $g^2 \in T_k$. Also, $g^2 \in T_{\mathbf{r}}$ if $g \in S_{\mathbf{r}}$.

3. The semigroups T, S, $T_{\mathbf{r}}$ and $S_{\mathbf{r}}$ have non-empty interior in $\mathrm{Sl}(d)$. This is well known and follows easily from fact that these semigroups are defined by polynomial inequalities on the entries of the matrices. Alternatively, one can give the following Lie theoretic argument: Recall that $A = (a_{ij})$ is called a Jacobi matrix if $a_{ij} = 0$ whenever $|i - j| > 1$. Let W be the cone formed by the diagonal matrices and the Jacobi matrices with nonnegative entries. Then $\exp(tX) \in T$ for all $t \geq 0$ and $X \in W$. Since W generates the Lie algebra of all matrices, it follows that T has nonempty interior (see Hilgert-Hofmann-Lawson [3] and Sussmann-Jurdjevic [14]).

3. Flag manifolds and control sets

The purpose of this section is to recall some results about semigroups and their actions on the flag manifolds. We mainly specialize the general results of [8], [9], [10], [11] and [13] to the case of semigroups in $\mathrm{Sl}(d)$.

3.1. Generalities

As before we denote by $\mathbb{F}(\mathbf{r})$ the manifold of flags whose type is the multi-index $\mathbf{r}$. Let $\beta = \{v_1, \ldots, v_d\}$ be an ordered basis of $\mathbb{R}^d$. If $\mathbf{r} = \{r_1, \ldots, r_k\}$ is a multi-index we denote by $b_{\mathbf{r}}(\beta)$ the flag of type $\mathbf{r}$ spanned by β, that is,

$$b_{\mathbf{r}}(\beta) = (V_1, \ldots, V_k)$$

where $V_i = \mathrm{span}\{v_1, \ldots, v_{r_i}\}$. For the complete sequence $\mathbf{r} = \mathbf{r}_c$ we suppress the subscript and write simply $b(\beta)$. Also, let $\mathcal{W}$ be the permutation group in d letters. Given $w \in \mathcal{W}$ and an ordered basis $\beta = \{v_1, \ldots, v_d\}$ we write $w\beta = \{v_{w(1)}, \ldots, v_{w(d)}\}$ for the basis in the permuted order. This construction gives rise to new flags $b_{\mathbf{r}}(w\beta)$ obtained from new orderings of β.

Of special importance for the study of semigroups in $\mathrm{Sl}(d)$ is the action of the diagonalizable matrices on the flag manifolds. We say that $h \in \mathrm{Sl}(d)$ is regular if its matrix with respect to some ordered basis $\beta = \{v_1, \ldots, v_d\}$ of $\mathbb{R}^d$ is

$$\mathrm{diag}[\lambda_1, \ldots, \lambda_d]$$

with $\lambda_1 > \cdots > \lambda_d > 0$. Of course, up to multiplication by scalars the ordered basis of eigenvectors of h is unique. Any such basis is denoted by $\beta(h)$. Then the flag $b_{\mathbf{r}}(\beta(h))$ is well defined. We write $b_{\mathbf{r}}(h)$ for $b_{\mathbf{r}}(\beta(h))$ or simply $b(h)$ if $\mathbf{r}$ is the complete multi-index.

The action of a regular h in $\mathbb{F}(\mathbf{r})$ is described as follows. Firstly a flag $b \in \mathbb{F}(\mathbf{r})$ is fixed by h, i.e., $hb = b$, if and only if $b = b_{\mathbf{r}}(w\beta(h))$ for some permutation $w \in \mathcal{W}$. In the sequel we say that $b_{\mathbf{r}}(w\beta)$ is the fixed-point of type w of h, and write $\mathrm{fix}_{\mathbf{r}}(h,w)$ for $b_{\mathbf{r}}(w\beta(h))$. Note that in particular, $\mathrm{fix}_{\mathbf{r}}(h,1) = b_{\mathbf{r}}(h)$, where 1 is the identity permutation.

The association $w \mapsto \mathrm{fix}_{\mathbf{r}}(h,w)$ maps W onto the set of fixed points of h. For general $\mathbf{r}$ this map is not injective (e.g. if $\mathbf{r} = \{1\}$ then $\mathbb{F}(\mathbf{r})$ is the projective space and $b_{\mathbf{r}}(w\beta) = b_{\mathbf{r}}(\beta)$ for every w with $w(1) = 1$). However, for the full flag $\mathbb{F}$ this map is injective defining a bijection between the h-fixed points and W. In general, the projection $\pi_{\mathbf{r}} : \mathbb{F} \to \mathbb{F}(\mathbf{r})$ maps $\mathrm{fix}(h,w)$ into $\mathrm{fix}_{\mathbf{r}}(h,w)$.

Now, let N_{β}^{-} be the subgroup of $\mathrm{Sl}(d)$ whose matrices with respect to the basis β are lower triangular with ones on the main diagonal. The so called Bruhat decomposition of $\mathbb{F}(\mathbf{r})$ says that the orbits

$$N_{\beta}^{-} b_{\mathbf{r}}(w\beta) = \{ n b_{\mathbf{r}}(w\beta) : n \in N_{\beta}^{-} \}$$

cover the flag manifold $\mathbb{F}^{d}(\mathbf{r})$ and are disjoint to each other. Just one of these orbits is open and dense in $\mathbb{F}^{d}(\mathbf{r})$, namely $N_{\beta}^{-} b_{\mathbf{r}}(\beta)$. Moreover, let h be a regular matrix with $\beta(h) = \beta$. Then it is easily checked that $h^{k} n h^{-k} \to 1$ as $k \to +\infty$, if $n \in N_{\beta}^{-}$. Hence, for any flag $b = n b_{\mathbf{r}}(w\beta) \in N_{\beta}^{-} b_{\mathbf{r}}(w\beta)$ it holds that $h^{k} b = h^{k} n h^{-k} b_{\mathbf{r}}(w\beta) \to b_{\mathbf{r}}(w\beta)$ as $k \to +\infty$. In other words the orbit $N_{\beta}^{-} b_{\mathbf{r}}(w\beta)$ is the stable manifold of the h-fixed point $b_{\mathbf{r}}(w\beta)$.

Among the orbits $N_{\beta}^{-} b_{\mathbf{r}}(w\beta)$ only $N_{\beta}^{-} b_{\mathbf{r}}(\beta)$ is open and dense. Hence, $b_{\mathbf{r}}(\beta)$ is the only attractor of the h-action on $\mathbb{F}(\mathbf{r})$. We denote its stable manifold $N_{\beta}^{-} b_{\mathbf{r}}(\beta)$ simply by $\sigma_{\mathbf{r}}(h)$. Also, we often write $\mathrm{att}_{\mathbf{r}}(h)$ for $\mathrm{fix}_{\mathbf{r}}(h,1)$.

Analogous considerations hold if we take instead the subgroup N_{β}^{+} of upper triangular matrices. Now, the orbits $N_{\beta}^{+} b_{\mathbf{r}}(w\beta)$ become unstable manifolds because $h^{k} n h^{-k} \to \infty$, as $k \to +\infty$, if $n \in N_{\beta}^{+}$. Only one of these orbits is open, which is the unstable manifold of a unique repeller, denoted by $\mathrm{rp}_{\mathbf{r}}(h)$. Explicitly, $\mathrm{rp}_{\mathbf{r}}(h) = \mathrm{fix}_{\mathbf{r}}(h, w_0)$, where $w_0 = (1,d)(2,d-1)\cdots$ is the permutation which completely inverts the ordering.

In the sequel we say that a set like $\sigma_{\mathbf{r}}(h)$ or $N_{\beta}^{-} b_{\mathbf{r}}(\beta)$ (where h is a regular element and β is a basis of $\mathbb{R}^{d}$) is an *open cell* in $\mathbb{F}(\mathbf{r})$. Note that the group $\mathrm{Sl}(d)$ acts transitively on the set of open cells, since $\sigma_{\mathbf{r}}(ghg^{-1}) = g\sigma_{\mathbf{r}}(h)$.

The existence of attractor fixed points is ensured also for matrices which are partially regular in the following sense: Let us say that a diagonalizable matrix h is $\mathbf{r}$-regular in case its eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0$, put in decreasing order, satisfy $\lambda_{r_i} > \lambda_{r_i+1}$, where $\mathbf{r} = \{r_1, \ldots, r_k\}$, and h is strictly $\mathbf{r}$-regular in case equality occurs for the other eigenvalues. In a manner similar to the case of regular elements $b_{\mathbf{r}}(\beta)$ is an attractor for the $\mathbf{r}$-regular matrix h, where β is some basis which diagonalizes h. To see this let $N_{\beta,\mathbf{r}}^{-}$ be the subgroup of matrices of the form

$$\begin{pmatrix} 1_{r_1} & & 0 \\ \vdots & \ddots & \\ * & \cdots & 1_{(d-r_k)} \end{pmatrix},$$

where the identity matrices in the diagonal blocks have size $r_i - r_{i-1}$. An easy computation shows that $N^-_{\beta,\mathbf{r}} b_{\mathbf{r}}(\beta) = N^-_{\beta} b_{\mathbf{r}}(\beta)$. Since $h^k n h^{-k} \to 1$ for all $n \in N^-_{\beta,\mathbf{r}}$, it follows that $b_{\mathbf{r}}(\beta)$ is an attractor of h. This shows the existence of attractors for $\mathbf{r}$-regular matrices in the specific flag manifold $\mathbb{F}(\mathbf{r})$. This implies that $b_{\mathbf{r}'}(\beta) = \pi^{\mathbf{r}}_{\mathbf{r}'} b_{\mathbf{r}}(\beta)$ is the attractor of h in $\mathbb{F}(\mathbf{r}')$ if $\mathbf{r}' \subset \mathbf{r}$. As for the regular matrices we denote by $\mathrm{att}(h)$ and $\sigma(h)$ the attractor and its stable manifold for an $\mathbf{r}$-regular matrix h.

The following lemma describes the contraction of subsets under the action of $\mathbf{r}$-regular elements. It shall be used frequently in the sequel.

Lemma 3.1. *Let h be a $\mathbf{r}$-regular matrix and $C \subset \mathbb{F}(\mathbf{r})$ a compact subset with $\mathrm{int}C \neq \emptyset$. Suppose that $C \subset \sigma(h)$ and $\mathrm{att}(h) \subset \mathrm{int}C$. Then for any neighborhood U of $\mathrm{att}(h)$ contained in C there exists a positive integer n_0 such that $h^n C \subset U$ for all $n \geq n_0$.*

Proof. Since $C \subset \sigma(h)$, it follows that $h^n x \to \mathrm{att}(h)$, as $n \to +\infty$, for all $x \in C$. But C is compact. Hence this convergence is uniform in C, as claimed in the lemma. ∎

3.2. Control sets

We let Γ be a subsemigroup of $\mathrm{Sl}(d)$ with $\mathrm{int}\Gamma \neq \emptyset$ where the interior is taken with respect to the standard topology of $\mathrm{Sl}(d)$. By restricting the $\mathrm{Sl}(d)$-action to Γ it becomes a semigroup of transformations of any flag manifold $\mathbb{F}(\mathbf{r})$. Given $b \in \mathbb{F}(\mathbf{r})$ we denote with Γb its orbit under Γ, that is, $\Gamma b = \{gb : g \in \Gamma\}$.

A *control set* of Γ in $\mathbb{F}(\mathbf{r})$ is a subset $C \subset \mathbb{F}(\mathbf{r})$ satisfying

1. $C \subset \mathrm{cl}(\Gamma b)$ for all $b \in C$, where cl means closure.
2. $gb = b$ for some $b \in C$ and $g \in \mathrm{int}\Gamma$.
3. C is maximal with respect to set inclusion.

A control set is said to be invariant if $\mathrm{cl}(C) = \mathrm{cl}(\Gamma b)$ for all $b \in C$. The condition $\mathrm{int}\Gamma \neq \emptyset$ implies that invariant control sets are closed, so that we have actually that $C = \mathrm{cl}(\Gamma b)$ for all $b \in C$ if C is an invariant control set (see e.g. [13]).

The *set of transitivity* (or *core*) C_0 of a control set C is defined by

$$C_0 = \{b \in C : \exists g \in \mathrm{int}\Gamma,\ gb = b\}.$$

It is known that C_0 is open and dense in C (see [13]).

The control sets of the semigroup Γ in a flag manifold $\mathbb{F}(\mathbf{r})$ were described in [13]. In order to describe the main result we denote by $\mathrm{reg}(\Gamma)$ the set of regular matrices in $\mathrm{int}\Gamma$.

Theorem 3.2. *Let $\Gamma \subset \mathrm{Sl}(d)$ be a semigroup with $\mathrm{int}\,\Gamma \neq \emptyset$ and consider its action in $\mathbb{F}(\mathbf{r})$. Then for each $w \in W$, the set*

$$D_{\mathbf{r}}(w)_0 = \{\mathrm{fix}_{\mathbf{r}}(h, w) : h \in \mathrm{reg}(\Gamma)\}$$

is the core of a unique control set denoted by $D_{\mathbf{r}}(w)$. Any control set is $D_{\mathbf{r}}(w)$ for some $w \in W$. There exists a unique invariant control set, namely $D_{\mathbf{r}}(1)$, whose core is given by $\mathrm{att}_{\mathbf{r}}(h)$, $h \in \mathrm{reg}(\Gamma)$. Also, the set of repellers $D_{\mathbf{r}}(w_0)$ is the only control set which is Γ^{-1}-invariant.

Following our pattern of notation we suppress the subscript for the control sets in the full flag manifold and write them as $D(w)$, $w \in W$.

The above theorem is complemented by the following result, also proved in [13].

Proposition 3.3. *For multi-indices $\mathbf{r}_1 \subset \mathbf{r}_2$ let $\pi^{\mathbf{r}_2}_{\mathbf{r}_1} : \mathbb{F}(\mathbf{r}_2) \to \mathbb{F}(\mathbf{r}_1)$ be the canonical projection. Then $\pi(D_{\mathbf{r}_2}(w)) = D_{\mathbf{r}_1}(w)$.*

3.3. Parabolic type

Theorem 3.2 shows the existence of a mapping $w \mapsto D(w)$ onto the control sets. The study of its level sets yields the concept of parabolic type of a semigroup. Our purpose here is to discuss this concept for semigroups in $\mathrm{Sl}(d)$.

To begin with we shall write partitions of $\delta = \{1, \ldots, d\}$ using the same kind of multi-indices appearing before. Thus, any $\mathbf{r} = \{r_1, \ldots, r_k\}$, with $1 \leq r_1 < \cdots < r_k < d$, gives rise to the partition

$$\delta = [1, r_1] \cup [r_1 + 1, r_2] \cup \cdots \cup [r_k + 1, d]$$

of δ into non-overlapping intervals. From such a partition we get the following subgroup of the permutation group

$$W(\mathbf{r}) = \Pi[1, r_1] \cdot \Pi[r_1 + 1, r_2] \cdots \Pi[r_k + 1, d],$$

where $\Pi[a, b]$ stands for the permutation group of the elements of the interval $[a, b] = \{a, a + 1, \ldots, b\}$. In the Lie theoretic literature these groups are known as parabolic subgroups of W.

Now, for a semigroup Γ with $\mathrm{int}\,\Gamma \neq \emptyset$, put

$$W(\Gamma) = \{w \in W : D(w) = D(1)\}.$$

It turns out that $W(\Gamma)$ is a parabolic subgroup of W, that is, $W(\Gamma) = W(\mathbf{r})$ for some multi-index $\mathbf{r} = \mathbf{r}(\Gamma)$.

We call the *parabolic type* of Γ either the multi-index $\mathbf{r}(\Gamma)$ or the subgroup $W(\mathbf{r}(\Gamma))$. In case $\mathbf{r} = \{k\}$ is a singleton we say simply that k is the parabolic type of Γ or that Γ is of type k.

The next propositiontion relates the parabolic type of a semigroup to the possible $\mathbf{r}$-regular elements that exists inside the interior of the semigroup. In its statement we say that a matrix with real eigenvalues is of type $\mathbf{r}$ in case its semi-simple component is strictly $\mathbf{r}$-regular.

Proposition 3.4. *The following conditions are equivalent for a semigroup $\Gamma \subset \mathrm{Sl}(d)$ with $\mathrm{int}\Gamma \neq \emptyset$.*

1. *The parabolic type of Γ is $\mathbf{r}$.*
2. *$\mathbf{r}$ is maximal with the property $D(1) = \pi_{\mathbf{r}}^{-1}(D_{\mathbf{r}}(1))$.*
3. *There exists $h \in \mathrm{int}\Gamma$ of type $\mathbf{r}$, and conversely if $g \in \mathrm{int}\Gamma$ has real eigenvalues then its type is $\mathbf{r}' \supset \mathbf{r}$.*
4. *$\mathrm{int}\Gamma$ contains a strictly $\mathbf{r}$-regular element and if $g \in \mathrm{int}\Gamma$ is $\mathbf{r}'$-regular then $\mathbf{r}' \supset \mathbf{r}$.*

Proof. See [13] for a proof which works for semigroups in general semi-simple Lie groups. ■

Afterwards we shall check that the parabolic type of both semigroups $S_{\mathbf{r}}$ and $T_{\mathbf{r}}$ is precisely $\mathbf{r}$.

If $\Gamma_1 \subset \Gamma_2$ and $\mathbf{r}$ is the parabolic type of Γ_2 then the parabolic type of Γ_1 contains $\mathbf{r}$.

For later reference we state the following properties of the invariant control sets related to the parabolic type of a semigroup. Let us say that a subset $C \subset \mathbb{F}(\mathbf{r})$ is Γ-admissible in case it is contained in $\sigma(h)$ for all $h \in \mathrm{reg}(\Gamma)$.

Proposition 3.5. *For a semigroup $\Gamma \subset \mathrm{Sl}(d)$ with non-empty interior the following properties hold.*

1. *Suppose that the parabolic type of Γ is $\mathbf{r}$, and let $\mathbf{r}_1$ be a multi-index containing $\mathbf{r}$. Let $\pi : \mathbb{F}(\mathbf{r}_1) \to \mathbb{F}(\mathbf{r})$ be the projection. Then $D_1(\mathbf{r}_1) = \pi^{-1}(D_1(\mathbf{r}))$.*
2. *Suppose that the parabolic type of Γ is $\mathbf{r}$. Then its invariant control set $D_1(\mathbf{r})$ is Γ-admissible.*
3. *If $D_1(\mathbf{r})$ is Γ-admissible in $\mathbb{F}(\mathbf{r})$ then the parabolic type of Γ contains $\mathbf{r}$.*

Proof. The first two properties are the contents of Theorem 4.3 and Proposition 4.8 in [13], respectively. The last statement is a consequence of previous two. ■

Note that the first property stated above allows the determination of the invariant control set of the semigroup Γ in any flag manifold as soon as one knows the invariant control set $D_1(\mathbf{r})$ if its parabolic type is $\mathbf{r}$. In fact, the invariant control set in the full flag $\mathbb{F}(\mathbf{r}_c)$ is $\pi^{-1}(D_1(\mathbf{r}))$. Hence the invariant control set in a flag manifold $\mathbb{F}(\mathbf{r}_1)$ is $\pi_{\mathbf{r}_1}(\pi^{-1}(D_1(\mathbf{r})))$.

The third condition of Proposition 3.4 shows in particular that for a semigroup Γ whose parabolic type is the complete multi-index $\mathbf{r}_c$, any $h \in \operatorname{int}\Gamma$ with real eigenvalues is diagonalizable. We can improve this fact by showing that any $g \in \operatorname{int}\Gamma$ has real eigenvalues and is diagonalizable.

Corollary 3.6. *Let Γ be a semigroup with non-empty whose parabolic type is the complete multi-index $\mathbf{r}_c$. Let $g \in \operatorname{int}\Gamma$. Then g has real distinct eigenvalues.*

Proof. Let $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_s$ the decomposition of $\mathbb{R}^d$ into primary subspaces of g. We must show that $\dim V_i = 1$ for all $i = 1, \ldots, s$. First observe that we can perturb g and get $g_1 \in \operatorname{int}\Gamma$ which has the same primary components of g and such that its eigenvalues are of the form e^{a+ib} with $b = q\pi$, $q \in \mathbb{Q}$. Hence by taking some power of g_1 we arrive at $g_2 \in \operatorname{int}\Gamma$, having positive eigenvalues and whose primary decomposition, say $\mathbb{R}^d = W_1 \oplus \cdots \oplus W_r$, satisfies $V_i \subset W_j$. However $\mathcal{W}(\Gamma) = \{1\}$, hence the eigenvalues of g_2 are simple. This implies that $\dim W_i = 1$ and hence that g is diagonalizable. ■

In the next section we shall check that the semigroups T of totally positive matrices and S of sign regular matrices have parabolic type $\mathbf{r}_c$, and hence satisfy the condition of the above corollary.

4. Maximal semigroups

In [11] the maximal semigroups in a semi-simple Lie group were described as compression semigroups of certain subsets in the flag manifolds of the group. We shall specialize here the results of [11] to the case of $\mathrm{Sl}(d)$ and the flag manifolds $\mathbb{F}(\mathbf{r})$.

Recall that a subsemigroup Γ of a group G is said to be maximal provided Γ is not a group and if U is a semigroup satisfying $\Gamma \subset U \subset G$ then $\Gamma = U$ or $U = G$. In our context we have the following generalization of the notion of maximality (see [11]).

Definition 4.1. A semigroup $\Gamma \subset \mathrm{Sl}(d)$ with $\operatorname{int}\Gamma \neq \emptyset$ is said to be $\mathbf{r}$-maximal if $\mathbf{r}$ is its parabolic type and Γ is not properly contained in any semigroup of parabolic type $\mathbf{r}$.

It is proved in [11] that a semigroup Γ with $\operatorname{int}\Gamma \neq \emptyset$ is maximal in case it is $\mathbf{r}$-maximal with $\mathbf{r}$ a singleton. We prove below that a sign-regular semigroup $S_{\mathbf{r}}$ is $\mathbf{r}$-maximal. In particular it will produce that $\mathbf{r}$ is the parabolic type of $S_{\mathbf{r}}$ which, in turn, implies that $\mathbf{r}$ is also the parabolic type of $T_{\mathbf{r}}$. This has some nice consequences about the eigenvalues of a matrix which belongs to the interior of one of these semigroups.

We start with the notion of duality between flag manifolds. Given a multi-index $\mathbf{r} = \{r_1, \ldots, r_k\}$, with $1 \leq r_1 < \cdots < r_k < d$, let $\mathbf{r}^* = \{s_1, \ldots, s_k\}$ be such that $s_1 = d - r_k, \ldots, s_k = d - r_1$. Following [11] (see Section 3.1) we say that the flag manifolds $\mathbb{F}(\mathbf{r})$ and $\mathbb{F}(\mathbf{r}^*)$ are dual to each other. The point about this duality is that the possible open cells in $\mathbb{F}(\mathbf{r})$ (as well as in $\mathbb{F}(\mathbf{r}^*)$) are given by incidence as follows: Fix $b = (U_1, \ldots, U_k)$ in $\mathbb{F}(\mathbf{r}^*)$ and put

$$\sigma_b = \{(V_1, \ldots, V_k) \in \mathbb{F}(\mathbf{r}) : V_i \cap U_{k-i+1} = 0\}.$$

Then σ_b is an open cell in $\mathbb{F}(\mathbf{r})$ and every such cell is σ_b for some $b \in \mathbb{F}(\mathbf{r}^*)$. Therefore, the flag manifold $\mathbb{F}(\mathbf{r}^*)$ identifies with the set of open cells in $\mathbb{F}(\mathbf{r})$. Of course, since $\mathbf{r} = (\mathbf{r}^*)^*$, we have an analogous identification between $\mathbb{F}(\mathbf{r})$ and the open cells in $\mathbb{F}(\mathbf{r}^*)$.

These identifications give rise to the following notion of duality between subsets of $\mathbb{F}(\mathbf{r})$ and $\mathbb{F}(\mathbf{r}^*)$. Given $C \subset \mathbb{F}(\mathbf{r})$ let $C^* \subset \mathbb{F}(\mathbf{r}^*)$ be defined by

$$C^* = \{b \in \mathbb{F}(\mathbf{r}^*) : C \subset \sigma_b\}.$$

Clearly, if $D \subset \mathbb{F}(\mathbf{r}^*)$ we can write the same way $D^* \subset \mathbb{F}(\mathbf{r})$, so that it makes sense to write $(C^*)^* \subset \mathbb{F}(\mathbf{r})$. According to [11] we say that a subset $C \subset \mathbb{F}(\mathbf{r})$ is *B-convex* in case $C = (C^*)^*$. A subset $C \subset \mathbb{F}(\mathbf{r})$ is said to be *admissible* if $C^* \neq \emptyset$.

The following result shows that the $\mathbf{r}$-maximal semigroups are essentially the compression semigroups of B-convex sets.

Theorem 4.2. *A semigroup $\Gamma \subset \mathrm{Sl}(d)$ with $\mathrm{int}\Gamma \neq \emptyset$ is $\mathbf{r}$-maximal if and only if there exists an admissible B-convex set $D \subset \mathbb{F}(\mathbf{r})$ such that Γ is the compression semigroup of $K = \mathrm{cl}(\mathrm{int}D)$, that is,*

$$\Gamma = \{g \in \mathrm{Sl}(d) : gK \subset K\}.$$

Proof. See [11], Theorem 5.4. ∎

Our objective now is to prove that the subsets $C_{\mathbf{r}}$ defined in (6) are B-convex. This will imply that the semigroup $S_{\mathbf{r}}$ is $\mathbf{r}$-maximal. This fact was claimed without proof in [11] (see Section 6.3) and a proof was offered for the case $\mathbf{r} = \{k\}$ is a singleton. In this case the proof is based in the following lemma.

Lemma 4.3. *Let $C_k \subset \mathrm{Gr}_k$ be as before. Then*

$$C_k^* = \{V^\perp : V \in \mathrm{int}(C_k)\}$$

where $V^\perp$ stands for the ortho-complement in $\mathbb{R}^d$ of the k-dimensional subspace V.

Proof. See [11], Lemma 6.7 and comments after the proof. ∎

This lemma shows in particular that the subsets C_k are admissible. Now we can check the B-convexity of C_k:

Corollary 4.4. *For any k we have $C_k^{**} = C_k$, that is C_k is B-convex.*

Proof. It follows from general facts that $C_k \subset C_k^{**}$. For the reverse inclusion, suppose that the k-dimensional subspace $V \in C_k^{**}$ does not belong to C_k. By the lemma and the definitiontion of duality, it follows that there exists $W \in \mathrm{int} C_k$ such that $V \cap W^{\perp} \neq 0$. To see that this is a contradiction choose a basis $\{v_1, \ldots, v_k\}$ of V and a basis $\{w_1, \ldots, w_k\}$ of W such that $v_1 \in W^{\perp}$. Then if we put $\xi = v_1 \wedge \cdots \wedge v_k$ and $\eta = w_1 \wedge \cdots \wedge w_k$ it follows that $(\xi, \eta) = 0$ (c.f. formula (1)). But this contradicts the fact that $W \in \mathrm{int} C_k$, that is, $\eta \in \pm \mathrm{int} O_k$.

■

Now it is a matter of playing successively with the definitiontions to check B-convexity in general.

Proposition 4.5. *Each set $C_{\mathbf{r}}$ is B-convex in $\mathbb{F}(\mathbf{r})$.*

Proof. First recall that if $\mathbf{r}$ is given by $1 \leq r_1 < \cdots < r_k < d$ then

$$C_{\mathbf{r}} = \left(\pi^{\mathbf{r}}_{r_1}\right)^{-1}(C_{r_1}) \cap \cdots \cap \left(\pi^{\mathbf{r}}_{r_k}\right)^{-1}(C_{r_k}).$$

This definitiontion implies immediately that $(V_1, \ldots, V_k) \in \mathbb{F}(\mathbf{r})$ belongs to $C_{\mathbf{r}}$ if and only if each $V_i \in C_{r_i}$, $i = 1, \ldots, k$. Now, take $b = (U_1, \ldots, U_k) \in \mathbb{F}(\mathbf{r}^*)$. By definitiontion of the duality operator, it follows that $b \in C_{\mathbf{r}}^*$ if and only if $V_i \cap U_{k-i+1} = 0$, for all $i = 1, \ldots, k$ and $(V_1, \ldots, V_k) \in C_{\mathbf{r}}$. But this is equivalent to saying that each U_{k-i+1}, $i = 1, \ldots, k$, belongs to $C_{r_i}^*$. This shows that

$$C_{\mathbf{r}}^* = \left(\pi^{\mathbf{r}}_{r_1}\right)^{-1}(C_{r_1}^*) \cap \cdots \cap \left(\pi^{\mathbf{r}}_{r_k}\right)^{-1}(C_{r_k}^*).$$

If we repeat the same reasoning, we arrive that $C_{\mathbf{r}}^{**}$ is given by the intersection of the pre-images of $C_{r_i}^{**}$ under the projections. Therefore, the above corollary implies that $C_{\mathbf{r}}^{**} = C_{\mathbf{r}}$, concluding the proof.

■

During the proof of this propositiontion we got the following description of $C_{\mathbf{r}}^*$.

Corollary 4.6. *Given $b = (V_1, \ldots, V_k) \in \mathbb{F}(\mathbf{r})$ put $b^{\perp} = (V_k^{\perp}, \ldots, V_1^{\perp}) \in \mathbb{F}(\mathbf{r}^*)$. Then*

$$C_{\mathbf{r}}^* = \{b^{\perp} \in \mathbb{F}(\mathbf{r}^*) : b \in \mathrm{int}\,(C_{\mathbf{r}})\}.$$

The B-convexity of $C_{\mathbf{r}}$ together with the fact that $\mathrm{cl}\,(\mathrm{int} C_{\mathbf{r}}) = C_{\mathbf{r}}$ and Theorem 4.2 yield the desired results about the semigroups $S_{\mathbf{r}}$ and $T_{\mathbf{r}}$.

Theorem 4.7. *The following facts hold true.*

1. *Given a multi-index, the parabolic type of $S_{\mathbf{r}}$ is $\mathbf{r}$ and $S_{\mathbf{r}}$ is $\mathbf{r}$-maximal. For each $k = 1, \ldots, d-1$, the semigroup S_k is maximal in $\mathrm{Sl}\,(d)$.*
2. *The invariant control set of both semigroups $S_{\mathbf{r}}$ and $T_{\mathbf{r}}$ in $\mathbb{F}(\mathbf{r})$ is $C_{\mathbf{r}}$.*

3. *If* $\mathbf{r}' \subset \mathbf{r}$ *then the invariant control set of both semigroups* $S_{\mathbf{r}}$ *and* $T_{\mathbf{r}}$ *in* $\mathbb{F}(\mathbf{r}')$ *is* $\left(\pi^{\mathbf{r}}_{\mathbf{r}'}\right)^{-1} C_{\mathbf{r}}$.

4. *If* $h \in \mathrm{int} S_{\mathbf{r}}$ *or* $h \in \mathrm{int} T_{\mathbf{r}}$ *has real eigenvalues then* h *has type* $\mathbf{r}_1$ *with* $\mathbf{r}_1 \subset \mathbf{r}$.

5. *If* $h \in \mathrm{int} S$ *then* h *is diagonalizable.*

6. *If* $h \in \mathrm{int} T$ *then its eigenvalues are* > 0.

Proof.

1. Is a consequence of Theorem 4.2 and Proposition 4.5.

2. The invariant control set of $S_{\mathbf{r}}$ in $\mathbb{F}(\mathbf{r})$ is $C_{\mathbf{r}}$ by Theorem 4.2. As to $T_{\mathbf{r}}$ we note that any $x \in \mathrm{int} C_{\mathbf{r}}$ is the attractor fixed point of some regular element $h \in \mathrm{int} S_{\mathbf{r}}$. Then $h^2 \in \mathrm{int} T_{\mathbf{r}}$ also has x a attractor. This implies that x belongs to the interior of the invariant control set of $T_{\mathbf{r}}$. Taking closures it follows that $C_{\mathbf{r}}$ is the invariant control set of $T_{\mathbf{r}}$ as well.

3. Since $\mathbf{r}$ is the parabolic type of both $S_{\mathbf{r}}$ and $T_{\mathbf{r}}$, the result follows by Proposition 3.5.

4. Follows from Proposition 3.4.

5. Is a consequence of Corollary 3.6.

6. The subspace of $\bigwedge^k$ spanned by principal eigenvector of $\wedge^k g$ belongs to the invariant control set of T in Gr_k. Hence, g has a principal eigenvector in O_k. Since $(\wedge^k g)\, O_k \subset O_k$, it follows that the highest eigenvalue, say μ_k, of $\wedge^k g$ is positive. But $\mu_k = \lambda_1 \cdots \lambda_k$ where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of g ordered by $|\lambda_1| > \cdots > |\lambda_d|$. Since $\mu_k > 0$ for every $k = 1, \ldots, d$, it follows that $\lambda_i > 0$, $i = 1, \ldots, d$. ■

Remark. The last two statements in the above theorem are well known results about sign-regular and totally positive matrices (see [1]).

5. Some lemmas

For the description of the control sets of the semigroups S, T and more generally $S_{\mathbf{r}}$ and $T_{\mathbf{r}}$ the following basic facts about sign changes of vectors in $\mathbb{R}^d$ shall be required. By an orthant in $\mathbb{R}^d$ we understand a closed set of the form $O^+_\varepsilon = \{(x_1, \ldots, x_d) : \varepsilon_i x_i \geq 0\}$, where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)$ is a sign vector, that is, $\varepsilon_i = \pm 1$, $i = 1, \ldots, d$. In our notation the superscript $+$ is intended to distinguish the orthant in $\mathbb{R}^d$ from the corresponding orthant $O_\varepsilon \subset \mathbb{P}$, which is the set lines in $\mathbb{R}^d$ contained in $O^+_\varepsilon \cup O^-_\varepsilon$, $O^-_\varepsilon = -O^+_\varepsilon$. Clearly, in $\mathbb{P}$ the orthants satisfy $O_{-\varepsilon} = O_\varepsilon$.

Given a sign vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)$, we let $V(\varepsilon)$ denote the number of sign changes of ε, that is, $V(\varepsilon)$ is the number of indices $i = 1, \ldots, d-1$

such that $\varepsilon_i \varepsilon_{i+1} = -1$. The number of sign changes of the orthant O_ε^+ is by definition $V(\varepsilon)$. Note that $V(-\varepsilon) = V(\varepsilon)$, so that it makes sense to define the number of sign changes of an orthant O_ε in $\mathbb{P}$.

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we say that $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)$ is a sign sequence of x if $x \in O_\varepsilon$. Let $V_+(x)$ [respectively $V_-(x)$] stand for the maximum [minimum] of $V(\varepsilon)$ with ε running through the sign sequences of x. Note that $V_+(x) = V_-(x)$ if and only if x belongs to the interior of an orthant (or equivalently if x has nonvanishing coordinates). In this case we put $V(x) = V_+(x) = V_-(x)$, which is the number of indices i such that $x_i x_{i+1} < 0$ (c.f. [1]).

For $k = 0, 1, \ldots, d-1$, put

$$O_{\leq k} = \bigcup_{V(\varepsilon) \leq k} O_\varepsilon \qquad \Sigma_k = O_{\leq k} \setminus O_{\leq (k-1)} \tag{7}$$

where $O_{\leq -1} = \emptyset$. The subspace $[x]$ spanned by a vector $0 \neq x \in \mathbb{R}^d$ belongs to $O_{\leq k}$ if and only if some of its sign sequences have at most k sign changes, that is, if and only if $V_-(x) \leq k$. It follows that $[x] \in \Sigma_k$ if and only if $V_-(x) = k$. For later reference we prove the following lemma which describes the interior and the boundary of these sets.

Lemma 5.1. *Let $0 \neq x \in \mathbb{R}^d$ and denote by $[x] \in \mathbb{P}$ the subspace it spans. Then*

1. *$[x] \in \operatorname{int}(\Sigma_k)$ if and only if $V_-(x) = V_+(x) = k$. Hence, $[x] \in \Sigma_k \cap \partial\Sigma_k$ if and only if $k = V_-(x) < V_+(x)$.*

2. *$[x] \in \operatorname{int}(O_{\leq k})$ if and only if $V_+(x) \leq k$. Hence, $[x] \in O_{\leq k} \cap \partial O_{\leq k}$ if and only if $V_-(x) \leq k < V_+(x)$.*

Proof. Suppose that $V_-(x) = V_+(x) = k$, and take a vector $\delta = (\delta_1, \ldots, \delta_d)$ small enough so that $(x_i + \delta_i) x_i > 0$ if $x_i \neq 0$. Note that the only restriction for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)$ to be a sign sequence of $y = (y_1, \ldots, y_d)$ is that $\varepsilon_i = y_i / |y_i|$ if $y_i \neq 0$. Hence the sign sequences of $x + \delta$ are sign sequences of x. Hence $V_-(x + \delta) = V_+(x + \delta) = k$, showing that $x + \delta \in \Sigma_k$. Therefore $x \in \operatorname{int}\Sigma_k$. Conversely, suppose that $k = V_-(x) < V_+(x)$. Then x belongs to an orthant with $V_+(x)$ sign changes, hence $x \notin \operatorname{int}\Sigma_k$.

For $O_{\leq k}$ the proof is similar. If $V_+(x) \leq k$ then the sign sequences of a neighboring point $x + \delta$ have at most k sign changes, so that $x + \delta \in O_{\leq k}$. Reciprocally, if $k < V_+(x)$ then x belongs to an orthant with more than k sign changes. ∎

We conclude this section with the following lemmas on k-positive subspaces. The notation $(1, \varepsilon)$ stands for a vector $(1, \varepsilon_2, \ldots, \varepsilon_d)$ with $\varepsilon_i = \pm 1$, $i = 2, \ldots, d$.

Lemma 5.2. *Let V be a k-positive [respectively, strictly k-positive] subspace containing a vector $(1,x)$. Then $V \cap e_1^\perp$ is a $(k-1)$-positive [respectively strictly $(k-1)$-positive] subspace in the orthocomplement of e_1, with respect to the basis $\{e_2,\dots,e_d\}$.*

Proof. The assumption that $(1,x) \in V$ implies that V is not contained in $e_1^\perp$, so that $\dim(V \cap e_1^\perp) = k-1$. Hence, there are $v_2,\dots,v_k \in V \cap e_1^\perp$ such that $\{(1,x), v_2,\dots,v_k\}$ is a basis of V. Since V is k-positive we can assume without loss of generality that

$$\xi = (1,x) \wedge v_2 \wedge \cdots \wedge v_k \in C_k^+.$$

Let $\mathbf{r}$ be a multi-index of $\{2,\dots,d\}$ and form the multi-index $(1,\mathbf{r})$, by adjoining 1. Since e_1 is orthogonal to $v_2,\dots,v_k$, it follows that

$$\langle \xi, e_{(1,\mathbf{r})} \rangle = \langle v_2 \wedge \cdots \wedge v_k, e_{\mathbf{r}} \rangle.$$

Hence $\langle v_2 \wedge \cdots \wedge v_k, e_{\mathbf{r}} \rangle \geq 0$ (respectively > 0) for every multi-index $\mathbf{r}$ if V is k-positive (respectively strictly k-positive), proving the lemma. ■

Clearly the subspace $V \cap e_1^\perp$ in this lemma is also $(k-1)$-positive in $\mathbb{R}^d$ with respect to the standard basis. Later on this fact will be used to show that every k-positive subspace contains positive subspaces with smaller dimensions. On the other hand the next lemma shows how to extend k-positive subspaces to positive subspaces of higher dimensions.

Lemma 5.3. *Let V be a $(k-1)$-positive subspace with $2 \leq k \leq d$. Then there exists a k-positive subspace $W \supset V$.*

Proof. Let $\{v_2,\dots,v_k\}$ be a basis of V such that $v_2 \wedge \cdots \wedge v_k \in C_{k-1}^+$. Then $e_1 \wedge v_2 \wedge \cdots \wedge v_k \in C_k^+$. In fact, the matrix whose columns are the coordinates of $e_1,\dots,v_k$ is

$$\begin{pmatrix} 1 & a \\ 0 & A \end{pmatrix}$$

with A a $(k-1) \times (k-1)$ matrix. Expanding determinants by the first column one sees that a k-minor of this matrix is either zero or a $(k-1)$-minor of A. Thus the subspace spanned by V and e_1 is k-positive. ■

In the next lemmas we relate k-positive subspaces with the number of sign changes of its elements.

Lemma 5.4. *Suppose that $(1,x)$ belongs to a k-positive subspace V. Then $V(1,x) \leq k-1$.*

Proof. Suppose that there are k sign changes. Then there exists a multi-index

$$\mathbf{r} = (2 \leq i_2 \leq \cdots \leq i_k \leq d)$$

such that $(\epsilon_{i_1}, \ldots, \epsilon_{i_k}) = (-1, 1, -1, \ldots)$. Let $v_2, \ldots, v_k \in e_1^\perp$ be such that

$$\xi = (1, \epsilon) \wedge v_2 \wedge \cdots \wedge v_k \in C_k^+.$$

By Lemma 5.2 $\langle v_2 \wedge \cdots \wedge v_k, e_{\mathbf{r}} \rangle > 0$. However,

$$\langle \xi, e_{\mathbf{r}} \rangle = \det \begin{pmatrix} \epsilon_{i_1} & (v_2)_{i_1} & \cdots & (v_k)_{i_1} \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon_{i_k} & (v_2)_{i_k} & \cdots & (v_k)_{i_k} \end{pmatrix}.$$

But this determinant is strictly negative because the subspace spanned by $v_2, \ldots, v_k$ is strictly negative in $e_1^\perp$. ■

Lemma 5.5. *Suppose that $v = (1, \epsilon)$, with $\epsilon_i = \pm 1$, has exactly $k-1$ sign changes. Then there exists a k-positive subspace V such that $(1, \epsilon) \in V$.*

Proof. The vector v has the form $v = \left(1_{m_1}, -1_{m_2}, \ldots, (-1)^{k+1}_{m_k}\right)$, where the subindex means repetition m_j times. Build v_2 from v by forgetting the last sign change, that is, $v_2 = \left(1_{m_1}, -1_{m_2}, \ldots, -(-1)^{k+1}_{m_k}\right)$. Now proceed successively: from v_{i-1} construct v_i by forgetting the last sign change. This construction ends at $v_k = (1, 1, \ldots, 1)$. Let A be the $d \times k$ matrix formed by the columns of these vectors. It has just k different rows, namely the first m_1 rows are equal, the rows from $m_1 + 1$ to m_2 are equal, and so on. Hence A has just one k-minor. An easy check shows that it is not zero. Hence the subspace spanned by $\{v, v_2, \ldots, v_k\}$ is k-dimensional and positive. ■

6. Invariant control sets

The invariant control sets of $S_{\mathbf{r}}$ and $T_{\mathbf{r}}$ in the flag manifolds $\mathbb{F}(\mathbf{r}')$, $\mathbf{r}' \subset \mathbf{r}$ were given by Theorem 4.7. The purpose of this section is to describe some further invariant control sets of $T_{\mathbf{r}}$ and $S_{\mathbf{r}}$.

We denote by $D_1(\mathbf{r}, \Gamma)$ the invariant control set in the flag manifold $\mathbb{F}(\mathbf{r})$ of the semigroup Γ. By Theorem 4.7 we have $D_1(\mathbf{r}', \Gamma) = \left(\pi_{\mathbf{r}}^{\mathbf{r}'}\right)^{-1} C_{\mathbf{r}}$ if $\mathbf{r}' \subset \mathbf{r}$ and Γ is $T_{\mathbf{r}}$ or $S_{\mathbf{r}}$. Applying this to the complete multi-index $\mathbf{r}' = \mathbf{r}_c$ we get the invariant control set $D_1(\mathbf{r}_c, \Gamma)$ at the full flag manifold, which in turn gives the invariant control set in any flag manifold by projections.

Proposition 6.1. *The invariant control set $D_1(\mathbf{r}', \Gamma)$ of either $\Gamma = S_{\mathbf{r}}$ or $\Gamma = T_{\mathbf{r}}$ in the flag manifold $\mathbb{F}(\mathbf{r}')$ is given by*

$$D_1(\mathbf{r}', \Gamma) = \pi_{\mathbf{r}'}\left(\pi_{\mathbf{r}}^{-1} C_{\mathbf{r}}\right). \tag{8}$$

More explicitly put $\mathbf{r} = \{r_1, \ldots, r_k\}$ and $\mathbf{r}' = \{s_1, \ldots, s_l\}$. Then $D_1(\mathbf{r}', \Gamma)$ is the set of flags $(V_1, \ldots, V_l)$ in $\mathbb{F}(\mathbf{r}')$ which can be completed to a flag $(W_1, \ldots, W_{d-1}) \in \mathbb{F}$ such that W_{r_i} is r_i-positive for any $r_i \in \mathbf{r}$.

For later reference we specialise this result to $\mathbf{r} = \{k\}$ and $\mathbf{r}' = \{1\}$, that is, to invariant control sets on $\mathbb{P}$.

Corollary 6.2. *The invariant control set $D_1(1,\Gamma)$ of either $\Gamma = S_k$ or $\Gamma = T_k$ on $\mathbb{P}$ is the union of k-positive subspaces:*

$$D_1(1,\Gamma) = \bigcup_{V \in C_k} \mathbb{P}(V).$$

Proof. Follows from the fact that a line $x \in \mathbb{P}$ belongs to $\pi_1(\pi_k^{-1}C_k)$ if and only if it is contained in a k-positive subspace. ■

Now we look at the invariant control sets of the semigroups S and T in Gr_k, $k = 1,\ldots,d-1$. Note that for a given k, the invariant control set for both semigroups in Gr_k coincide and is given by $\pi_k(C)$, where $C := C_{\mathbb{F}}$, is the common invariant control set in $\mathbb{F}$ (see Theorem 4.7). We write $D_1(k)$ for the invariant control set $\pi_k(C) \subset \mathrm{Gr}_k$.

Since $S \subset S_k$ and by Theorem 4.7, C_k is the invariant control set of S_k, it follows that $D_1(k) \subset C_k$. We shall prove next that the reverse inclusion also holds.

We have

$$C = (\pi_1)^{-1}(C_1) \cap \cdots \cap (\pi_{d-1})^{-1}(C_{d-1}),$$

which means that C is the subset of flags $(V_1,\ldots,V_{d-1})$ such that V_k is k-positive subspace for all $k = 1,\ldots,d-1$. Therefore, to prove that $C_k \subset D_1(k)$ we must check that every k-positive subspace belongs to a complete flag formed by positive subspaces.

Now, by Lemma 5.3 every k-positive subspace, $k \leq d-1$, is contained in a $(k+1)$-positive subspace. Applying this lemma successively, we see that if V is a k-positive subspace then there exists a flag $V \subset V_{k+1} \subset \cdots \subset V_{d-1}$, with $\dim V_i = i$ such that V_i is i-positive $i = k+1,\ldots,d-1$.

On the other hand to prove the existence of positive subspaces of smaller dimension contained in V we use an argument which combines the results on control sets and the determinantal lemmas of Section 5.

Proposition 6.3. *Let V be a strictly k-positive subspace. Then for any $i \leq k$ there exists an i-positive subspace $W \subset V$.*

Proof. It is enough to prove the statement for $i = k-1$. Put $\mathbf{r} = (k-1,k)$ and consider the semigroup $T_{\mathbf{r}} = T_{k-1} \cap T_k$. By Theorem 4.7 the invariant control set of $T_{\mathbf{r}}$ in the flag manifold $\mathbb{F}(\mathbf{r})$ is $C_{\mathbf{r}} = \pi^{-1}(C_{k-1}) \cap \pi^{-1}(C_k)$, where $\pi = \pi_k^{\mathbf{r}}$ or $\pi = \pi_{k-1}^{\mathbf{r}}$. We claim that this set projects onto C_k. In fact, let U be a strictly k-positive subspace. Then U contains a vector $x = (x_1,\ldots,x_d)$ with $x_i \neq 0$ for all $i = 1,\ldots,d$, i.e., x belongs to the interior of some orthant in $\mathbb{R}^d$. Now, the subgroup D of diagonal matrices with positive entries acts transitively on the set of rays contained in an orthant. Hence there exists $h \in D$ such that hU contains a vector $(1, x)$ with $x = (x_2, \ldots, x_d)$, $x_i = \pm 1$. Since h is totally positive, it follows that hU is strictly k-positive. Thus Lemma 5.2 ensures that hU contains a $(k-1)$-positive subspace W_1. But $h^{-1} \in D$, so that $W_2 = h^{-1}(W_1) \subset U$ is also $(k-1)$-positive. Hence the flag (W_2, U) belongs to $C_{\mathbf{r}}$. Since $\pi(W_2, U) = U$ it follows that $U \in \pi(C_{\mathbf{r}})$. Therefore $C_{\mathbf{r}}$ projects onto C_k and onto C_{k-1}, so that these are the invariant control sets of $T_{\mathbf{r}}$ in Gr_{k-1} and Gr_k, respectively.

This implies that $\mathrm{int} C_{\mathbf{r}}$ projects onto $\mathrm{int} C_{k-1}$ and $\mathrm{int} C_k$. Finally let V be a strictly k-positive subspace. Then $V \in \mathrm{int} C_k$, so that there exists $(W, V) \in \mathrm{int} C_{\mathbf{r}}$, implying that $W \in \mathrm{int} C_{k-1}$, that is, W is strictly $(k-1)$-positive.

■

So far we have proved that any strictly k-positive subspace belongs to a complete positive flag. This means that $\mathrm{int} C_k$ is contained in $D_1(k)$. But $\pi(C)$ is closed and C_k is the closure of $\mathrm{int} C_k$. Since $\pi(C) \subset C_k$, it follows that this inclusion is an equality. In other words, we have proved that

Theorem 6.4. *The invariant control set of both T and S in Gr_k is C_k.*

Corollary 6.5. *For any $k = 1, \ldots, d-1$ and any multi-index $\mathbf{r}$, the invariant control set $D_1(k, S_{\mathbf{r}})$ of $S_{\mathbf{r}}$ in Gr_k contains C_k. Furthermore if $k \in \mathbf{r}$ then $D_1(k, S_{\mathbf{r}}) = C_k$.*

Proof. The first statement follows from the fact that the invariant control sets of $S_{\mathbf{r}}$ contain the invariant control sets of S. To see the equality in the second statement note that since $k \in \mathbf{r}$, the projection $\pi^{\mathbf{r}}_k$ is well defined and maps $C_{\mathbf{r}}$ into C_k. But $\pi^{\mathbf{r}}_k(C_{\mathbf{r}})$ is the invariant control set of $S_{\mathbf{r}}$ in Gr_k, so that $C_k = \pi^{\mathbf{r}}_k(C_{\mathbf{r}}) = D_1(k, S_{\mathbf{r}})$.

■

Our next objective is to write down the invariant control sets of S_k on $\mathbb{P}$ in terms of sign changes. Given a k-dimensional subspace V, the fiber $\pi^{-1}\{V\}$ is the set of complete flags whose k-dimensional subspace is V. Thus the projection of this fiber into $\mathbb{P}$ is the set of one dimensional subspaces contained in V, i.e., $\mathbb{P}(V)$. Therefore the invariant control set of T_k and S_k in $\mathbb{P}$ is the set of those lines which are contained in k-positive subspaces.

On the other hand the semigroup T_k contains the subgroup D of diagonal matrices with positive entries. This group is transitive on the interior of any orthant in $\mathbb{P}$ so that the control sets of T_k (and S_k) are given by union of orthants in $\mathbb{P}$. Hence to find the invariant control sets explicitly it is enough to look at the vectors of the form $(x_1, \ldots, x_d)$, with $x_i = \pm 1$, and decide whether they belong or not to a k-positive subspace. Of course there is no loss in generality in assuming that $x_1 = 1$.

Now, by Lemma 5.4 we have that the number of sign changes of $(1, x)$ is $\leq k-1$ if $(1, x)$ of belongs to a k-positive subspace. On the other hand Lemma 5.5 shows that if the number of sign changes of $(1, x)$ is $i \leq k-1$

then $(1, x)$ belongs to an $(i+1)$-positive subspace and hence to a k-positive subspace. Summarizing, we obtain

Theorem 6.6. *The invariant control set of T_k and S_k in $\mathbb{F}$ is $O_{\le k}$, i.e., the closure of the union of the orthants containing elements with at most $k-1$ sign changes.*

The pre-image of C_k under the projection $\pi_k : \mathbb{F} \to \mathrm{Gr}_k$ is the set of complete flags $(V_1, V_2, \cdots)$ such that the k dimensional subspace V_k is k-positive. Hence we get the following characterization of $O_{\le k}$.

Corollary 6.7. *The set $O_{\le k} \subset \mathbb{F}$ is the set of lines which are contained in some k-positive subspace.*

Finally we have the following invariant control sets on the oriented Grassmannians.

Proposition 6.8. *The semigroup T_k has two invariant control sets in Gr_k^+, namely C_k^+ and C_k^-, while $C_k^+ \cup C_k^-$ is the only invariant control set of S_k in Gr_k^+.*

Proof. By Theorem 4.7 the invariant control set of T_k in Gr_k is C_k. On the other hand both $C_k^{\pm} \subset \mathrm{Gr}_k^+$ project onto C_k and are T_k-invariant. Hence they are the invariant controls sets on Gr_k^+.

To see that $C_k^+ \cup C_k^-$ is the S_k-invariant control set it is enough to exhibit $g \in S_k$ such that $gC_k^+ \subset C_k^-$. For this start with a regular matrix h such that $\mathrm{att}(h) \subset \mathrm{int}C_k^+$ and $C_k^+ \subset \sigma(h)$, and choose $h_1 \in \mathrm{Sl}(d)$ such that $h_1 \mathrm{att}(h) \subset \mathrm{int}C_k^-$. Let U be a neighborhood of $\mathrm{att}(h)$ such that $U \subset h_1^{-1}(\mathrm{int}C_k^-)$. For some positive integer n, $h^n C_k^+ \subset U$. Hence $g = h_1 h^n$ satisfies $gC_k^+ \subset C_k^-$, concluding the proof. ∎

7. Inverse semigroups

Let J be the diagonal $d \times d$ matrix

$$J = \mathrm{diag}\{1, -1, \ldots, (-1)^{d+1}\}. \tag{9}$$

It is well known that a matrix g is totally positive if and only if $Jg^{-1}J^{-1}$ is totally positive (see [1], Theorem 3.3). This means that the semigroup $T^{-1} = \{g^{-1} : g \in T\}$ is conjugate to T: $T^{-1} = JTJ^{-1}$. In this section we exploit the fact that S_k and T_k are compression semigroups to get analogous expressions for T_k^{-1} and S_k^{-1}. This way an alternative proof for the equality $T^{-1} = JTJ^{-1}$ will emerge.

Recall that the dual $C^* \subset \mathrm{Gr}_{d-k}$ of a subset $C \subset \mathrm{Gr}_k$ is defined by

$$C^* = \{V \in \mathrm{Gr}_{d-k} : V \cap W = 0 \text{ for all } W \in C\}.$$

If $g \in \mathrm{Sl}(d)$ leaves invariant C then g^{-1} leaves invariant C^*. In fact, for every pair of subspaces V, W it holds $V \cap g^{-1}W = g^{-1}(gV \cap W)$, so that if $W \in C$

then $g^{-1}W$ is transversal to all $V \in C$, which means that C^* is invariant under g^{-1}.

In particular, C_k^* is invariant under S_k^{-1}. It will be proved in a moment that conversely, if g^{-1} leaves C_k^* invariant then $g \in S_k$. For this recall that by Corollary 4.4, $(C_k^*)^* = C_k$.

Proposition 7.1. *S_k^{-1} is the compression semigroup of C_k^*.*

Proof. If $g \in S_k$ then $g^{-1}C_k^* \subset C_k^*$, so that S_k^{-1} is contained in

$$\mathrm{Comp}\,(C_k^*) = \{h \in Sl(d) : hC_k^* \subset C_k^*\}.$$

On the other hand suppose that $g \in \mathrm{Comp}\,(C_k^*)$. Then g^{-1} leaves invariant $(C_k^*)^* = C_k$, that is, $g^{-1} \in S_k$. ■

Now we recognize the dual C_k^* as an orthant in $\bigwedge^{d-k}$. For this we note first the following consequence of Corollary 4.4.

Proposition 7.2. *Let $D_1(1, S_k)$ be the invariant control set of S_k in $\mathbb{P}$. Suppose that V is a k-dimensional subspace whose projective space $\mathbb{P}(V)$ is contained in $D_1(1, S_k)$. Then V is k-positive.*

Proof. By Corollary 6.2 the invariant control set $D_1(1, S_k)$ is the union of $\mathbb{P}(W)$ with W running through the k-positive subspaces. This implies that $\mathbb{P}(U) \cap D_1(1, S_k) = \emptyset$ if $U \in C_k^*$. In fact if the intersection were not empty then U would have a nontrivial intersection with some k-positive subspace, contradicting the definitization of C_k^*. Therefore if $\mathbb{P}(V)$ is contained in the invariant control set then $\mathbb{P}(V) \cap \mathbb{P}(U) = \emptyset$ for all $U \in C_k^*$. This means that $V \in (C_k^*)^*$. Hence, $V \in C_k$ by Corollary 4.4. ■

This propositiontion implies that $\mathbb{P}(V) \subset \mathbb{P} \setminus D_1(1, S_k)$ if $V \in C_k^*$. Now, take the basis $\beta^- = J\beta_1$ of $\mathbb{R}^d$, that is,

$$\beta^- = \{e_1, -e_2, \ldots, (-1)^{d+1} e_d\}. \tag{10}$$

The coordinates of a vector $x = (x_1, \ldots, x_d)$ with respect to the basis β^- are $(-1)^{i+1} x_i$. Hence an orthant has j sign changes with respect to the basis β_1 if and only if there are $d - j - 1$ sign changes with respect to β^-. Therefore a $(n-k)$-subspace belongs to C_k^* if and only if it is contained in the orthants with at most $d - k - 1$ sign changes with respect to β^-.

Now we can prove the desired expressions for the inverse semigroups.

Proposition 7.3. *$S_k^{-1} = JS_{d-k}J^{-1}$; hence $S = JSJ^{-1}$. Analogous result holds for the semigroups T_k^{-1} and T^{-1}.*

Proof. For each j denote by $S_j(\beta^-)$ the compression semigroup of the set of subspaces which are j-positive with respect to β^-. Clearly, $S_j(\beta^-) = JS_jJ^{-1}$

because $\beta^- = J\beta_c$. Now, apply Theorem 6.6 to both S_k and $S_{d-k}(\beta^-)$ by taking for the later semigroup sign changes with respect to β^-. Comparing the sign changes we see that the invariant control set of $S_{d-k}(\beta^-)$ is the closure of $\mathbb{P} \setminus D_1(1, S_k)$. Combining this with Proposition 7.2, we get that a subspace $V \in C_k^-$ if and only if it is strictly $(d-k)$-positive with respect to β^-. Therefore $S_{d-k}(\beta^-)$ is the compression semigroup of C_k^-, showing that $S_{d-k}(\beta^-) = S_k^{-1}$, concluding the proof. ∎

Another consequence of Proposition 7.2 is the alternative characterization of S_k as the compression semigroup of $D_1(1, S_k)$. Of course, S_k is contained in this compression semigroup. On the other hand suppose that $g \in Sl(d)$ satisfies $gD_1(1, S_k) \subset D_1(1, S_k)$. If V is a k-positive subspace then $V \subset D_1(1, S_k)$. Thus $gV \subset D_1(1, S_k)$. However if a k-dimensional subspace is contained in $D_1(1, S_k)$ then it is k-positive. Then $gV \in C_k$ showing that $g \in S_k$. Therefore, we have

Corollary 7.4. *S_k is the compression semigroup of its invariant control set $D_1(1, S_k)$.*

8. Control sets

In this section we discuss the control sets of S and T in the flag manifolds. The main point is their description in the full flag manifold $\mathbb{F}$. Once the control sets are known in $\mathbb{F}$, they can be projected to the other flag manifolds to obtain the complete picture.

We start with the minimal control set. From it and the invariant control set, already determined, the other control sets are obtained.

In what follows we say that a subspace V is [strictly] k-negative in case JV is [strictly] k-positive, where J is the diagonal matrix defined in (9). The same way a flag is [strictly] k-negative provide its subspaces are [strictly] k-negative. For ξ a subspace or a flag we write $\xi \preceq 0$ and $\xi \prec 0$ to indicate that ξ is negative and strictly negative, respectively.

By Proposition 7.3 it follows that $S_{d-k}^{-1} = JS_kJ^{-1}$ is the semigroup of k-positive matrices with respect to the basis β^-. Thus applying Theorem 4.7 to the basis β^- instead of the standard one, we see that the invariant control set of S^{-1} in the maximal flag manifold $\mathbb{F}$, is the set of k-negative flags.

Now, the minimal control set of S in $\mathbb{F}$ is the interior of the invariant control set of S^{-1}. Thus we obtain

Proposition 8.1. *The minimal control set of S in $\mathbb{F}$ is the set of strictly negative flags.*

The other control sets are obtained by the following construction of flags from positive and negative ones.

Two complete flags $x = (V_1, \ldots, V_{d-1})$ and $y = (W_1, \ldots, W_{d-1})$ are said to be opposed or transversal if $V_k \cap W_{d-k} = \{0\}$ for all $k = 1, \ldots, d-1$. In symbols $x \pitchfork y$ if x is opposed to y. The transversality condition implies that

$\dim(V_k \cap W_{d-k+1}) = 1$ for all $k = 1, \ldots, d$, where $V_d = W_d = \mathbb{R}^d$. Thus we can construct a basis adapted to (x, y), say $\beta(x, y) = \{f_1, \ldots, f_d\}$, with $f_k \in V_k \cap W_{d-k+1}$. Of course, there are different adapted basis, but any one will do. For a permutation $w \in W$, put $b_w(x, y) = b(w\beta(x, y))$. It follows easily that this complete flag is independent of the specific basis adapted to (x, y). Observe that if h is a regular matrix with eigenvalues $\lambda_1 > \cdots > \lambda_d > 0$ with corresponding eigenvectors $f_1, \ldots, f_d$, then $\mathrm{fix}(h, w)$ in the full flag $\mathbb{F}$, is just $b_w(x, y)$.

Now, let $x = (V_1, \ldots, V_{d-1})$ be a strictly positive complete flag and $y = (W_1, \ldots, W_{d-1})$ a strictly negative one. From the previous section it follows that W_{d-k} belongs to C_k^*, so that it is transversal to V_k, hence x is opposed to y, giving sense to $b_w(x, y)$ for all permutations w. We shall prove below that the flags $b_w(x, y)$, with x running through the strictly positive flags and y through the strictly negative ones, cover the core of the control set $D_w(\mathbf{r}_c, S)$. This requires the next lemma.

Lemma 8.2. *Let $x \in \mathbb{F}$ be strictly positive and $y \in \mathbb{F}$ strictly negative. Then there exists a regular matrix h having attractor x and repeller y and such that $h \in \mathrm{int}S$.*

Proof. Put $x = (V_1, \ldots, V_{d-1})$, $y = (W_1, \ldots, W_{d-1})$ and choose a basis $\beta(x, y) = \{f_1, \ldots, f_d\}$ adapted to (x, y). Let h_1 be a matrix which is diagonal with respect to $\beta(x, y)$:

$$h_1 = \mathrm{diag}\{\lambda_1, \ldots, \lambda_d\}$$

with $\lambda_1 > \cdots > \lambda_d > 0$. By construction the subspace V_k is the attractor of h_1 in Gr_k, whose stable manifold is the subset of Gr_k consisting of the subspaces transversal to $\mathrm{span}\{f_{k+1}, \ldots, f_d\}$, i.e. to W_{d-k}. Since W_{d-k} is strictly k-negative, it follows that C_k is contained in the stable manifold of h_1. Then Lemma 3.1 implies that $h_1^n \in \mathrm{int}S_k$ for large enough n. Hence some power of h_1 belongs to $\mathrm{int}S$. ∎

Let h be as in the lemma. Then its fixed point of type w is the flag $b_w(x, y)$, so that by Theorem 3.2 this flag belongs to the core of $D_w(\mathbf{r}_c, S_k)$ for all $k = 1, \ldots, d-1$. Since k is arbitrary, it follows that $b_w(x, y) \in D_w(\mathbf{r}_c, S)_0$. Conversely, take $z \in D_w(\mathbf{r}_c, S)_0$. By Theorem 3.2 again, $z = \mathrm{fix}(h, w)$ for some regular matrix $h \in \mathrm{int}S$. Its attractor $\mathrm{att}(h)$ belongs to the core of the invariant control set, that is, is a strictly positive flag. The same way its repeller $\mathrm{att}(h^{-1})$ is a strictly negative flag. Therefore, $z = b_w(\mathrm{att}(h), \mathrm{att}(h^{-1}))$ is built from strictly positive and negative flags. Thus we have proved the

Proposition 8.3. *The core of the control set $D_w(\mathbf{r}_c, S)$ is given by*

$$D_w(\mathbf{r}_c, S)_0 = \{b_w(x, y) : x > 0, y < 0\}.$$

9. Control sets in $\mathbb{P}$

The invariant control set $D_1(1, S_k)$ of S_k in $\mathbb{P}$ was seen to be the union of orthants with at most $(k-1)$-sign changes. On the other hand the closure of $\mathbb{P} \setminus D_1(1, S_k)$ is the invariant control set of S_k^{-1}. The interior of this last invariant control set, namely the complement $\mathbb{P} \setminus D_1(1, S_k)$ is a control set of S_k. Therefore, we have

Theorem 9.1. *The semigroup S_k has just two control sets in $\mathbb{P}$, the invariant and the minimal. Their union is $\mathbb{P}$.*

As an application of Theorem 9.1 we can give an alternative proof of the following well known variation-diminishing characterization of strictly sign-regular square matrices.

Theorem 9.2. *A necessary and sufficient condition for a $d \times d$ matrix g to be strictly sign-regular is that $V_+(gx) \leq V_-(x)$ for all $0 \neq x \in \mathbb{R}^d$.*

Proof. Suppose that $V_+(gx) \leq V_-(x)$ for all $x \neq 0$. In particular, for $[x] \in O_{\leq k}$,

$$V_-(gx) \leq V_+(gx) \leq V_-(x) \leq k.$$

Hence, by Lemma 5.1, $gx \in \operatorname{int} O_{\leq k}$. It follows that $g \in \operatorname{int} S_k$ for all k. Therefore, g is strictly sign-regular.

Reciprocally, suppose g strictly sign-regular. If $x \neq 0$ then $[x] \in O_{\leq k}$, where $k = V_-(x)$. Since $g \in \operatorname{int} S$, $gx \in \operatorname{int} O_{\leq k}$. Applying Lemma 5.1 again we get that $V_+(gx) \leq k$, that is, $V_+(gx) \leq V_-(x)$. ∎

The objective now is to get all the control sets of S in $\mathbb{P}$.

Before stating the result we recall that there is a unique element $w_0 \in W$ of maximal length, which is the permutation $w_0 = (1, d)(2, d-1)\cdots$.

Recall that for an arbitrary semigroup Γ the control sets in $\mathbb{P}$ are projections of the control sets in the full flag $\mathbb{F}$. More precisely, for $i = 1, \ldots, d$ consider the permutation $(1, i)$. Then any control set in $\mathbb{P}$ is of the form $\pi(D_{(1,i)}(\pi_i))$. For the semigroup S of sign-regular matrices, the fact that $W(S) = \{1\}$ implies that these control sets are different from each other, so that there are d control sets in $\mathbb{P}$. As before let $O_{\leq k}$ be the union of orthants with at most k sign changes, and $\Sigma_k = O_{\leq k} \setminus O_{\leq (k-1)}$.

We intend to show that $\Sigma_{k-1} = \pi(D_{(1,k)}(\pi_i))$, so that the control sets are determined exactly by the number of sign changes.

Now, let $\Gamma \subset \mathrm{Sl}(d)$ be a semigroup with non-empty interior. Denote by C and C^- respectively the invariant control set and the minimal control set of Γ in $\mathbb{P}$. Also, for $w \in W$ denote by D_w the control set of type w. Denote by $\mathcal{A}_w$ and $\mathcal{R}_w$ the domain of attraction and the domain or repulsion of D_w, respectively (see [20]). These sets were studied in [20]. In order to state the required results we introduce the following notation.

For $i = 1, \ldots, d-1$, consider the cycle $r_i = (i, i+1) \in W$. Any permutation $w \in W$ can be written as $w = r_{i_k} \cdots r_{i_1}$ with $r_{i_j} = (i_j, i_j + 1)$.

The number s of cycles is called the length of w if this expression is reduced, i.e., minimal. Corresponding to r_i let $\mathbb{F}_i = \mathbb{F}(1,\ldots,\hat{i},\ldots,d-1)$ be the flag manifold which avoids the dimension i. Denote by $\pi_i : \mathbb{F} \to \mathbb{F}_i$ the canonical projection, and by γ_i the operation of exhausting a subset of $\mathbb{F}$ with the fibers of π_i:

$$\gamma_i(X) = \pi_i^{-1}\pi_i(X), \qquad X \subset \mathbb{F}.$$

The sets $\mathcal{A}_w$ and $\mathcal{R}_w$ are given by successive applications of γ_i as follows.

Proposition 9.3. *With the above notations, the domain of attraction of D_w is*

$$\mathcal{A}_w = \gamma_{i_1}\cdots\gamma_{i_s}(C^-)$$

where the indices are given by a reduced expression $w_0 w = r_{i_1}\cdots r_{i_s}$. Analogously

$$\mathcal{R}_w = \gamma_{j_1}\cdots\gamma_{j_k}(C_0)$$

where C_0 is the set of transitivity of C and the indices come from a reduced expression $w = r_{j_1}\cdots r_{j_k}$.

Proof. The formula for $\mathcal{A}_w$ is proved in [10], Theorem 6.3. On the other hand the formula for $\mathcal{R}_w$ is an application of the first one to Γ^{-1}. In fact, $\mathcal{R}_w$ is the domain of attraction of the control set of Γ^{-1} of type $w_0 w$. ■

We get now explicit descriptions of some of the exhaustions $\gamma_{i_1}\cdots\gamma_{i_s}(X)$. First let $x = (V_1, V_2, \ldots, V_{d-1})$ be a complete flag. Then $\pi_i x \in \mathbb{F}_i$ is the flag having same subspaces, but avoiding the i-dimensional one. Hence the fiber

$$\pi_i^{-1}\pi_i\{x\} = \{(V_1,\ldots,V_{i-1},W,V_{i+1},\ldots,V_{d-1}) : \dim W = i\}.$$

Lemma 9.4. *Let U be a k-dimensional subspace with $k \geq 3$, and take a flag $x = (V_1,\ldots,V_{d-1})$ with $V_{k-1} \subset U$. Consider the reduced expression*

$$(1,k) = (k-1,k)\cdots(23)(12)(23)\cdots(k-1,k),$$

with the associated sequence of exhaustions $\gamma_1,\ldots,\gamma_r$, $r = 2k-3$. Then

$$\pi\gamma_1\cdots\gamma_r\{x\} = \mathbb{P}(U)$$

where $\pi : \mathbb{F} \to \mathbb{P}$ is the projection.

Proof. It is enough to show that $\gamma_2\cdots\gamma_r\{x\}$ projects onto $\mathbb{P}(U)$ because $\gamma_1(B) \supset B$ for any subset B. We use induction on d. If $k = 3$, $\gamma_3\{x\}$ is the set of flags (V_1, W_2) with $W_2 \subset U$ and $\dim W_2 = 2$, because γ_3 corresponds to the permutation $r_3 = (23)$. Applying γ_2 to this set we get

$$\gamma_2\gamma_3\{x\} = \{(W_1, W_2) : V_1 \subset W_2 \subset U\}.$$

In this set W_1 is an arbitrary subspace contained in U, hence it projects onto $\mathbb{P}(U)$.

For general k, the last exhaustion corresponds to $(k-1,k)$, so that

$$\gamma_k\{x\} = \{x_W = (V_1,\ldots,V_{k-2},W) : V_{k-2} \subset W \subset U, \dim W = k-1\}.$$

On the other hand $\gamma_2\cdots\gamma_{k-1}$ corresponds to the permutation $(1,k-1)$. Fix W satisfying the definitioning property of this set and apply the induction hypothesis to W and the flag $x' = (V_1,\ldots,V_{k-2})$. It follows that $\gamma_2\cdots\gamma_{k-1}\{x'\}$ projects onto $\mathbb{P}(W)$. But $\gamma_2\cdots\gamma_{k-1}\{x'\} = \gamma_2\cdots\gamma_{k-1}\{x_W\}$. Hence the projection of $\gamma_2\cdots\gamma_k\{x\}$ onto $\mathbb{P}$ contains $\mathbb{P}(W)$ with $V_2 \subset W \subset U$ and $\dim W = k-1$. Since the set of these $\mathbb{P}(W)$ cover $\mathbb{P}(U)$, it follows that $\gamma_2\cdots\gamma_k\{x\}$ projects onto $\mathbb{P}(U)$, as claimed. ■

Corollary 9.5. *For $k = 2,\ldots,d$ take the permutation*

$$(1,k) = (k-1,k)\cdots(23)(12)(23)\cdots(k-1,k),$$

and let $\gamma_1,\ldots,\gamma_r$ be the corresponding set of exhaustions. Then for a complete flag $x = (V_1,\ldots,V_{d-1})$,

$$\gamma_1\cdots\gamma_r\{x\}$$

projects onto $\mathbb{P}(V_k)$.

Proof. It remains only to observe that if $k = 2$, the exhaustion γ corresponding to $(1,2)$ gives $\gamma\{x\} = \{(W,\ldots,V_{d-1}) : W \subset V_2\}$, so that it projects onto $\mathbb{P}(V_2)$. ■

Combining these facts about exhaustions with the characterization of $\mathcal{R}_w$ given in Proposition 9.3, we get easily that

Theorem 9.6. *The control sets of S in $\mathbb{P}$ are $\Sigma_0,\ldots,\Sigma_{d-1}$.*

Proof. The control sets in $\mathbb{P}$ are the projections of the control sets $D_{(1,k)}(\pi_e,S)$, $k = 1,\ldots,d-1$. For simplicity denote by D_k the projection of $D_{(1,k)}(\pi_e,S)$. By Proposition 9.3 the repulsion domain $\mathcal{R}_{(1,k)}$ of $D_{(1,k)}(\pi_e,S)$ is the union of $\gamma_1\cdots\gamma_r\{x\}$ where x is a strictly positive complete flag and $\gamma_1,\ldots,\gamma_r$ corresponds to

$$(1,k) = (k-1,k)\cdots(23)(12)(23)\cdots(k-1,k).$$

By the previous corollary $\gamma_1\cdots\gamma_r\{x\}$ projects onto a strictly k-positive subspace. Since any strictly k-positive subspace is contained in a strictly positive complete flag, it follows that $\pi(\mathcal{R}_{(1,k)})$ is the interior of $O_{\leq(k-1)}$. Now we can repeat the same argument to the semigroup $S^{-1} = JSJ^{-1}$, to get that the domain of attraction of D_k is the interior of $\bigcup_{\mathrm{sgn}(\sigma)\geq k-1} O_\sigma$. Hence the core $(D_k)_0$ of D_k is $\mathrm{int}\Sigma_{k-1}$.

It remains to check the boundary points. Take $x \in E_{k-1} \cap \partial E_{k-1}$. Then $Sx \subset O_{\leq(k-1)}$ because this subset is S-invariant. On the other hand $x \notin O_{\leq(k-2)}$. Since $O_{\leq(k-2)}$ is closed, there is an open set $U \ni x$ such that $U \cap O_{\leq(k-2)} = \emptyset$. Now int$S$ meets any neighborhood of the identity in $\mathrm{Sl}(d)$. Hence $((\mathrm{int}S)x) \cap U$ is not empty. Since $(\mathrm{int}S)x \subset O_{\leq(k-1)}$, it follows that Sx meets intΓ_k. Hence x belongs to the control set D_k, concluding the proof. ■

The determination of the control sets given in this theorem allows alternative proofs for two known facts about sign-regular matrices. In first place, we note that for a general semigroup Γ, the core of its control sets in the projective space contain the eigenvectors for any $g \in \mathrm{int}\Gamma$. Precisely, if $g \in \mathrm{int}\Gamma$ is regular then its fixed point of type w in $\mathbb{F}$ is contained in the core of the control set $D_w(\mathfrak{r}_c, \Gamma)$. Now, a g-fixed point in $\mathbb{F}$ projects into a fixed point in $\mathbb{P}$, that is, to an eigenspace of g. Hence the eigenspaces of a regular $g \in \mathrm{int}\Gamma$ belongs to a control set in $\mathbb{P}$.

In particular let g be a strictly sign-regular matrix, i.e., $g \in \mathrm{int}S$. By Corollary 3.6, g is regular. Let $\beta = \{f_1, \ldots, f_d\}$ be a basis of eigenvectors of g. Then for a permutation w, the fixed point of type w of g in $\mathbb{F}$ is the flag $\mathfrak{b}_{\mathbb{F}}(w; \beta)$. By Theorem 3.2 this fixed point belongs to the core of the control set $D_w(\mathfrak{r}_c, S)$. A direct inspection of the construction of this flag shows that its projection into $\mathbb{P}$ is the line spanned by $f_{w(1)}$. Taking $w = (1, k)$ we see that the eigenvector f_k belongs to the $\pi(D_{(1,k)}(\mathfrak{r}_c, S))$, which by the above theorem is intΣ_k. Therefore we get from Theorem 9.6 the following classical result by Gantmacher and Krein (see [1], Section 6):

Theorem 9.7. *Let g be a strictly sign-regular matrix and order its (real) eigenvalues by $|\lambda_1| > \cdots > |\lambda_d|$. Let f_k be an eigenvector corresponding to the eigenvalue λ_k. Then $[f_k] \in \mathrm{int}\Sigma_k$, that is, $V_-(f_k) = V_+(f_k) = k - 1$.*

References

[1] Ando, T., *Totally positive matrices*, Linear Algebra and its Applications **90** (1987), 165-219.

[2] Colonius, K. and W. Kliemann, "Dynamics and Control", Birkhäuser, 2000.

[3] Hilgert, J., K. H. Hofmann, and J. Lawson, "Lie Groups, Convex Cones, and Semigroups", Oxford University Press, 1989.

[4] Hilgert, J. and K.-H. Neeb, "Lie Semigroups and their Applications", Lecture Notes in Math., Vol. 1552, Springer-Verlag, 1993.

[5] Hilgert, J. and K.-H. Neeb, *Maximality of compression semigroups*, Semigroup Forum **50** (1995), 205-222.

[6] Lawson, J. D., *Maximal semigroups of Lie groups that are total*, Proc. Edinburgh Math. Soc. **30** (1987) 479-501.

[7] Lusztig, G., *Introduction to total positivity*. In "Positivity in Lie Theory: Open Problems" (J. Hilgert, J. D. Lawson, K.-H. Neeb, and E. B. Vinberg, eds.). De Gruyter Expositions in Mathematics, Vol. 26, pp. 133–145, 1998.

[8] San Martin, L. A. B., *Invariant control sets on flag manifolds*, Mathematics of Control, Signals, and Systems **6** (1993), 41–61.

[9] San Martin, L. A. B., *Homogeneous Spaces Admitting Transitive Semigroups*, J. of Lie Theory **8** (1998), 111–128.

[10] San Martin, L. A. B., *Order and domains of attraction of control sets in flag manifolds*, J. of Lie Theory **8** (1998), 335–350.

[11] San Martin, L. A. B., *Maximal semigroups in semi-simple Lie groups*, Trans. Amer. Math. Soc. **353** (2001), 5165–5184.

[12] San Martin, L. A. B., *A family of maximal noncontrollable Lie wedges with empty interior*, Systems Control Lett. **43** (2001), 53–57.

[13] San Martin, L. A. B. and P. A. Tonelli, *Semigroup actions on homogeneous spaces*, Semigroup Forum **50** (1995), 59–88.

[14] Sussmann, H. J. and V. Jurdjevic, *Controllability of nonlinear systems*, J. Differential Equations **12** (1972), 95–116.

[15] Warner, G., "Harmonic analysis on semi-simple Lie groups I", Springer-Verlag, 1972.

Departamento de Matemáticas
Universidad Católica del Norte
Casilla 1280, Antofagasta, Chile
[illegible]

Department of Mathematics
Iowa State University
400 Carver Hall,
Ames, Iowa 50011, USA
[illegible]

Instituto de Matemática
Universidade Estadual de Campinas
Cx.Postal 6065
13081-970 Campinas SP, Brasil
[illegible]

Received March 12, 2003
and in final form January 23, 2004
Online publication May 5, 2004