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CONTROLLABILITY OF TWO-DIMENSIONAL BILINEAR SYSTEMS : RESTRICTED CONTROLS AND DISCRETE-TIME

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Abstract

Given a bilinear control system $\dot{x} = (X + uY)x$ with restricted control range, necessary and sufficient conditions for controllability are given under the assumption that the group of the system is Sl(2). *These conditions extend known conditions for systems with unrestricted controls and work also for the discrete-time version of the system.*

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1. Introduction

Necessary and sufficient conditions for the controllability of a bilinear system

$$\dot{x} = (X + uY)x \tag{1.1}$$

in R^2, with unrestricted controls, were given by Barros, Gonçalves, do Rocio and San Martin [1] and Joó and Tuan [3]. The purpose of this paper is to exploit the methods of [1] to extend the given conditions to systems with restricted controls and to the discrete-time version of (1.1), either with restricted or unrestricted controls.

Recall that the bilinear system (1.1) is said to be controllable if for any pair of vectors $x, y \in R^2 \setminus \{0\}$, there exists a trajectory of the system starting at x and finishing at y. Equivalently, denote by S the semigroup generated by (1.1):

$$S = \{e^{t_1(X+u_1Y)} \cdots e^{t_k(X+u_kY)} : t_i \geq 0, u_i \in R \text{ for arbitrary } k\}.$$

Then the system is controllable if for every $x, y \in R^2 \setminus \{0\}$, there exists $g \in S$ such that $gx = y$. This is the same as to say that $Sx = R^2 \setminus \{0\}$ for all $x \in R^2 \setminus \{0\}$, where

$$Sx = \{gx : g \in S\}$$

is the orbit of S through x.

Let G be the subgroup of invertible matrices generated by S. It is well known that G is a connected Lie group, so that it is completely determined by its Lie algebra. In the case at hand the Lie algebra g of G is the algebra of matrices generated by X and Y. This fact implies that, with respect to the topology of G, S has nonempty interior.

According to [1] there are three possibilities for G, namely $\mathrm{Gl}^+(2)$, $\mathrm{Sl}(2)$ and the cylinder group C^*. In other words, the possibilities for the Lie algebra g of G are $gl(2)$, the Lie algebra of 2×2 matrices; $sl(2)$, the subalgebra of matrices in $gl(2)$ having zero trace; and C, the abelian Lie algebra of 2×2 real matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

We consider in this paper only those systems which generate the group $\mathrm{Sl}(2)$ of 2×2 matrices with $\det = 1$. For these systems it was proved before that controllability is equivalent to $S = \mathrm{Sl}(2)$ (see [9] for a general proof and [1] for a proof specific to $\mathrm{Sl}(2)$.) Moreover, $S = \mathrm{Sl}(2)$ if and only if it acts transitively in the projective line **P** (c.f. Section 3, below.) Based on these facts the following algebraic criterion for controllability was proved in [1], Theorem 5.3.

Theorem 1.1. *Suppose that $X, Y \in sl\,(2)$. Then (1.1) is controllable $R^2 - \{0\}$ if and only if* $\det[X,Y] < 0$.

In this paper we interpret this criterion in terms of the location of the straight line $X + uY$, $u \in R$, contained in $sl\,(2)$, thus providing a geometric necessary and sufficient condition for controllability. This condition turns out to hold also for control systems with restricted controls and for discrete-time control systems.

2. Unrestricted controls

In this section we translate the algebraic condition of Theorem 1.1 into geometric terms.

Before starting we note that for $X, Y \in sl\,(2)$ the condition $\det[X,Y] \neq 0$ is equivalent to the fact that the Lie algebra generated by X and Y is $sl\,(2)$ (see [1], Corollary 5.2.) This means that the conditions of Theorem 1.1) imply that the group G generated by the system is $\mathrm{Sl}\,(2)$.

Given a Lie algebra g and $X \in g$ let $\mathrm{ad}\,(X) : g \to g$ stand for its adjoint: $\mathrm{ad}\,(X)\,(Y) = [X,Y]$. Recall that the Cartan-Killing form $\mathcal{K}$ of g is defined by $\mathcal{K}\,(X,Y) = \mathrm{tr}\,(\mathrm{ad}\,(X)\,\mathrm{ad}\,(Y))$. In $sl\,(2)$ the Cartan-Killing form is a multiple of the trace form

$$\langle X,Y\rangle = \mathrm{tr}\,(XY)\,.$$

In fact $\mathcal{K}\,(X,Y) = 4\langle X,Y\rangle$. The trace form gives rise to the quadratic form $Q\,(Z) = \mathrm{tr}\,(Z^2)$. It is nondegenerate, and its matrix with respect to the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of $sl\,(2)$ is the diagonal matrix

$$\begin{pmatrix} 2 & & \\ & 2 & \\ & & -2 \end{pmatrix}.$$

The zero set of Q is a double circular cone $\mathcal{C}$ whose axis is the line generated by the matrix A. Any generating ray of $\mathcal{C}$ makes an angle of 45° with A.

The elements of $\mathcal{C}$ are nilpotent matrices. We denote by $\mathcal{C}_{\rm int}$ the interior of $\mathcal{C}$, that is, the region $\{Z : Q(Z) < 0\}$, and by $\mathcal{C}_{\rm ext}$, the exterior $\mathcal{C}_{\rm ext} = \{Z : Q(Z) > 0\}$. The matrices in the interior of $\mathcal{C}$ have purely imaginary eigenvalues, whereas those in the exterior have real eigenvalues and are diagonalizable.

If a 2×2 matrix Z satisfies ${\rm tr} Z = 0$ then its characteristic polynomial is $\lambda^2 + \det Z$, so that $Z^2 + \det Z = 0$ where $\det Z$ stands for the scalar matrix $\det Z \cdot 1$ with 1 the identity 2×2 matrix. Taking traces in this equality we get $\det Z = -\frac{1}{2}{\rm tr} Z^2$, showing that the quadratic form Q is essentially the determinant.

Now, we relate the condition of Theorem 1.1 to the geometric position of the straight line $X+uY$ with respect to the cone $\mathcal{C}$. Clearly, $\det(X+uY) = -\frac{1}{2}{\rm tr}(X+uY)^2$ is a polynomial in u. Since ${\rm tr}(X+uY) = 0$ this polynomial is given explicitly by

$$\begin{aligned} \det(X+uY) &= -(X+uY)^2 \\ &= -Y^2u^2 - (XY+YX)u - X^2 \end{aligned}$$

However $-Y^2 = \det Y$ and $-X^2 = \det X$. Hence

(2.1) $$\det(X+uY) = (\det Y)u^2 - (XY+YX)u + \det X.$$

From this expression it follows that $XY+YX$ is a scalar matrix, so in the sequel we treat it as a real number. Note that $\det(X+uY)$ is quadratic if and only if $\det Y \neq 0$. In this case its discriminant is

(2.2) $$(XY+YX)^2 - 4\det X \det Y.$$

In order to evaluate the real number $(XY+YX)^2$ use the characteristic polynomial of $[X,Y]$. Then by developing the commutator it follows

$$\begin{aligned} -\det[X,Y] &= [X,Y]^2 \\ &= XYXY + YXYX - XY^2X - YX^2Y \\ &= XYXY + YXYX - 2\det X \det Y. \end{aligned}$$

On the other hand

$$\begin{aligned} (XY+YX)^2 &= XYXY + YXYX + XY^2X + YX^2Y \\ &= XYXY + YXYX + 2\det X \det Y. \end{aligned}$$

Joining together these two equalities we get

(2.3) $$(XY+YX)^2 = -\det[X,Y] + 4\det X \det Y.$$

Hence taking into account the expression (2.2) for the discriminant of $\det(X+uY)$, we get

Proposition 2.1. *The discriminant of the quadratic polynomial* $\det(X+uY)$ *is*

$$-\det[X,Y].$$

Of course, $u_0 \in R$ is a root of $\det(X+uY) = -\frac{1}{2}\operatorname{tr}(X+uY)^2$ if and only if $X+u_0Y \in C$. Hence the above proposition shows that $\det[X,Y]$ measures the number of crossings of the line $X+uY$, $u \in R$, with C. Precisely, these computations provide the following geometric picture.

Proposition 2.2. *Suppose that* $\det[X,Y] \neq 0$. *Then the straight line* $l = \{X+uY : u \in R\}$ *meets the interior region of the double cone* C *if and only if* $\det[X,Y] < 0$.

Proof: In case $\det Y \neq 0$, the polynomial

$$\det(X+uY) = (\det Y)\,u^2 - (XY+YX)\,u + \det X$$

is quadratic so that the result is a consequence of the previous proposition. Assume that $\det Y = 0$. Then the equality in (2.3) implies that $XY+YX \neq 0$ if and only if $\det[X,Y] < 0$. In this case $\det(X+u_0Y) > 0$ for some u_0, that is, l meets the interior of C. □

Therefore the criterion of Theorem 1.1 is translated into the following geometric condition for controllability.

Theorem 2.3. *Suppose that* $\det[X,Y] \neq 0$. *Then the system (1.1) with unrestricted controls is controllable if and only the straight line* $l = \{X+uY : u \in R\}$ *meets the interior to* C. *In other words, under the Lie algebra rank condition in* $sl(2)$, *controllability is equivalent the existence of* $u_0 \in R$ *such that* $X+u_0Y$ *has purely imaginary eigenvalues.*

3. Control sets

The method developed in [1] for studying controllability of bilinear systems is based on the action of its semigroup S in the projective line $\mathbf{P} = \mathbf{P}^1$. We recall here some facts about this action. The assumption that the group of the system G is $\mathrm{Sl}(2)$ implies that S is a subsemigroup with nonempty interior of $\mathrm{Sl}(2)$. Since the results to be described here hold for any such semigroup, here we assume only that $S \subset \mathrm{Sl}(2)$ is a semigroup with $\mathrm{int}S \neq \emptyset$.

The group $\mathrm{Sl}(2)$ acts on $\mathbf{P}$ turning it into a group of diffeomorphisms of $\mathbf{P}$. We denote the action by gx, $g \in \mathrm{Sl}(2)$, $x \in \mathbf{P}$. Plainly, gx is the image under g of the one-dimensional subspace x of R^2. This action is transitive, that is, for any $x, y \in \mathbf{P}$ there exists $g \in \mathrm{Sl}(2)$ such that $gx = y$.

Consider the restriction to S of the $\mathrm{Sl}(2)$-action on $\mathbf{P}$. It can be proved that the action of S in $\mathbf{P}$ is not transitive unless $S = \mathrm{Sl}(2)$ (this is a particular case of a general result of [7]. See [1] for a proof specific to $\mathrm{Sl}(2)$.) In order to look at the transitivity of S on $\mathbf{P}$ we recall that a control set of S is a nonempty subset $D \subset \mathbf{P}$ which satisfies

1. $D \subset \mathrm{cl}(Sx)$ for all $x \in D$.
2. There exists $g \in \mathrm{int}S$ and $x \in D$ such that $gx = x$.
3. D is maximal with the above properties.

Before describing the control sets of S in $\mathbf{P}$, let us draw some comments about the action on $\mathbf{P}$ of a specific matrix $g \in \mathrm{Sl}(2)$. The Jordan canonical form of g has one of the following forms:

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \lambda \neq 1, \qquad \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

In the first case g has two fixed points in $\mathbf{P}$, namely its eigenspaces. One of the fixed points, say x^+, which corresponds to the principal eigenvalue, is an attractor, i.e., $g^n x \to x^+$ as $n \to +\infty$, for x in a dense subset of $\mathbf{P}$, while the other, say x^-, is a repeller in the sense that $g^n x \to x^-$ as $n \to -\infty$, again for x in a dense subset of $\mathbf{P}$. For the second Jordan form there is just one fixed point if $a \neq 0$, which is the eigenspace of g. In this case $g^n x$ converges to the fixed point as $n \to \pm\infty$ for any $x \in \mathbf{P}$. Finally, if g is equivalent to a rotation then there are no fixed points in $\mathbf{P}$.

With these remarks in mind we can state the description of the control sets in $\mathbf{P}$.

Proposition 3.1. *Suppose that S is a proper subsemigroup of $\mathrm{Sl}(2)$ with $\mathrm{int}S \neq \emptyset$. Then there are exactly two control sets in $\mathbf{P}$. Denote then by $C^\pm$. They satisfy the following properties:*

1. *$C^+ \cap C^- = \emptyset$, C^+ is closed and C^- is open.*

2. *The subset $C_0^{\pm} = \{x \in C^{\pm} : gx = x$ for some $g \in \mathrm{int}S\}$ is open and dense in $C^{\pm}$. We call $C_0^{\pm}$ the set of transitivity of $C^{\pm}$. It satisfies: for all $x, y \in C_0^{\pm}$ there exists $g \in S$ such that $gx = y$.*

3. *C^+ is invariant, i.e., $Sx \subset C^+$ for all $x \in C^+$. On the other hand C^- is S^{-1}-invariant.*

4. *If $g \in \mathrm{int}S$ then g is diagonalizable.*

5. *If $g \in S$ is diagonalizable then its attractor belongs to C^+ and its repeller to the closure of C^-.*

4. The adjoint representation of Sl (2)

In the analysis of the control sets in the projective line it is worth to interpret our geometrical objects inside $sl\,(2)$, subjected to the action of Sl (2), given by conjugation.

As before let $\mathcal{C}$ be the double cone $\langle Z, Z\rangle = \mathrm{tr}\,(Z^2) = 0$. Then $\mathcal{C}$ is invariant under conjugation by invertible matrices, i.e., $gZg^{-1} \in \mathcal{C}$ if $Z \in \mathcal{C}$. The set $\mathcal{C} \setminus \{0\}$ has two connected components. We distinguish them by putting $\mathcal{C}^+$ for the one which contains the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

The other one is denoted by $\mathcal{C}^-$. Each of these components is an orbit of Sl (2). For instance, to see that Sl (2) acts transitively on $\mathcal{C}^+$, note that the rotation group turns around $\mathcal{C}^+$ while the group diagonal matrices is transitive along the ray of upper triangular matrices in $\mathcal{C}^+$.

Note that $\mathcal{C} \setminus \{0\}$ is the set of rank one 2×2 matrices having trace zero. If u and v are orthogonal 2×1 matrices in R^2 then $u^t v$ belongs to $\mathcal{C}$. Moreover, $u^t v \in \mathcal{C}^+$ if $\{u, v\}$ is positively oriented with respect to the standard basis of R^2. Thus the map

$$\phi : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} -y & x \end{pmatrix}$$

induces a map from $\mathbf{P}$ into the set $[\mathcal{C}^+]$ of rays of $\mathcal{C}^+$. It is easy to see that the action of Sl (2) on $[C^+]$ orbit is equivalent to the action of Sl (2) in the projective line $\mathbf{P}$, through the map ϕ.

For $B \in sl\,(2)$ its adjoint $\mathrm{ad}\,(B)$ defines a linear differential equation in $sl\,(2)$ whose trajectories are $\exp\,(t\,\mathrm{ad}\,(B))\,Z$, $Z \in sl\,(2)$, $t \in R$. Next we describe these trajectories in $\mathcal{C}^+$ as the intersections of this set with the planes

$$B^{\perp} = \{Z \in \mathcal{C} : \langle Z, B \rangle = c\}.$$

orthogonal to B with respect to the trace form.

For this we recall first that the map $\mathrm{Ad}(\exp(B)) = \exp(\mathrm{ad}(B))$ is an isometry of the trace form $\langle \cdot, \cdot \rangle$, that is,

$$\langle \exp(\mathrm{ad}\, B) Z, \exp(\mathrm{ad}\, B) W \rangle = \langle Z, B \rangle.$$

for all $Z, W \in \mathrm{Sl}(2)$. In particular, $\langle \exp(t\,\mathrm{ad}\, B) Z, B \rangle = \langle Z, B \rangle$ hence, the $\mathrm{ad}(B)$-trajectory starting at Z stays in a plane $\{W : \langle W, B \rangle = \langle Z, B \rangle\}$.

Define the map $f_B(Z) = \langle B, Z \rangle$, $Z \in \mathcal{C}$. Let v be a tangent vector to $\mathcal{C}$ at Z. Then

$$(df_B)_Z(v) = \langle B, v \rangle. \tag{4.1}$$

Since $\mathcal{C}^{\pm}$ are adjoint orbits of $\mathrm{Sl}(2)$, the tangent vector is of the form $v = [W, Z]$, with $W \in sl(2)$. So we have

$$\begin{aligned} (df_B)_Z(v) &= \langle B, [W, Z] \rangle = \langle [Z, B], W \rangle \\ &= -\langle [B, Z], W \rangle. \end{aligned}$$

Hence $\mathrm{ad}(B) Z = 0$ if and only if $(df_B)_Z = 0$.

By a direct computation it follows that the tangent plane $T_Z\mathcal{C}$ of $\mathcal{C}$ at $Z \in \mathcal{C}$ is given by

$$T_Z\mathcal{C} = \{W \in sl(2) : \langle Z, W \rangle = 0\}.$$

Joining this with the expression for df_B in (4.1) we see that df_B degenerates at $Z \in \mathcal{C}$ if and only if B is orthogonal to Z. Summarizing.

Proposition 4.1. *The trajectory $\exp(t\,\mathrm{ad}\, B) Z$, $Z \in \mathcal{C}$, $t \in R$, is given by*

1. $\{W \in \mathcal{C} : \langle W, B \rangle = \langle Z, B \rangle\}$ *if* $\langle B, Z \rangle \neq 0$.
2. $\{Z\}$ *if* $\langle B, Z \rangle = 0$, *so that* $B^{\perp} \cap \mathcal{C}$ *is a union of fixed points.*

Remark: In practice the plane orthogonal to B, with respect to the trace form, can be seen with the aid of the inner product $(\cdot, \cdot)$ in $sl(2)$ defined by

$$(Z, W) = \langle Z, W^t \rangle = \mathrm{tr}\left(ZW^t\right).$$

Thus if we denote by $B^{(\perp)}$ the plane orthogonal to B, with respect to $(\cdot, \cdot)$ then $B^{\perp} = \left(B^{(\perp)}\right)^t$. On the other hand the operation of transposition is obtained by a reflection through the plane s of symmetric matrices (spanned by $\{H, S\}$). Therefore $B^{\perp}$ is the reflection through s of the plane orthogonal to B with respect to $(\cdot,\cdot)$.

From the above proposition we get the following description of the trajectories.

Proposition 4.2. *The trajectories in $\mathcal{C}$ are:*

1. *If $B \in \mathcal{C}_{int}$: ellipses around $\mathcal{C}$.*
2. *If $B \in \mathcal{C}$: points in the ray of $\mathcal{C}$ orthogonal to B or the parabolas $\{\langle B, Z\rangle = c\} \cap \mathcal{C}$.*
3. *If $B \in \mathcal{C}_{ext}$: the two rays in $B^{\perp} \cap \mathcal{C} \setminus \{0\}$ or the hyperbolas $\{\langle B, Z\rangle = c\} \cap \mathcal{C}$.*

In particular if $B \in \mathcal{C}_{ext}$ then it has different real eigenvalues and the eigenvectors projected in $\mathbf{P}$ – identified with the set $[\mathcal{C}^{+}]$ of rays of $\mathcal{C}^{+}$ – are given by the intersection of $B^{\perp}$ with $\mathcal{C}^{+}$. From this geometry it is possible to detect also the attractor and the repeller for B in $\mathbf{P}$. In fact, if B is diagonalizable then $B = \mathrm{Ad}\,(g)\,(cH)$ for some $c > 0$ and $g \in \mathrm{Sl}\,(2)$, where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The attractor and the repeller for H in $\mathbf{P}$ are Z and W respectively, where

$$Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad W = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Note that the basis $\{Z, H, W\}$ has the same orientation as the canonical basis $\{S, H, A\}$. Also, $Z, W \in \mathcal{C}^{+}$. Since $\det \mathrm{Ad}\,(g) = 1$, it follows that the basis formed by the attractor of B, B and the repeller is positively oriented w.r.t. $\{S, H, A\}$. From this it is clear which is the attractor and the repeller of B:

Proposition 4.3. *Take $B \in \mathcal{C}_{ext}$. Under the identification of $\mathbf{P}$ with the set of rays of $\mathcal{C}^{+}$, the fixed points of B in $\mathbf{P}$ are the rays in $B^{\perp} \cap \mathcal{C}^{+}$. Let Z and W in $\mathcal{C}^{+}$ correspond to the attractor and repeller respectively. Then the basis $\{Z, B, W\}$ has the same orientation as the standard basis $\{S, H, A\}$.*

Now, we consider a segment

$$\sigma = \{B + uC : u \in [a, b]\},$$

where $[a, b] \subset R$ is an interval. We assume that σ is contained in $\mathcal{C}_{ext}$ and look at the set of attractors and repellers in $\mathbf{P}$ of the matrices in σ. According to the above proposition these fixed points are given by the intersection with $\mathcal{C}$ of the plane $W^{\perp}$ with $W \in \sigma$.

Suppose that $Z \neq 0$ spans the line $\sigma^{\perp}$. In what follows we assume the generic situation in which $Z \notin \mathcal{C}$. The planes $W^{\perp}$, $W \in \sigma$, are better visualized in case σ is put in one of the following normal forms.

1. Suppose that $Z \in \mathcal{C}_{int}$. Then there exists $g \in \mathrm{Sl}(2)$ such that gZg^{-1} is skew-symmetric. Since the conjugation preserves the trace form, we can assume without loss of generality that
$$Z = A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
In this case σ is contained in the subspace s of symmetric matrices. Therefore is $W \in \sigma$, $W^{\perp}$ is the plane spanned by Z and $W^{\perp} \cap s$.

2. In case $Z \in \mathcal{C}_{ext}$, it is conjugate to a symmetric matrix. So we can assume that
$$Z = S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
In this case σ is contained in the plane spanned by $\{H, A\}$. In this case $W \in \sigma$ has the form
$$W = \begin{pmatrix} x & -y \\ y & -x \end{pmatrix}$$
with $\det W = -x^2 + y^2 < 0$. Thus $W^{\perp}$ is the plane spanned by S and
$$\begin{pmatrix} y & -x \\ x & -y \end{pmatrix} \in \mathcal{C}_{int}.$$

From this picture of $W^{\perp}$, $W \in \sigma$, the set of attractors and repellers are easily given.

Proposition 4.4. *Let σ be a segment contained in C_{ext}. A $W \in \sigma$ has an attractor and one repeller in* $\mathbf{P}$. *Making W runs through σ, denote by $\mathcal{A}(\sigma)$ and $\mathcal{R}(\sigma)$ respectively the set of attractors and repellers thus obtained. Then $\mathcal{A}(\sigma)$ and $\mathcal{R}(\sigma)$ are nonoverllaping intervals in* $\mathbf{P}$.

Furthermore, $\mathcal{A}(\sigma)$ is invariant under the one-parameter semigroup

$$\exp(tW),\ t \geq 0$$

for all $W \in \sigma$.

Proof: The first statement follows by directly from the above description of $W^{\perp}$, $W \in \sigma$ and the identification of $\mathbf{P}$ with $[C^{+}]$. As to the invariance of $\mathcal{A}(\sigma)$, note that any interval in $\mathbf{P}$ containing the attractor of W but not its repeller is invariant under $\exp(tW)$, $t \geq 0$. □

5. Restricted controls

In this section we consider bilinear systems with restricted controls. The objective is to extend to this case the controllability result given for unrestricted controls. Instead of looking plainly at controllability we consider a bilinear system with varying control range

$$(5.1) \qquad \dot{x} = (X + uY)x \qquad\qquad u \in U^{\rho} = [-\rho, \rho], \rho \geq 0.$$

The problem is to detect the values of ρ for which the corresponding system is controllable. Actually, we consider the bifurcation scenario posed by Colonius and Kliemann [2], in what regards the transitivity properties of the system in the projective line $\mathbf{P}$.

In order to state precisely this scenario we note that it is assumed, as above that $\operatorname{tr} X = \operatorname{tr} Y = 0$ and that $\det[X, Y] \neq 0$. These conditions ensure that the Lie algebra generated by X and Y is $sl(2)$. Since $\det[X, X + uY] = u^2 \det[X, Y]$ this condition also ensures that the system group is $\mathrm{Sl}(2)$ for any value of $\rho > 0$.

Denote by S_{ρ} the semigroup of the system (5.1^{ρ}), that is, the system (5.1) with control range U^{ρ}. Since $\operatorname{int} S_{\rho} \neq \emptyset$, $\rho > 0$, the control sets $C_{\rho}^{\pm} \subset \mathbf{P}$ are well defined. Clearly $S_{\rho_1} \subset S_{\rho_2}$ if $\rho_1 \leq \rho_2$. This implies that the maps $\chi^{\pm} : \rho \mapsto C_{\rho}^{\pm}$ are increasing. Consider in particular χ^{+} which associates with ρ the invariant control set of (5.1^{ρ}) in $\mathbf{P}$. We look at its continuity properties when the set of compact of subsets of $\mathbf{P}$ is endowed with the Hausdorff topology. It was proved in [2], Chapter 3, that χ^{+} is lower semicontinuous (see [2], Appendix B, for the definition.) With this in mind we define the bifurcation points of (5.1^{ρ}).

Definition 5.1. *A $\rho > 0$ is said to be a bifurcation point of (5.1^{ρ}) if χ^+ is not continuous at ρ (with respect to the Hausdorff topology on compact subsets.)*

Remark: Before proceeding let us mention that the above continuity concept is the right one for the general theory in [2]. However, in our context we do not need to consider the Hausdorff topology in $\mathbf{P}$. In fact, the invariant control set C_ρ^+ is connected because S_ρ is connected. Since $\mathbf{P}$ is one-dimensional C_ρ^+ is an interval in $\mathbf{P}$, and hence given by its endpoints, so that instead of the Hausdorff topology on all compact sets we can look at the simpler topology given by pairs of points in $\mathbf{P}$.

We look now at the bifurcation points of (5.1^{ρ}). Recall that $\{X, Y\}$ generates $sl(2)$ if and only if $\det[X, Y] \neq 0$.

Theorem 5.2. *Suppose that $\det[X, Y] \neq 0$. Then the controllability of the system (5.1^{ρ}) with restricted control range $U^{\rho} = [-\rho, \rho]$ is given by the relative position of the segment*

$$\sigma_\rho = \{X + uY : u \in U^{\rho}\}$$

as follows:

1. *If $\det X \geq 0$ then $\sigma_\rho \cap \mathcal{C}_{\text{int}} \neq \emptyset$ and the system is controllable for any $\rho > 0$, that is, $\chi^+(\rho) = \mathbf{P}$ for all $\rho > 0$.*
2. *If $\det X < 0$, there are the possibilities:*
   - (a) *If $\det[X, Y] < 0$ then the line $X + uY$, $u \in R$, crosses the interior of $\mathcal{C}$ and (5.1^{ρ}) is controllable if and only if $\sigma_\rho \cap \mathcal{C}_{\text{int}} \neq \emptyset$. The only bifurcation point is $\rho^* = \inf\{\rho : \sigma_\rho \cap \mathcal{C}_{\text{int}} = \emptyset\}$.*
   - (b) *In case $\det[X, Y] > 0$ the system is not controllable for any $\rho > 0$. Furthermore, χ^+ is continuous in $(0, +\infty)$.*

The rest of this section is devoted to the case by case proof of this theorem.

I) If $\det X > 0$ then the eigenvalues of X are complex. Hence is no compact proper subset of $\mathbf{P}$ invariant under $\exp(tX)$, $t > 0$. This implies that (5.1^{ρ}) is controllable for any $\rho > 0$, that is, $\chi^+(\rho) = \mathbf{P}$ for all $\rho > 0$.

II) If $\det X = 0$, the system is also controllable for any $\rho > 0$. In fact, $u = 0$ is a real root of $\det(X + uY)$. Hence if $\det Y \neq 0$ the discriminant of the quadratic polynomial is $-\det[X, Y] \geq 0$. Since we are assuming that $\{X, Y\}$ generates $sl(2)$, it follows that $\det[X, Y] < 0$. This implies that for any $\rho > 0$ there exists $u_0 \in [-\rho, \rho]$ such that $X + u_0 Y$ belongs to the interior of $\mathcal{C}$. Therefore (5.1^ρ) is controllable for any $\rho > 0$. On the other hand if $\det Y = 0$ then

$$\det(X + uY) = -(XY + YX)\,u + \det X.$$

But $XY + YX \neq 0$, as follows from (2.3). This implies again that $\det(X + u_0 Y) > 0$ for some $u_0 \in [-\rho, \rho]$ for all $\rho > 0$, ensuring that the system is controllable.

III) If $\det X < 0$ then there are two cases to be considered:

1. The unrestricted system is controllable ($\det[X, Y] < 0$). This holds if and only if the straight line $X + uY$, $u \in R$, meets $\mathcal{C}_{\mathrm{int}}$. Since $\det X < 0$, $X \in \mathcal{C}_{\mathrm{ext}}$. Hence there exists $\rho_* > 0$ such that the segment $X + uY$, $u \in [-\rho_*, \rho_*]$ is the smallest one meeting $\mathcal{C}$. In other words, $X + \rho_* Y$ or $X - \rho_* Y$ is the first hitting $\mathcal{C}$ of the line $X + uY$, starting from X.

   We claim that (5.1^ρ) is controllable if $\rho > \rho_*$ and not controllable otherwise.

   To see this note first that if $\rho > \rho_*$ then some point $X + u_0 Y$ belongs to the interior of the double cone $\mathcal{C}$ for some $u_0 \in [-\rho, \rho]$, so that the system is controllable, because no proper subset of $\mathbf{P}$ is invariant under $X + u_0 y$.

   Now, suppose that $\rho < \rho_*$. Let $Z \neq 0$ be orthogonal (w.r.t. the trace form) to the plane π spanned by X and Y. Then $Z \in \mathcal{C}_{\mathrm{ext}}$ because π intersects the interior of $\mathcal{C}$ and the plane orthogonal to any line in $\mathcal{C}_{\mathrm{int}}$ is contained in the exterior of $\mathcal{C}$. Therefore by Proposition 4.4, the set $\mathcal{A}(\sigma_\rho)$ of attractors of the matrices in the segment

   $$\sigma_\rho = \{X + uY : u \in [-\rho, \rho]\},$$

   is invariant under the semigroup generated by (5.1^ρ). Therefore the system is not controllable. Moreover, the invariance of $\mathcal{A}(\sigma_\rho)$ together with the fact that it is contained in the invariant control set of (5.1^ρ) (see Proposition 3.1) implies that $\chi^+(\rho) = \mathcal{A}(\sigma_\rho)$. Therefore, the characterization of $\mathcal{A}(\sigma_\rho)$ by intersections of planes with $\mathcal{C}^+$ shows that χ^+ is continuous at ρ.

Finally, for $\rho = \rho_*$ the system is not controllable. This can be seen either by the lower semicontinuity of χ^+ ($\chi^+(\rho)$ is proper in case $\rho < \rho^*$) or directly, as follows: Suppose without loss of generality that $X + \rho_* Y$ is the first hitting in $\mathcal{C}$. The intersection with $\mathcal{C}^+$ of $(X + \rho_* Y)^\perp$ is the ray defined by $X + \rho_* Y$ and the interval of the attractors for $X + uY$, $u \in [-\rho_*, \rho_*]$ ends in this ray. So as above the invariant control set is the closure of the interval of the attractors, and the system is not controllable.

Since the system is controllable for $\rho > \rho_*$ it follows that ρ_* is the only point of discontinuity.

2. The unrestricted system is not controllable ($\det[X, Y] > 0$).

   Let $Z \neq 0$ span the line orthogonal to the plane spanned X and Y. Then $Z \in \mathcal{C}_{\text{int}}$ and the segment σ_ρ defined by the system is contained in $\mathcal{C}_{\text{ext}}$ for all $\rho > 0$. Applying Proposition 4.4 we see as above that the invariant control set of (5.1^ρ) is $\mathcal{A}(\sigma_\rho)$ the set of attractors of the matrices in σ_ρ. Therefore $\chi^+(\rho) = \mathcal{A}(\sigma_\rho)$ implying that χ^+ is continuous at every $\rho > 0$. A similar argument shows that χ^- is continuous as well.

6. Discrete-time

In this section we consider the discrete-time version of (1.1), namely

$$x_{n+1} = \exp(X + uY)\, x_n \tag{6.1}$$

where X and Y are 2×2 matrices. Analogous to the continuous-time case we can consider the control range U to be unrestricted ($U = R$) or restricted ($U^\rho = [-\rho, \rho]$). Our purpose here is to show that controllability is given again by the intersection of $X + uY$ with the cone $\mathcal{C}$ of nilpotent matrices.

The semigroup generated by system (6.1) is defined by

$$S_d = \{\exp(X + u_1 Y) \cdots \exp(X + u_k Y) : k \geq 1\}$$

where u_k varies in the control range. Also, the group of the system G is the group generated by S_d. The system is said to be controllable if $R^2 \setminus \{0\}$ if for any pair of nonzero vectors x, y there exists $g \in S_d$ such that $gx = y$.

In the sequel we consider only systems whose group is $\mathrm{Sl}(2)$. As happens to the continuous-time case there exists a simple criterion for ensuring that $G = \mathrm{Sl}(2)$. First, we must have $\mathrm{tr}X = \mathrm{tr}Y = 0$ to have that $S \subset \mathrm{Sl}(2)$. On the other hand, if this condition holds, then the fact that $\mathrm{Sl}(2)$ is a simple group implies that $\mathrm{int}S \neq \emptyset$ in case the Lie algebra generated by X and Y is $sl(2)$ (see 8, Section 4.) Therefore we have

Proposition 6.1. *Suppose that $\mathrm{tr}X = \mathrm{tr}Y = 0$ and $\det[X,Y] \neq 0$. Then the semigroup S generated by the system (6.1) is contained in $\mathrm{Sl}(2)$ and has nonempty interior in $\mathrm{Sl}(2)$.*

For systems satisfying the assumptions of this proposition we can apply the results about control sets to analyze controllability. In particular, the system is controllable if and only if $S = \mathrm{Sl}(2)$ which in turn holds if and only if S is transitive in $\mathbf{P}$.

In the following lemma we check the system is controllable if $X + uY$ crosses the interior of $\mathcal{C}$.

Lemma 6.2. *Consider the system (6.1) with control range $U^\rho = [-\rho, \rho]$. Assume that $X + u_0 Y$ belongs to the interior of $\mathcal{C}$ for some $u_0 \in U^\rho$. Then the system is controllable.*

Proof: Put $Z_0 = X + u_0 Y$. By assumption $\det(Z_0) > 0$. Moreover, the eigenvalues of $X + uY$ are $\pm\sqrt{\det(X + uY)}$. Then we can change u_0 slightly and assume without loss of generality that the eigenvalues of Z_0 are $\pm\varepsilon\sqrt{-1}$ with ε/π irrational. This being so, put $g_0 = \exp(Z_0)$. Then for any $x \in \mathbf{P}$, the orbit $\{g_0^k x : k \geq 0\}$ is dense in $\mathbf{P}$. Clearly, $g_0^k \in Sd$ if $u \geq 0$. Hence S does not leave invariant any compact subset $C \subset \mathbf{P}$, showing that the invariant control set in $\mathbf{P}$ is not proper, that is, S_d is transitive in $\mathbf{P}$.
□

Reciprocally, assume that the segment $X + uY$, $u \in U^\rho$, does not meet the interior of $\mathcal{C}$. Then the discrete-time control system (6.1) is not controllable. In fact, by definition of these semigroups it follows that $S_d \subset S_c$. Now, segment $X + uY$, $u \in U^\rho$, does not meet the interior of $\mathcal{C}$, then S_c is a proper semigroup, as was proved before. Therefore $S_d \neq \mathrm{Sl}(2)$, showing that the discrete-time system is not controllable. Summarizing, we have

Theorem 6.3. *Given the discrete-time system (6.1) with restricted control range U^ρ. Assume that $\mathrm{tr}X = \mathrm{tr}Y = 0$ and $\det[X,Y] \neq 0$. Then the system is controllable if and only if the segment $X + uY$, $u \in U^\rho$, crosses the interior $\mathcal{C}_{\mathrm{int}}$ of $\mathcal{C}$. This geometric condition holds for some ρ only if $\det[X,Y] < 0$.*

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