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Comparation between some approaches to solve fuzzy differential equations☆

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Abstract

In this paper, we study the class of fuzzy differential equations where the dynamics is given by a continuous fuzzy mapping which is obtained via Zadeh's extension principle. We get a fuzzy solution for this class of fuzzy differential equations and several illustrative examples are presented. We also give some properties and we show the relationships with others interpretation. Finally, we propose a numerical procedure for obtaining the fuzzy solution.

Keywords: Extension principle; Fuzzy differential equation; Fuzzy differential inclusions

1. Introduction

Fuzzy-valued mappings (fuzzy functions) were initially developed by Puri and Ralescu [37]. They generalized and extended the concept of Hukuhara differentiability (H-derivative) for set-valued mappings to a class of fuzzy mappings. Subsequently, by using the H-derivative, Kaleva [25] started to develop a theory for fuzzy differential equations (FDE).

In the last few years, many works have been done by several authors in theoretical and applied fields for FDE considering the H-derivative (see [13,16,15,25–27,30–33,38,39,41–46]). Now, in some cases this approach suffers certain disadvantages since the diameter $\text{diam}(x(t))$ of the solution $x(t)$ of an FDE is unbounded as time t increases [16,14,18,27]. This problem demonstrates that in some cases this interpretation is not a good generalization of the associated crisp case and we assume that this problem is due to the fuzzification of the derivative utilized in the formulation of the FDE. As a consequence, alternative formulations have been suggested.

One of the alternatives is to replace the FDE by a family of differential inclusions generated from the function involved in FDE, see [15,14,20,19,22]. Another alternative is given by Bede et al. in [5] and Chalco-Cano et al. in [10], where they introduce a more general definition of derivative for fuzzy functions enlarging the class of differentiable fuzzy mappings by considering fuzzy lateral H-derivatives. With this interpretation of derivatives new solutions are obtained for FDE (see also [6,11,9]). Also, there are various papers devoted to the study of differential equation with

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fuzzy parameters, see [28,34]. Under some conditions and for some classes of fuzzy functions, we can see relationships between different interpretations earlier, for instance see [10,26].

In this paper, we study FDE where the fuzzy function is obtained via Zadeh's extension principle. We get a fuzzy solution for this class of FDE. We give some properties and we show the relationships with fuzzy differential inclusions and fuzzy differential equations considering generalized derivative. Also, we propose a numerical procedure for obtaining the fuzzy solution.

2. Basic concepts

We denote by $\mathcal{K}(X)$ the family of all nonempty compact convex subsets of a Banach space X. If $A, B \in \mathcal{K}(X)$ and $\lambda \in \mathbb{R}$, then the operations of addition and scalar multiplication are defined as

$$A + B = \{a + b/a \in A, b \in B\}, \quad \lambda A = \{\lambda a/a \in A\}.$$

A fuzzy set u on a universe set X is a mapping $u : X \to [0, 1]$. We think u as assigning to each element $x \in X$ a degree of membership, $0 \leqslant u(x) \leqslant 1$. If X is a Banach space and u is a fuzzy set on X, we define $[u]^\alpha = \{x \in X/\ u(x) \geqslant \alpha\}$ the α-level of u, with $0 < \alpha \leqslant 1$. For $\alpha = 0$ the support of u is defined as $[u]^0 = \mathrm{supp}(u) = \overline{\{x \in X/u(x) > 0\}}$, where $\overline{A}$ denotes the clousure of $A \subset X$.

A fuzzy set u on X is called compact if $[u]^\alpha \in \mathcal{K}(X)$, $\forall \alpha \in [0, 1]$. Also, u is called convex if $[u]^\alpha$ is a convex set for all $\alpha \in [0, 1]$. We will denote by $\mathcal{F}(X)$ the space of all compact and convex fuzzy sets on X.

If $u \in \mathcal{F}(\mathbb{R})$, then u is called a fuzzy interval and the α-level set $[u]^\alpha$ is a nonempty compact interval for all $\alpha \in [0, 1]$. We denote by (a, b, c) the triangular fuzzy number with support $[a, c]$.

The operations of addition and the scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ are defined as

$$(u + v)(x) = \sup_{y \in \mathbb{R}^n} \{u(y) \wedge v(x - y)\} \quad \text{and} \quad (\lambda \cdot u)(x) = \begin{cases} u\left(\dfrac{x}{\lambda}\right) & \text{if } \lambda \neq 0, \\ \chi_{\{0\}}(x) & \text{if } \lambda = 0, \end{cases} \tag{1}$$

where $\chi_{\{0\}}$ is the characteristic function of $\{0\}$. If $u, v \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$, then the following properties are true:

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha \quad \text{and} \quad [\lambda \cdot u]^\alpha = \lambda [u]^\alpha, \quad \forall \alpha \in [0, 1].$$

Also, we can extend the Hausdorff metric H on $\mathcal{K}^n$ to $\mathcal{F}(\mathbb{R}^n)$ by means of

$$D(u, v) = \sup_{\alpha \in [0,1]} H([u]^\alpha, [v]^\alpha), \quad \forall u, v \in \mathcal{F}(\mathbb{R}^n).$$

In [47], Zadeh proposed the extension principle, which has become an important tool in fuzzy theory and its applications. The idea is that each function $f : X \to Y$ induces another function $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$, defined for each fuzzy set u in X by

$$\hat{f}(u)(y) = \begin{cases} \sup\limits_{x \in X, f(x) = y} u(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

The function $\hat{f}$ is obtained from f by the extension principle.

In general, the computation of $\hat{f}$ is a rather difficult task. An exception occurs when f is a linear function. On the other hand, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, then $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is a well-defined function, and (see [40])

$$[\hat{f}(u)]^\alpha = f([u]^\alpha), \quad \forall \alpha \in [0, 1], \ \forall u \in \mathcal{F}(\mathbb{R}^n), \tag{2}$$

where $f(A) = \{f(a)/a \in A\}$.

3. Solving FDE

In this section, we give a method for solving the fuzzy initial value problem

$$X'(t) = f(t, X(t)), \quad X(0) = X_0, \tag{3}$$

where $f : [0, T] \times \mathcal{F}(U) \to \mathcal{F}(\mathbb{R}^n)$ is obtained by Zadeh's extension principle from a continuous function $g : [0, T] \times U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^n$. Note that f is continuous because g is continuous (see [40]) and by (2) we have

$$[f(t, X)]^\alpha = g(t, [X]^\alpha),$$

where $g(t, A) = \{g(t, a)/a \in A\}$.

Associated with FDE (3) we can consider the deterministic differential equation (DDE):

$$x'(t) = g(t, x(t)), \quad x(0) = c, \tag{4}$$

where $x'(t)$ is the derivative (crisp) of a function $x : [0, T] \to \mathbb{R}^n$.

We obtain a fuzzy solution for (3) derived from (4): suppose that problem (4) has the solution $x(t, c)$. Then, applying the Zadeh's extension principle to $x(t, c)$ in relation to the parameter c, we obtain the extension $X(t) = \widehat{x}(t, X_0)$, for each t fixed, which is a fuzzy solution of problem (3) (in the sequel, this extended solution will be called the fuzzy solution for problem (3)). The idea of this proposal to obtain a fuzzy solution is not new, for instance see [7,24,28].

This procedure is more precisely established in the following result:

Theorem 1. *Let U be an open set in $\mathbb{R}^n$ and $[X_0]^\alpha \subset U$. Suppose that g is continuous, and that for each $c \in U$ there exists a unique solution $x(\cdot, c)$ of the deterministic problem (4) and that $x(t, \cdot)$ is continuous on U for each $t \in [0, T]$ fixed. Then, there exists a unique fuzzy solution $X(t) = \hat{x}(t, X_0)$ of the FDE (3).*

Proof. Since problem (4) has a unique solution $x(t, c)$ and it is continuous on U, $x(t, \cdot) : U \to \mathbb{R}^n$ is well defined and it is continuous for each $t \in [0, T]$ fixed. Then, by (2), $\hat{x}(t, \cdot) : \mathcal{F}(U) \to \mathcal{F}(\mathbb{R}^n)$ is a continuous function and it is well defined. Therefore there exists a unique solution of the form $X(t) = \hat{x}(t, X_0)$ for the FDE (3). □

Example 1. Consider the fuzzy initial value problem

$$\begin{cases} X' = -X(t), \\ X(0) = C, \end{cases} \tag{5}$$

where the initial condition C is any fuzzy interval.

This problem has the following deterministic associated problem:

$$x'(t) = -x(t), \quad x(0) = c,$$

which possesses the exact solution

$$x(t, c) = c\mathrm{e}^{-t}.$$

Note that $x(t, c)$ is continuous in $c \in \mathbb{R}$ for each $t \geqslant 0$ fixed. We apply the Zadeh's extension principle to $x(t, c)$ in relation to c, for each $t \geqslant 0$ fixed. Then we obtain the unique fuzzy solution $X(t) = \hat{x}(t, C)$ of problem (5) for any initial condition C, with C a fuzzy interval, which is given by

$$X(t) = C \cdot \mathrm{e}^{-t}, \quad t \geqslant 0,$$

where $\cdot$ is the scalar multiplication (1).

Now, if we consider the fuzzy initial value problem

$$\begin{cases} X' = X(t), \\ X(0) = C, \end{cases} \tag{6}$$

where C is any fuzzy interval. Then we have the following deterministic associated problem:

$$x'(t) = x(t), \quad x(0) = c,$$

which possesses the exact solution

$$x(t, c) = c\mathrm{e}^{t}.$$

Using the proposal method above, for any fuzzy interval C, we obtain the fuzzy solution

$$X(t) = C \cdot e^t, \quad t \geqslant 0,$$

where $\cdot$ is the scalar multiplication (1).

Example 2. Consider the following fuzzy initial value problem:

$$\begin{cases} X' = -X(t) + t + 1, \\ X(0) = C. \end{cases} \tag{7}$$

Then the deterministic problem associated to (7) is

$$x'(t) = -x(t) + t + 1, \quad x(0) = c.$$

Thus, the deterministic solution for this problem is

$$x(t, c) = t + ce^{-t},$$

which is linear in relation to c for each $t \geqslant 0$. Therefore, there exists a unique fuzzy solution of problem (7) for any fuzzy interval C and it is given by

$$X(t) = t + C \cdot e^{-t}.$$

Example 3. Consider the fuzzy initial value problem

$$\begin{cases} X'(t) = \lambda X^2(t), \\ X(0) = X_0 \end{cases}$$

with $\lambda > 0$. Then, the deterministic solution $x(t, c)$ is continuous with respect to c on the open interval $J = (0, z)$ for each $t \in [0, 1/\lambda z)$ fixed. Therefore, for any fuzzy interval X_0 there is the fuzzy solution $X(t) = \widehat{x}(t, X_0)$ for each $t \in [0, 1/\lambda z)$, where $z = \sup_{k \in [X_0]^0} k$.

In [25] Kaleva studied the previous problem for $\lambda = 1$, i.e.,

$$X'(t) = X^2(t), \quad X(0) = C, \tag{8}$$

where the initial condition X_0 is the triangular fuzzy number

$$X_0(y) = \begin{cases} 3 - y & \text{if } 2 \leqslant y \leqslant 3, \\ y - 1 & \text{if } 1 \leqslant y \leqslant 2, \\ 0 & \text{elsewhere.} \end{cases}$$

The deterministic problem associated with (8) is

$$x'(t) = x^2(t), \quad x(0) = c,$$

and the solution to this problem is

$$x(t, c) = \frac{c}{1 - tc}.$$

For each $t \in [0, \frac{1}{3})$ fixed, the function $x(t, c)$ is continuous with respect to c. Therefore there exists a unique fuzzy solution $X(t) = \hat{x}(t, X_0), t \in [0, \frac{1}{3})$ for problem (8). Also, for each $t \in [0, \frac{1}{3})$ fixed the function $x(t, c)$ is nondecreasing with respect to c. Then, for each $\alpha \in [0, 1]$ we have

$$\begin{aligned} [X(t)]^\alpha &= [\hat{x}(t, X_0)]^\alpha \\ &= x(t, \ [X_0]^\alpha) \\ &= x(t, \ [1 + \alpha, 3 - \alpha]) \\ &= [x(t, 1 + \alpha), x(t, 3 - \alpha)] \\ &= \left[\frac{1 + \alpha}{1 - t - t\alpha}, \frac{3 - \alpha}{1 - 3t + t\alpha} \right]. \end{aligned}$$

Remark 1. Example 1 was studied in various papers [16,14,27,25]. As we see, the fuzzy solution obtained in Example 1 is coincident with the solution obtained by fuzzy differential inclusions in [14], and it is also coincident with the solution obtained in [10] by considering $X'(t)$ the generalized derivative. Example 2 was studied in [4], and we can also verify that the fuzzy solution obtained in Example 2 is coincident with the solution obtained by fuzzy differential inclusions and is coincident with the solution obtained by considering $X'(t)$ the generalized derivative [10]. The fuzzy solution of Example 3 is coincident with the solution obtained by fuzzy differential inclusion and considering $X'(t)$ the H-derivative, for instance see [26,28]. In this direction, in the following Section 4 we will study the relationships between fuzzy solutions of FDE obtained via fuzzy differential inclusions and FDE with generalized derivative.

4. Relationships with others interpretations

4.1. Solutions via differential inclusions

Following Hüllermeier [22] and Diamond [15,14], we interpret the fuzzy initial value problem (3) as a family of differential inclusions

$$y'_\alpha(t) = g(t, y_\alpha(t)), \quad y_\alpha(0) \in [X_0]^\alpha, \ \ 0 \leqslant \alpha \leqslant 1. \tag{9}$$

Under suitable assumptions, the attainable sets

$$\mathcal{A}_\alpha(t) = \{y_\alpha(t)/y_\alpha \text{ is a solution of (9)}\}$$

are α-levels of a fuzzy set $\mathcal{A}(t)$, which we call a solution of (3).

Theorem 2. *Let U be an open set in $\mathbb{R}^n$ and $X_0 \in \mathcal{F}(U)$. Suppose that g is continuous, that for each $c \in U$ there exists one unique solution $x(\cdot, c)$ of problem (4) and that $x(t, \cdot)$ is continuous in U for each $t \in [0, T]$. Then, the fuzzy solution $X(t)$ of problem (3) and the attainable set for problem (9) are coincidents, i.e.,*

$$X(t) = \mathcal{A}(t)$$

for all $0 \leqslant t \leqslant T$.

Proof. In order to prove this result we must show that

$$[X(t)]^\alpha = [\hat{x}(t, X_0)]^\alpha = \mathcal{A}_\alpha(t), \quad \forall\, \alpha \in [0, 1].$$

By the hypotheses we have that if g is continuous, then f is continuous. Thus, by Theorem 1, there exists a unique solution of problem (3) $X(t) = \hat{x}(t, X_0)$ and

$$[X(t)]^\alpha = [\hat{x}(t, X_0)]^\alpha = x(t, [X_0]^\alpha), \quad \forall\, \alpha \in [0, 1].$$

Therefore, given that $\alpha \in [0, 1]$, we have

$$[X(t)]^\alpha = x(t, [X_0]^\alpha) = \{x(t, c)/c \in [X_0]^\alpha\}. \tag{10}$$

On the other hand, the α-levels of the attainable set for problem (9) are given by

$$\mathcal{A}_\alpha(t) = \{x(t, c)/c \in [X_0]^\alpha\}. \tag{11}$$

From (10) and (11) follows the result. □

4.2. Solutions using generalized derivative

Following Bede and Gal [5] and Chalco and Román [10], we have the following definition of generalized derivative:

Definition 1. Let $u, v \in \mathcal{F}(\mathbb{R}^n)$ be. If there exists $w \in \mathcal{F}(\mathbb{R}^n)$ such that $u = v + w$, then w is called the *H*-difference of u and v and it is denoted by $u \ominus v$.

Definition 2. Let $x : T \to \mathcal{F}(\mathbb{R}^n)$ be and $t_0 \in T$. We say that x is differentiable at t_0 if:

(I) there exists an element $x'(t_0) \in \mathcal{F}(\mathbb{R}^n)$ such that, for all $h > 0$ sufficiently near to 0, there are $x(t_0 + h) \ominus x(t_0)$, $x(t_0) \ominus x(t_0 - h)$ and the limits (in D-metric)

$$\lim_{h \to 0^+} \frac{x(t_0 + h) \ominus x(t_0)}{h} = \lim_{h \to 0^+} \frac{x(t_0) \ominus x(t_0 - h)}{h} = x'(t_0)$$

or

(II) there is an element $x'(t_0) \in \mathcal{F}(\mathbb{R}^n)$ such that, for all $h < 0$ sufficiently near to 0, there are $x(t_0 + h) \ominus x(t_0)$, $x(t_0) \ominus x(t_0 - h)$ and the limits

$$\lim_{h \to 0^-} \frac{x(t_0 + h) \ominus x(t_0)}{h} = \lim_{h \to 0^-} \frac{x(t_0) \ominus x(t_0 - h)}{h} = x'(t_0).$$

Note that derivative in the first form (I) is coincident with the H-derivative. Also, if x is differentiable in the first form (I) and $x'(t_0) \in \mathscr{F}(\mathbb{R}^n/\mathbb{R}^n$ then, from Definition 2, it is not differentiable in the second form (II) and vice versa.

Theorem 3 (*Chalco-Cano et al. [10]*). *Let $x : T \to \mathcal{F}(\mathbb{R})$ be a function and denote*

$$[x(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$$

for each $\alpha \in [0, 1]$. Then:

(i) *If x is differentiable in the first form (I), then f_α and g_α are differentiable functions and*

$$[x'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]. \tag{12}$$

(ii) *If x is differentiable in the second form (II), then f_α and g_α are differentiable functions and*

$$[x'(t)]^\alpha = [g'_\alpha(t), f'_\alpha(t)]. \tag{13}$$

This last result (Theorem 3) gives us a procedure to solve the FDE (3) where $X'(t)$ is the generalized derivative (in the first form (I) or second form (II)) in the sense of Definition 2, see [10,26]. In the following result we will show the relationships between the fuzzy solution for problem (3) proposed in Section 3 and the solution of problem (3) when $X'(t)$ is the generalized derivative.

Theorem 4. *Let U be an open set in $\mathbb{R}^n$ and $X_0 \in \mathcal{F}(U)$. Suppose that g is continuous, that for each $c \in U$ there exists one unique solution $x(\cdot, c)$ of problem (4) and that $x(t, \cdot)$ is continuous in U for each $t \in [0, T]$. Then*:

(i) *If g is nondecreasing with respect to the second argument then, the fuzzy solution of (3) and the solution of (3) via the derivative in the first form (I), are identical.*

(ii) *If g is nonincreasing with respect to the second argument, then the fuzzy solution of (3) and the solution of (3) via the derivative in the second form (II), if it exists, are identical.*

Proof. If g is nondecreasing, then from Theorem 3 in [26], the solution of (3) and the solution by differential inclusions are identical. Thus, from Theorem 2 we have the result. In the same way, if g is nonincreasing, then from Theorem 4 in [10], the solution of (3) and the solution by differential inclusions are identical. Therefore, from Theorem 2 we have the result. □

Remark 2. There exist some results on the existence of solution for FDE with generalized derivative, for instance see Theorem 22 in [5].

5. Numerical procedure

In this section we present a numerical procedure for solving the FDE (3). This numerical procedure is based on the calculation of the extension principle given in [12,8], where the authors obtained an approximation for the extension principle of a continuous function. We will see two cases: when the deterministic equation has an explicit solution and when the deterministic equation has no an explicit solution.

In [8], the authors present a proposal to calculate $\hat{g}(u)$ using decomposition of a fuzzy interval u and the linear spline function G_n associated with g. For this, let u be a fuzzy interval with support $[a, b]$ and let $g : A \to \mathbb{R}$ be a continuous function, where $A \subseteq \mathbb{R}$ is an open set containing the support of u.

Following the proposal for decomposition of fuzzy intervals given in [12,8], we decompose u in n compact fuzzy sets, i.e., we divide $[a, b]$ in n subintervals $[x_i, x_{i+1}]$ and we consider the fuzzy set u_i defined by $u_i(x) = u(x)$ for all $x \in [x_i, x_{i+1}]$ and $u_i(x) = 0$ in another case. Consequently, we have

$$u(x) = \bigvee_{i=1,\ldots,n} u_i(x), \tag{14}$$

where $\vee$ denotes the maximum.

Using a linear spline function we can approximate the function g, doing as follows:

For each $i = 1, \ldots, n$ we define the linear function g_i by

$$g_i(x) = \begin{cases} 0 & \text{if } x \notin [x_i, x_{i+1}], \\ f(x_i)\dfrac{x_{i+1} - x}{h_i} + f(x_{i+1})\dfrac{x - x_i}{h_i} & \text{if } x \in [x_i, x_{i+1}], \end{cases}$$

where $h_i = x_{i+1} - x_i$ is the length of each subinterval.

We take $G_n : \mathbb{R} \to \mathbb{R}$ defined by

$$G_n(x) = \begin{cases} g_1(x) & \text{if } x \in [x_1, x_2], \\ g_2(x) & \text{if } x \in [x_2, x_3], \\ \vdots & \vdots \qquad \vdots \\ g_n(x) & \text{if } x \in [x_n, x_{n+1}]. \end{cases} \tag{15}$$

Then G_n is the linear spline function and it is a bounded and continuous function. Therefore, we obtain $\widehat{G_n}$, given by

$$\widehat{G_n}(u) = \bigvee_{i=1,\ldots,n} \widehat{g_i}(u_i),$$

which is the approximation for $\hat{g}$ as we can see in the next theorem:

Theorem 5 (*Chalco-Cano et al. [8]*). *Let $g : A \to \mathbb{R}$ be a continuous function such that $g \in C^2([a, b])$. Then, for every $n \in \mathbb{N}$*

$$D(\widehat{G_n}(u), \hat{g}(u)) \leqslant \tfrac{1}{8}\|g''\|_\infty h^2$$

with $h = \max\{h_i/1 \leqslant i \leqslant n\}$.

5.1. The case when the deterministic equation has an explicit solution

In this case, we obtain an fuzzy solution of (3) using the extension principle. For this, let $U \subset \Omega$ be an open set in $\mathbb{R}$ such that there exists a unique solution $x(\cdot, c)$ of (4) with $c \in U$ in the interval $[0, T]$, and for all $t \in [0, T]$, $x(t, \cdot)$ is continuous on U. Then, we can define the operator

$$x(t, \cdot) : U \longrightarrow \mathbb{R}.$$

Utilizing the extension principle for $x(t, \cdot)$, we obtain

$$\widehat{x}(t, \cdot) : \mathcal{F}(U) \to \mathcal{F}(\mathbb{R}),$$

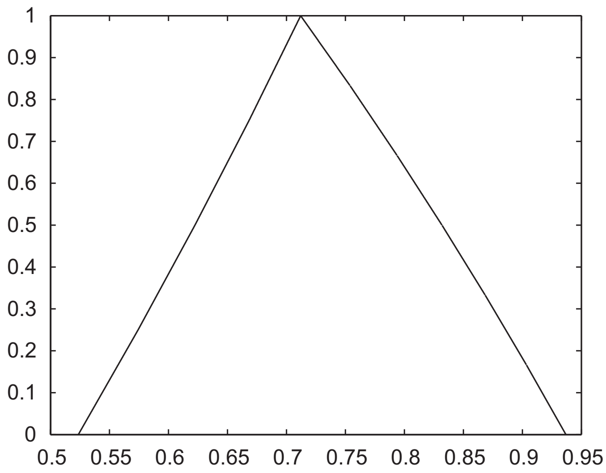

Fig. 1. The fuzzy number $\hat{G}_{10}(C)$.

which is the extension of the solution of (4). Therefore a solution of problem (3), with initial condition X_0, is given by $X(t) = \widehat{x}(t, X_0)$ for all $t \in [0, T]$. To obtain $X(t)$ we apply the numerical procedure as described at the beginning of this section.

Example 4. Let us consider the normalized deterministic Verhulst population model

$$\begin{cases} X' = 0.5X(1-X), \\ X(0) = (0.4, 0.6, 0.9) = C. \end{cases} \tag{16}$$

The deterministic solution of (16) is

$$x(t,c) = \frac{c}{c-(c-1)\mathrm{e}^{-0.5t}}.$$

Let $U = (0, 1)$ be. Then U is an open set in $\mathbb{R}$ and $x(t, c)$ is continuous on U for each $t > 0$ fixed. Since $x(t, c)$ is continuous with respect to c on U, there is $X(t) = \widehat{x}(t, X_0)$ for X_0 with support contained in U.

For $t = 1$ for instance and decomposing the triangular fuzzy number $C = (0.4, 0.6, 0.9)$ into n fuzzy sets, we obtain the approximation $\widehat{G_n}(C)$ of $X(1) = \hat{x}(1, C)$, as described at the beginning of this section. This approximation converges very quickly to $X(1)$ because $D(\widehat{G_n}(C), X(1)) \leqslant 0.0516/n^2$.

Fig. 1 displays $\widehat{G_{10}}(C)$, the approximation to $X(1)$, which is not a triangular fuzzy number.

Fig. 2 shows the deformation of the initial triangular fuzzy number X_0 on time.

5.2. The case when the deterministic equation has no explicit solution

In this case, we consider the FDE (3). In this context, given the fuzzy set X_0, we can decompose it in n fuzzy sets P_i with support $[p_{i-1}, p_i]$, see the beginning of this section (14). For each p_i we have the deterministic problem associated with (3), i.e.,

$$\begin{cases} x'(t) = g(t, x(t)), \\ x(t_0) = p_i. \end{cases} \tag{17}$$

For each $i = 0, 1, \ldots, n$ we solve problem (17), using any numerical method, for obtaining the solution $x(t, p_i)$.

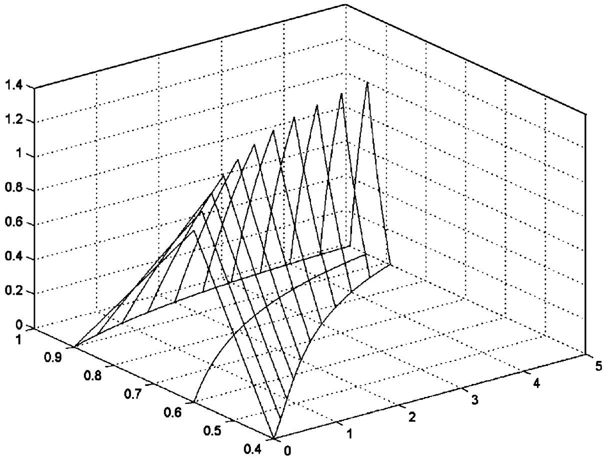

Fig. 2. The dynamics of the fuzzy differential equation (16).

Now, to obtain $X(t)$ with $t \in [0, T]$, we can define the function g_i as the line joining the points $x(t, p_{i-1}), x(t, p_i)$. So,

$$\widehat{G_n}(P_0) = \bigvee_{i=1,\dots,n} \hat{g}_i(P_i)$$

and this solution is an approximating solution for problem (3).

Example 5. Let us consider the following fuzzy initial value problem:

$$\begin{cases} X'(t) = \mathrm{e}^{-X^2(t)}, \\ X(0) = X_0, \end{cases} \tag{18}$$

where $X_0 = (0, 1, 2)$ is a triangular shape fuzzy number with support $[0, 2]$. We decompose the fuzzy interval X_0 in eight fuzzy sets P_i, $i = 1, 2, 3, \dots 8$ by mean (see (14))

$$X_0 = \bigvee_{i=0,\dots,8} P_i, \quad P_i(z) = \begin{cases} X_0(z) & \text{if } z \in [p_{i-1}, p_i], \\ 0 & \text{if } z \notin [p_{i-1}, p_i], \end{cases}$$

where

$$p_0 = 0 < p_1 = \tfrac{1}{4} < p_2 = \tfrac{1}{2} < \cdots < p_7 = \tfrac{7}{4} < p_8 = 2.$$

Therefore each fuzzy set P_i has support $[p_{i-1}, p_i] = [(i-1)/4, i/4]$, $i = 1, 2, \dots, 8$.

Then, for each p_i, we can use the Runge–Kutta numerical method for solving the following deterministic associated problem:

$$\begin{cases} x'(t) = \mathrm{e}^{-x^2(t)}, \\ x(0) = p_i. \end{cases} \tag{19}$$

Thus we have the following approximated values for the solutions $x(t = 1, p_i)$:

p_i	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
$x(1, p_i)$	0.7952	0.9174	1.0294	1.1435	1.2699	1.4171	1.5913	1.7933	2.0177

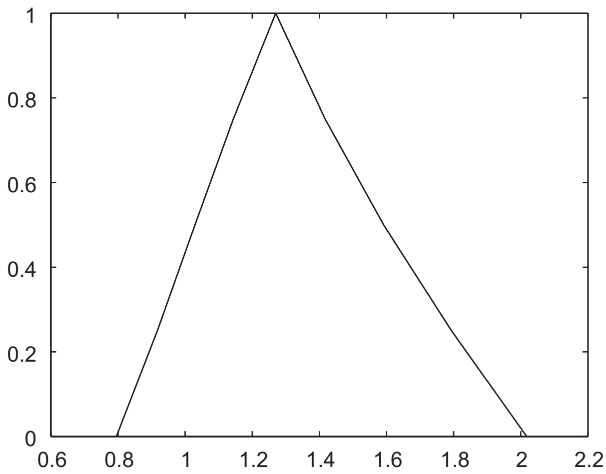

Fig. 3. The fuzzy number $\hat{G}_8(X_0)$.

Fig. 3 displays $\widehat{G_8}(X_0)$, the approximation to $X(1) = \widehat{x}(1, X_0)$.

Remark 3. Other proposals on numerical methods for FDE have been published. Some examples can be found in [1,2,23], where numerical methods for fuzzy differential inclusions are presented, and in [3,35,36], where numerical methods for FDE with H-derivative are presented. The approach followed in these papers is quite different from the one presented here.

6. Conclusions

In this paper we study fuzzy differential equations (FDE) where the fuzzy function f is obtained by applying the Zadeh's extension principle to a function g. We obtain a fuzzy solution for this class of FDE by applying the Zadeh's extension principle to the deterministic solution associated with the fuzzy problem (see Section 3). In Section 4.1, we show that the fuzzy solution is coincident with the solution obtained by fuzzy differential inclusions. Also, in the case that f is monotone, we show that the fuzzy solution is coincident with the solution obtained by considering X' the generalized derivative (in Section 4.2). We can say that there is a drawback for the Zadeh's extension and differential inclusions interpretation, that is, that we do not use a derivative for fuzzy function. But both interpretations, under some conditions, are equivalent to FDE with generalized derivative. Finally, in Section 5 we present a numerical procedure to obtain the fuzzy solution. As we can see this interpretation is much more intuitive for modeling FDE.

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