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Generalized derivative and π-derivative for set-valued functions ☆

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ABSTRACT

In this paper we study the generalized derivative and the π-derivative for interval-valued functions. We show the connections between these derivatives. Some illustrative examples and applications to interval differential equations and fuzzy functions are presented.

1. Introduction

The importance of the study of set-valued analysis from a theoretical point of view as well as from their application is well known [3,4]. Many advances in set-valued analysis have been motivated by control theory and dynamical games [5]. Optimal control theory and mathematical programming were a motivating force behind set-valued analysis since the sixties [5]. Interval Analysis is a particular case and it was introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. The first monograph dealing with interval analysis was given by Moore [32]. Moore is recognized to be the first to use intervals in computational mathematics, now called numerical analysis. He also extended and implemented the arithmetic of intervals to computers. One of his major achievements was to show that Taylor series methods for solving differential equations are not only more tractable, but also more accurate [33].

Other issues of control theory, dynamic economy and biological evolution theory, such as the regulation of control systems subjected to viability constraints, motivated the discovery of differential calculus of set-valued maps [5]. In this direction, several notions of derivative of a set-valued map are introduced. For instance, see [4–6,8,12,24,26,9,15,19,38]. The paper of Hukuhara [24] was the starting point for the topic of set differential equations and later on for Fuzzy Differential Equations. Set differential equations have recently attracted the attention of many researchers [7,13,28,29,23]. Fuzzy Calculus and Fuzzy Differential Equations are also two very important related fields [9,10,15,16,19,36].

A set-valued function is a function with values in $\mathcal{K}^n$ or $\mathcal{K}^n_C$, both defined below, the space of all nonempty compact subsets of $\mathbb{R}^n$ (the space of all nonempty compact convex subsets of $\mathbb{R}^n$). $\mathcal{K}^n$ and $\mathcal{K}^n_C$ are not linear spaces since they do not contain inverse elements for the addition, and therefore subtraction is not well defined. As a consequence, alternative

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formulations for subtraction have been suggested [24,44,31]. One of these alternatives is the H-difference [8,24]. Another one is based on embedding the space $\mathcal{K}^n$ in a linear space [39] that also comes with a norm.

Banks et. al. [8] and Hukuhara [24] introduced the concept of H-differentiability for set-valued functions by using the H-difference. In order to overcome some shortcomings of this approach, other types of derivatives for set-valued functions have been explored (see for example [12,26]).

New alternatives of derivatives for set-valued functions and its applications for solving fuzzy differential equations have been introduced by Bede et al. [9,10] and by Chalco-Cano et al. [14,15] (also see [17]). The strongly generalized differentiability (G-differentiability, for short) [9] was defined by considering lateral H-derivatives (four cases). The concept of differentiability in [15] is a particular case of G-differentiability, since it only considers two cases. Recently, Stefanini and Bede [43] have introduced the concept of gH-differentiability, which is based on a generalization of the H-difference between two intervals [43,44]. In the same paper, the authors also studied relationships between the G-differentiability and the gH-differentiability, showing the equivalence between these two concepts when the set of switching points of the interval-valued function is finite. Some recent papers containing applications of this differentiability concept are [1,2]. For the interval-valued case, the gH-differentiability coincides with the differentiability concept introduced in Markov [30].

Radström's embedding theorem [39], on the other hand, tell us that there is a real normed linear space $\mathcal{B}$ and an isometric mapping $\pi : \mathcal{K}^n \to \mathcal{B}$. $\mathcal{B}$ is a space of equivalence classes (see [8,37]). Then, taking advantage of this embedding theorem, a set-valued function F is said to be π-differentiable at t_0 if $\pi \circ F$ is differentiable at t_0. Some properties of this derivative and its connection with other derivatives for set-valued functions can be found in [8].

This paper studies relationships between G-differentiability, gH-differentiability, Markov-differentiability, and π-differentiability for interval-valued functions. With this aim, the paper is organized as follows. In Section 2 we present the basic notations. In Section 3 we study the G-derivative for interval-valued mappings. In Section 4 we see the equivalence among gH-differentiability, Markov differentiability, and π-differentiability and study their relationship to G-differentiability. In Section 5 we present an application to interval differential equations. Some illustrative examples are also presented. In the last Section we conclude and outline some subjects that, in our opinion, deserve further study.

2. Basic concepts

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. We denote by $\mathcal{K}^n$ the family of all nonempty compact subsets of $\mathbb{R}^n$ and $\mathcal{K}^n_C$ the family of all $A \in \mathcal{K}^n$ such that A is a convex set, that is,

$$\mathcal{K}^n = \{A \subset \mathbb{R}^n / A \neq \emptyset \text{ is compact}\} \quad \text{and} \quad \mathcal{K}^n_C = \{A \in \mathcal{K}^n / A \text{ is convex}\}.$$

The Pompeiu–Hausdorff metric H on $\mathcal{K}^n$ is defined by

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(x,A) = \inf_{a\in A}\|x - a\|$. It is well known that $(\mathcal{K}^n, H)$ is a complete metric space and that $\mathcal{K}^n_C$ is a closed subspace of $\mathcal{K}^n$ (see [3,4]). The Minkowski sum and scalar multiplication are defined by

$$A + B = \{a + b / a \in A, b \in B\} \quad \text{and} \quad \lambda A = \{\lambda a / a \in A\}. \tag{1}$$

The spaces $\mathcal{K}^n$ and $\mathcal{K}^n_C$ are not linear spaces since they do not contain inverse elements and therefore subtraction is not well defined [3,6].

The Minkowski difference is given by $A - B = A + (-1)B$ (1). In general, $A - A \neq 0$, where here 0 denotes the set $\{0\}$. It is well known that if $A = B + C$, then the Hukuhara difference of A and B, denoted by $A -_H B$, exists and it is equal to C (see [8,19,37,43]). The Hukuhara difference of A and B is also called the geometrical difference between the sets A and B [46]. The authors of [8] introduce subtraction in $\mathcal{K}^n_C$ by using Radström's embedding theorem [39], which tell us that there is a real normed linear space $\mathcal{B}$ and an isometry $\pi : \mathcal{K}^n_C \to \mathcal{B}$ such that $\pi(\mathcal{K}^n_C)$ is a convex cone in $\mathcal{B}$. To construct $\mathcal{B}$, let us consider the following equivalence relation in the space $\mathcal{K}^n_C \times \mathcal{K}^n_C$ (see [8,37])

$$(A,B) \sim (C,D) \text{ if and only if } A + D = B + C.$$

If $\langle A,B\rangle$ is the equivalence class of the pair (A,B), then $\mathcal{B}$ is the quotient space $\mathcal{K}^n_C \times \mathcal{K}^n_C / \sim$. Now, in $\mathcal{B}$ we define the operations of addition and scalar multiplication by

$$\langle A,B\rangle + \langle C,D\rangle = \langle A + C, B + D\rangle,$$

$$\lambda\langle A,B\rangle = \begin{cases} \langle \lambda A, \lambda B\rangle, & \lambda \geqslant 0, \\ \langle |\lambda| B, |\lambda| A\rangle, & \lambda < 0. \end{cases}$$

Given these definitions, $\mathcal{B}$ is a linear space.

The embedding $\pi : \mathcal{K}^n_C \to \mathcal{B}$ is defined as follows

$$\pi(A) = \langle A, 0\rangle, \quad A \in \mathcal{K}^n_C,$$

so that $\langle A, 0\rangle$ is the equivalence class $\{(A+D, D)/A, D \in \mathcal{K}_C^n\}$. The metric and the norm in $\mathcal{B}$ are defined by

$$\rho(\langle A,B\rangle,\langle C,D\rangle) = H(A+D, C+B)\|\langle A,B\rangle\| = \rho(\langle A,B\rangle,\langle 0,0\rangle).$$

If $A, B \in \mathcal{K}_C^n$, then the difference of the sets A and B is an element of $\mathcal{B}$ equal to $\langle A,B\rangle$. Since $\mathcal{B}$ is a linear space, this difference has all the properties of difference in linear spaces. However, in general, the difference of two sets in $\mathcal{K}_C^n$ is not necessarily an element of $\mathcal{K}_C^n$. On the other hand, if the Hukuhara difference is well defined for two sets A, B in $\mathcal{K}_C^n$, then

$$\langle A,B\rangle = \langle A -_H B, 0\rangle.$$

For our presentation, T represents an open interval of $\mathbb{R}$, that is, $T = (\tau_1, \tau_2),\ \tau_1,\ \tau_2 \in \mathbb{R}$. When $n = 1$, we simply denote $\mathcal{K}^1$ and $\mathcal{K}_C^1$ by $\mathcal{K}$ and $\mathcal{K}_C$, respectively.

3. Generalized derivative

The H-derivative (differentiability in the sense of Hukuhara) for set-valued functions was initially introduced in [24] and it is based on the H-difference of sets.

Definition 1 ([24,46]). Let $F : T \to \mathcal{K}^n$ be a set-valued function. We say that F is differentiable at $t_0 \in T$ if there exists an element $F'(t_0) \in \mathcal{K}^n$ such that the limits

$$\lim_{h\to 0^+} \frac{F(t_0+h) -_H F(t_0)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{F(t_0) -_H F(t_0-h)}{h}$$

exist and are equal to $F'(t_0)$.

Here the limits are taken in the metric space $(\mathcal{K}^n, H)$. Note that this definition of derivative is very restrictive. For example, if we consider a very simple set-valued function which should have a derivative, $F(t) = (1 - t^3)[-2,1]$, given that $F(0+h) -_H F(0) = (1 - h^3)[-2,1] -_H [-2,1]$, the H-difference $F(0+h) -_H F(0)$ does not exist as $h \to 0^+$. Therefore, the H-derivative of F does not exist at $t = 0$. In general, if $F(t) = C\cdot g(t)$ where C is an interval and $g : [a,b] \to \mathbb{R}^+$ is a function with $g'(t_0) < 0$, F is not differentiable at t_0 ([9,15]). To avoid this difficulty, the authors of [9] introduced a more general definition of derivative for set-valued functions enlarging the class of differentiable set-valued functions by considering lateral H-derivatives. A particular case was considered in [15].

Definition 2. Let $F : T \to \mathcal{K}_C$ be a set-valued function and let $t_0 \in T$. We say that F is strongly generalized differentiability (G-differentiable) at t_0 if

(*i*) there is an element $F'(t_0) \in \mathcal{K}_C$ such that there exist the H-differences and the limits

$$\lim_{h\to 0^+} \frac{F(t_0+h) -_H F(t_0)}{h} = \lim_{h\to 0^+} \frac{F(t_0) -_H F(t_0-h)}{h} = F'(t_0),$$

or

(*ii*) there is an element $F'(t_0) \in \mathcal{K}_C$ such that there exist the H-differences and the limits

$$\lim_{h\to 0^-} \frac{F(t_0+h) -_H F(t_0)}{h} = \lim_{h\to 0^-} \frac{F(t_0) -_H F(t_0-h)}{h} = F'(t_0),$$

or

(*iii*) there is an element $F'(t_0) \in \mathcal{K}_C$ such that there exist the H-differences and the limits

$$\lim_{h\to 0^+} \frac{F(t_0+h) -_H F(t_0)}{h} = \lim_{h\to 0^-} \frac{F(t_0+h) -_H F(t_0)}{h} = F'(t_0),$$

or

(*iv*) there is an element $F'(t_0) \in \mathcal{K}_C$ such that there exist the H-differences and the limits

$$\lim_{h\to 0^+} \frac{F(t_0) -_H F(t_0-h)}{h} = \lim_{h\to 0^-} \frac{F(t_0) -_H F(t_0-h)}{h} = F'(t_0).$$

We say that F is G-differentiable on T if F is G-differentiable at each point $t_0 \in T$.

The next result shows that if F is G-differentiable at $t_0 \in T$ in two or more of the cases in Definition 2, then F is G-differentiable at $t_0 \in T$ in all the cases. Thus, the definition of G-derivative is consistent.

Proposition 1. *Let $F : T \to \mathcal{K}_C$ be a set-valued function. If F is G-differentiable at $t_0 \in T$ in two or more of the cases in Definition 2, then $F'(t_0) = \{a\}$ for some $a \in \mathbb{R}$ and all limits (i) – (iv) exist and are equal.*

Proof. If F is simultaneously differentiable in the sense (i) and (ii), then for $h>0$ sufficiently small we have $F(t_0+h)=F(t_0)+A_1$, $F(t_0)=F(t_0-h)+A_2$, $F(t_0-h)=F(t_0)+B_1$ and $F(t_0)=F(t_0+h)+B_2$, with $A_1, A_2, B_1, B_2 \in \mathcal{K}_C$. Thus, $F(t_0)=F(t_0)+(A_2+B_1)$, i.e., $A_2+B_1=0$. This implies that either $A_2=B_1=\{0\}$ and then $F'(t_0)=\{0\}$, or $A_2=\{a\}$ and $B_1=\{-a\}$, for some $a\in\mathbb{R}-\{0\}$, and then $F'(t_0)\in\mathbb{R}$. Therefore, all limits in (i), (ii), (iii) and (iv) exist and they are equal. The same conclusion is reached for any other pair of cases. □

Example 1. Given any differentiable function $f: T\to\mathbb{R}$, the set-valued function $F(t)=\{f(t)\}$ is G-differentiable (in the form (i), (ii), (iii), (iv)) at each point $t\in T$ and $F'(t)=\{f'(t)\}$.

Example 2. Consider the set-valued function $F(t)=[-t^2,t^2]$. Since $F(0+h)=[-h^2,h^2]$, $F(0)=[0,0]=\{0\}$, there exists $F(0+h)-_HF(0)$ for any $h>0$ and $h<0$. Then F is G-differentiable at $t=0$ in the sense of Definition 2 (iii) and $F'(0)=\{0\}$. Moreover, the difference $F(0)-_HF(0+h)$ does not exist for any $h\in\mathbb{R}$, and thus F is not G-differentiable at $t=0$ in the sense of Definition 2 (i), (ii), (iv). On the other hand, F is G-differentiable at $t\in(0,\infty)$ in the sense of Definition 2 (i) and $F'(t)=[-2t,2t]$. But F is not G-differentiable in the sense of Definition 2 (ii), (iii), (iv) for $t\in(0,\infty)$. Also, F is differentiable at $t\in(-\infty,0)$ in the sense of Definition 2 (ii) and $F'(t)=[-2t,2t]$. But F is not differentiable in the sense of Definition 2 (i), (iii), (iv) for $t\in(-\infty,0)$.

Proposition 2. *Let $F: T\to\mathcal{K}_C$ be a set-valued function. If F is G-differentiable on T in the sense of Definition 2 (iii) (or (iv)), then $F'(t)=\{a_0\}$ for some $a_0\in\mathbb{R}$.*

Proof. The proof of this result is similar to that of Theorem 7 in [9]. □

Let $f: T\to\mathbb{R}$ be a real function and let $t_0\in T$. Let $f'_-(t_0)$ and $f'_+(t_0)$ denote the lateral derivatives of f at t_0,

$$f'_-(t_0)=\lim_{h\to 0^-}\frac{f(t_0+h)-f(t_0)}{h},\quad f'_+(t_0)=\lim_{h\to 0^+}\frac{f(t_0+h)-f(t_0)}{h}.$$

Theorem 1. *Let $F: T\to\mathcal{K}_C$ be a set-valued function such that $F(t)=[f(t),g(t)]$. If F is G-differentiable at $t_0\in T$, then*

(a) *If F is G-differentiable at $t_0\in T$ in the first form (i) then f and g are differentiable functions at t_0 and*

$$F'(t_0)=[f'(t_0),g'(t_0)]$$

(b) *If F is G-differentiable at $t_0\in T$ in the second form (ii) then f and g are differentiable functions at t_0 and*

$$F'(t_0)=[g'(t_0),f'(t_0)]$$

(c) *If F is G-differentiable at $t_0\in T$ in the third form (iii) then the lateral derivatives of f and g exist and satisfy*

$$F'(t_0)=\left[f'_+(t_0),g'_+(t_0)\right]=\left[g'_-(t_0),f'_-(t_0)\right]$$

(d) *If F is G-differentiable at $t_0\in T$ in the fourth form (iv) then the lateral derivatives of f and g exist and satisfy*

$$F'(t_0)=\left[g'_+(t_0),f'_+(t_0)\right]=\left[f'_-(t_0),g'_-(t_0)\right]$$

Proof

(a) Suppose that F is G-differentiable at t_0 in the first form (i). Then, for $h>0$ sufficiently small we have

$$F(t_0+h)-_HF(t_0)=[f(t_0+h)-f(t_0),g(t_0+h)-g(t_0)]$$

and, because h is a positive real number, multiplying by $1/h$ we get

$$\frac{F(t_0+h)-_HF(t_0)}{h}=\frac{1}{h}[f(t_0+h)-f(t_0),g(t_0+h)-g(t_0)]=\left[\frac{f(t_0+h)-f(t_0)}{h},\frac{g(t_0+h)-g(t_0)}{h}\right].$$

Passing to the limit as $h\to 0^+$, it follows that the lateral derivatives $f'_+(t_0)$ and $g'_+(t_0)$ exist and satisfy

$$\lim_{h\to 0^+}\frac{F(t_0+h)-_HF(t_0)}{h}=\left[f'_+(t_0),g'_+(t_0)\right].$$

Similarly,

$$\frac{F(t_0)-_HF(t_0-h)}{h}=\left[\frac{f(t_0)-f(t_0-h)}{h},\frac{g(t_0)-g(t_0-h)}{h}\right]=\left[\frac{f(t_0-h)-f(t_0)}{-h},\frac{g(t_0-h)-g(t_0)}{-h}\right].$$

Passing to the limit as $h \to 0^+$, it follows that the lateral derivatives $f'_-(t_0)$ and $g'_-(t_0)$ exist and satisfy

$$\lim_{h\to 0^+} \frac{F(t_0) -_H F(t_0-h)}{h} = [f'_-(t_0), g'_-(t_0)].$$

Thus, f and g are differentiable at t_0 and $F'(t_0) = [f'(t_0), g'(t_0)]$.

(b) Suppose that F is G-differentiable at t_0 in the sense (ii). Then, for $h < 0$ sufficiently small (in absolute value) we have

$$\frac{F(t_0+h) -_H F(t_0)}{h} = \left[\frac{g(t_0+h)-g(t_0)}{h}, \frac{f(t_0+h)-f(t_0)}{h}\right].$$

Thus, the lateral derivatives $f'_-(t_0)$ and $g'_-(t_0)$ exist and satisfy

$$\lim_{h\to 0^-} \frac{F(t_0+h) -_H F(t_0)}{h} = [g'_-(t_0), f'_-(t_0)].$$

Similarly,

$$\frac{F(t_0) -_H F(t_0-h)}{h} = \left[\frac{g(t_0-h)-g(t_0)}{-h}, \frac{f(t_0-h)-f(t_0)}{-h}\right].$$

The lateral derivatives $f'_+(t_0)$ and $g'_+(t_0)$ exist and satisfy

$$\lim_{h\to 0^-} \frac{F(t_0) -_H F(t_0-h)}{h} = [g'_+(t_0), f'_+(t_0)].$$

Therefore, f and g are differentiable at t_0 and $F'(t_0) = [g'(t_0), f'(t_0)]$.

(c) Suppose that F is G-differentiable at t_0 in the third form (iii). Then, for $h > 0$ sufficiently small we have

$$\frac{F(t_0+h) -_H F(t_0)}{h} = \left[\frac{f(t_0+h)-f(t_0)}{h}, \frac{g(t_0+h)-g(t_0)}{h}\right].$$

Passing to the limit as $h \to 0^+$, it follows that the lateral derivatives $f'_+(t_0)$ and $g'_+(t_0)$ exist and satisfy $F'(t_0) = [f'_+(t_0), g'_+(t_0)]$. On the other hand, for $h < 0$ sufficiently small (in absolute value) we have

$$\frac{F(t_0+h) -_H F(t_0)}{h} = \left[\frac{g(t_0+h)-g(t_0)}{h}, \frac{f(t_0+h)-f(t_0)}{h}\right].$$

Passing to the limit as $h \to 0^-$, it follows that the lateral derivatives $f'_-(t_0)$ and $g'_-(t_0)$ exist and satisfy $F'(t_0) = [g'_-(t_0), f'_-(t_0)]$. Therefore, ($c$) is shown.

(d) Suppose that F is G-differentiable at t_0 in the fourth form (iv). Then, for $h > 0$ sufficiently small we have

$$\frac{F(t_0) -_H F(t_0-h)}{h} = \left[\frac{f(t_0)-f(t_0-h)}{h}, \frac{g(t_0)-g(t_0-h)}{h}\right].$$

Passing to the limit as $h \to 0^+$, if follows that the lateral derivatives $f'_-(t_0)$ and $g'_-(t_0)$ exist and satisfy $F'(t_0) = [f'_-(t_0), g'_-(t_0)]$. On the other hand, for $h < 0$ sufficiently small (in absolute value) we have

$$\frac{F(t_0) -_H F(t_0-h)}{h} = \left[\frac{g(t_0)-g(t_0-h)}{h}, \frac{f(t_0)-f(t_0-h)}{h}\right].$$

Passing to the limit as $h \to 0^-$, it follows that the lateral derivatives $f'_+(t_0)$ and $g'_+(t_0)$ exist and satisfy $F'(t_0) = [g'_+(t_0), f'_+(t_0)]$. Therefore, ($d$) is shown. □

The following is an immediate consequence of Proposition 1 and Theorem 1.

Corollary 1. *Let $F : T \to \mathcal{K}_C$ be a set-valued function such that $F(t) = [f(t), g(t)]$. If F is G-differentiable at $t_0 \in T$ in two or more of the cases in Definition 2, then f and g are differentiable at t_0, all the limits in $(i)-(iv)$ exist and they are equal to $F'(t_0) = \{f'(t_0)\} = \{g'(t_0)\}$.*

Example 3. We consider the interval-valued function $F_1 : \mathbb{R} \to \mathcal{K}_C$ defined by $F_1(t) = [-|t|, |t|]$. Then F_1 is G-differentiable at 0 in the form (iii) and $F'_1(t) = [-1, 1]$. Since $f'_{1-}(0) = 1 > g'_{1-}(0) = -1$ then F_1 is not G-differentiable at 0 in the form (iv).

Now, if we consider the set-valued function $F_2 : (-1, 1) \to \mathcal{K}_C$ defined by $F_2(t) = [-(t+1), t+1]$, for $-1 < t \leqslant 0$ and $F_2(t) = [-(1-t), 1-t]$ for $0 \leqslant t < 1$. Then F_2 is G-differentiable at 0 in the form (iv) and $F'_2(t) = [-1, 1]$. But F_2 is not G-differentiable at 0 in the form (iii) because $f'_{2-}(0) = -1 < g'_{2-}(0) = 1$.

The G-derivative of a set-valued function F in forms (iii) and (iv) is linked to switching points (definition given below). In fact, from Example 3, we have that F_1 is G-differentiable only in the form (ii) on $(\infty, 0)$ and it is G-differentiable only in the form (i) on $(0, \infty)$, whereas F_1 is G-differentiable at 0 only in the form (iv). So the point 0 is a switching point for differentiability of F_1. Analogously, the point 0 is a switching point for differentiability of F_2.

Definition 3 ([43]). Let $F : T \to \mathcal{K}_C$ be a set-valued function. A point $t_0 \in T$ is said to be a switching point for the differentiability of F, if in any neighborhood V of t_0 there exist points $t_1 < t_0 < t_2$ such that

(type I) F is differentiable at t_1 in the sense (i) of Definition 2 while it is not differentiable in the sense (ii) of Definition 2, and F is differentiable at t_2 in the sense (ii) of Definition 2 while it is not differentiable in the sense (i) or Definition 2, or

(type II) F is differentiable at t_1 in the sense (ii) of Definition 2 while it is not differentiable in the sense (i) of Definition 2, and F is differentiable at t_2 in the sense (i) of Definition 2 while it is not differentiable in the sense (ii) of Definition 2.

Proposition 3. *Let $F : T \to \mathcal{K}_C$ be a G-differentiable set-valued function on T.*

(a) *If $t_0 \in T$ is a switching point for the differentiability of F of type I, then F is G-differentiable at t_0 in the form (iv).*
(b) *If $t_0 \in T$ is a switching point for the differentiability of F of type II, then F is G-differentiable at t_0 in the form (iii).*

Proof. For this proof, we will assume that $F(t) = [f(t), g(t)]$.

(a) If $t_0 \in T$ is a switching point for the differentiability of F type I, then the following limits exist

$$\lim_{h \to 0^+} \frac{F(t_0) -_H F(t_0 - h)}{h} = \left[f'_-(t_0), g'_-(t_0)\right], \tag{2}$$

$$\lim_{h \to 0^-} \frac{F(t_0) -_H F(t_0 - h)}{h} = \left[g'_+(t_0), f'_+(t_0)\right]. \tag{3}$$

Because F is G-differentiable at t_0 and the limit in (2) exists, it follows that F is G-differentiable at t_0 in the sense of Definition 2 (i) or (iv). Moreover, because F is G-differentiable at t_0 and the limit in (3) exists, it follows that F is G-differentiable at t_0 in the sense of Definition 2 (ii) or (iv). Thus, F is G-differentiable at t_0 in the sense of Definition 2 (iv).

(b) The proof of this part follows the same steps as those of part (a), so we omit it. □

From Theorem 1, it follows that if a set-valued function G-differentiable at a point t_0, then all lateral derivatives of the extreme functions exist at t_0. To ensure that the G-differentiability of a set-valued function at a point t_0 implies the differentiability of the extreme functions at t_0, we need to assume an additional condition, which is stated below.

Condition C1: If F is G-differentiable at $t_0 \in T$ in the third form (iii) or in the fourth form (iv), then $F'(t_0) \in \mathbb{R}$, that is, there exists $a_0 \in \mathbb{R}$ such that $F'(t_0) = \{a_0\}$.

Corollary 2. *Let $F : T \to \mathcal{K}_C$ be a set-valued function satisfying condition C1. If F is G-differentiable at $t_0 \in T$, then f and g are differentiable functions at t_0 and*

$$F'(t_0) = [\min\{f'(t_0), g'(t_0)\}, \max\{f'(t_0), g'(t_0)\}].$$

Corollary 2 shows that the G-differentiability of F implies the differentiability of the extreme functions f and g, assuming that condition C1 holds. In general, the converse is not true, that is, the differentiability of the extreme functions f and g does not imply the differentiability of F. To prove it we need to assume the following additional condition.

Condition C2: The length function of F, $len(F(t)) = g(t) - f(t)$, has isolated critical points.

The next result shows that, by assuming that condition C2 is satisfied, the converse of Corollary 2 holds. After proving this result, we will give an example showing that condition C2 cannot be removed.

Theorem 2. *Let $F : T \to \mathcal{K}_C$ be a set-valued function such that $F(t) = [f(t), g(t)]$. If f and g are differentiable functions in a neighborhood V of $t_0 \in T$ and F satisfies condition C2 in V, then F is G-differentiable at t_0 and*

$$F'(t_0) = [\min\{f'(t_0), g'(t_0)\}, \max\{f'(t_0), g'(t_0)\}].$$

Proof. Since f and g are differentiable functions at t_0, there are three possibilities: (I) $f'(t_0) < g'(t_0)$ or (II) $f'(t_0) > g'(t_0)$ or (III) $f'(t_0) = g'(t_0)$. Next we separately consider each of them.

Case (I): since $f'(t_0) < g'(t_0)$, we have $(g - f)'(t_0) > 0$. Then, for $h > 0$ sufficiently small $(g - f)(t_0 + h) > (g - f)(t_0)$ and $(g - f)(t_0 - h) < (g - f)(t_0)$. Therefore, there exist $[f(t_0 + h), g(t_0 + h)] -_H [f(t_0), g(t_0)]$ and $[f(t_0), g(t_0)] -_H [f(t_0 - h), g(t_0 - h)]$. Since f and g are differentiable functions there exit the limits

$$\lim_{h\to 0^+}\frac{F(t_0+h)-_H F(t_0)}{h}=\lim_{h\to 0^+}\frac{F(t_0)-_H F(t_0-h)}{h}.$$

Thus, F is G-differentiable at t_0 in the sense of Definition 2 (i) and $F'(t_0) = [f'(t_0), g'(t_0)]$.

Case (II): since $f'(t_0) > g'(t_0)$, we have $(f-g)'(t_0) > 0$. Then, for $h < 0$ sufficiently small (in absolute value) $(f-g)(t_0+h) < (f-g)(t_0)$ and $(f-g)(t_0-h) > (f-g)(t_0)$. Therefore, there exist $[f(t_0+h), g(t_0+h)] -_H [f(t_0), g(t_0)]$ and $[f(t_0), g(t_0)] -_H [f(t_0-h), g(t_0-h)]$. Since f and g are differentiable functions there exist the limits of the Definition 2 (ii). Thus F is G-differentiable at t_0 in the sense of Definition 2 (ii) and $F'(t_0) = [g'(t_0), f'(t_0)]$.

Case (III): First, let us suppose that t_0 is an extremum point of $g-f$. If t_0 is a minimum point, then for $h \in (-\delta, \delta)$, with δ sufficiently small, we have $(g-f)(t_0+h) \geqslant (g-f)(t_0)$. Thus, for $h > 0$ there exists $[f(t_0+h), g(t_0+h)] -_H [f(t_0), g(t_0)]$ and for $h < 0$ there exists $[f(t_0+h), g(t_0+h)] -_H [f(t_0), g(t_0)]$. Taking on account the differentiability of f and g, F is G-differentiable at t_0 in the sense of Definition 2 (iii) and $F'(t_0) = \{f'(t_0)\} = \{g'(t_0)\}$. If t_0 is a maximum point, then following a similar reasoning we get that F is G-differentiable at t_0 in the sense of Definition 2 (iv).

Now, if $(g-f)'(t_0) = 0$ and t_0 is not an extremum point then, taking into account that $len(F(t))$ has isolated critical points, we conclude that there exists an interval $(t_0-\delta, t_0+\delta) \subset T$, such that $(g-f)'(t) \geqslant 0$ for all $t \in (t_0-\delta, t_0+\delta)$ or $(g-f)'(t) \leqslant 0$ for all $t \in (t_0-\delta, t_0+\delta)$. If $(g-f)'(t) \geqslant 0$ then, for $h > 0$ sufficiently small, $(g-f)(t_0+h) \geqslant (g-f)(t_0)$ and $(g-f)(t_0-h) \leqslant (g-f)(t_0)$. Therefore there exist $[f(t_0+h), g(t_0+h)] -_H [f(t_0), g(t_0)]$ and $[f(t_0), g(t_0)] -_H [f(t_0-h), g(t_0-h)]$.

Thus F is G -differentiable at t_0 in the sense of Definition 2 (i) and $F'(t_0) = \{f'(t_0)\} = \{g'(t_0)\}$. Now, if $(g-f)'(t) \leqslant 0$ for all $t \in (t_0-\delta, t_0+\delta)$ then, for $h < 0$ sufficiently small (in absolute value), $(g-f)(t_0+h) \leqslant (g-f)(t_0)$ and $(g-f)(t_0-h) \geqslant (g-f)(t_0)$. Therefore, there exist $[f(t_0+h), g(t_0+h)] -_H [f(t_0), g(t_0)]$ and $[f(t_0), g(t_0)] -_H [f(t_0-h), g(t_0-h)]$. Thus F is G-differentiable at t_0 in the sense of Definition 2 (ii) and $F'(t_0) = \{f'(t_0)\} = \{g'(t_0)\}$. □

The following example shows that condition C2 cannot be removed from Theorem 2.

Example 4. Let us consider the function $F : \mathbb{R} \to \mathcal{K}_C$,

$$F(x)=\begin{cases}[0, 1+x^3\sin\frac{1}{x}], & \text{if } x\neq 0,\\ [0,1], & \text{otherwise.}\end{cases}$$

The functions f and g are differentiable at 0, which is a critical point of $g-f$, but it is not an isolated critical point. Note that F is not G-differentiable at 0 since there does not exist δ such that $F(h) -_H F(0)$ or $F(-h) -_H F(0)$ exists for all $h \in (0, \delta)$.

The following important result is an immediate consequence of Theorem 2.

Corollary 3. *If $h : T \to \mathbb{R}$ is a differentiable function on T which has isolated critical points in T and $C \in \mathcal{K}_C$, then $F(t) = C{\cdot}h(t)$ is G-differentiable on T and $F'(t) = C{\cdot}h'(t)$.*

Note that the condition "h has isolated critical points in T" cannot be removed from Corollary 3, as we showed in Example 4. The result in Corollary 3 has been proved in [9] assuming a stronger condition on h.

4. Generalized Hukuhara, Markov and π-derivatives

The authors of [43] have recently introduced the concept of gH-differentiability which is based on the following generalization of the H-difference between two intervals.

Definition 4 ([43]). The generalized Hukuhara difference of two intervals, A and B, (gH-difference) is defined as follows

$$A\ominus_g B = C \Longleftrightarrow \begin{cases}(a), & A = B + C,\\ \text{or } (b), & B = A + (-1)C.\end{cases}$$

This difference has many interesting new properties, for example $A \ominus_g A = \{0\}$. Also, the gH-difference of two intervals $A = [a,b]$ and $B = [c,d]$ always exists and it is equal to (see Proposition 4 in [43])

$$A\ominus_{gH} B = [\min\{a-c, b-d\}, \max\{a-c, b-d\}].$$

Thus, for the interval case, the gH-difference of two intervals coincides with the difference defined in Markov [30].

We examine the π-difference of two intervals next. With this aim we first study the equivalence classes. Each equivalence class can be represented in the form $\langle [x, x+\delta], 0\rangle$, $\delta \geqslant 0$, or $\langle x, [0,\delta]\rangle$, $\delta > 0$. This is called the *canonical representation* (see [37]). In fact, an arbitrary element $\langle [a,b],[c,d]\rangle$ belongs to the first class if $b-a \leqslant d-c$ (with $x = a-c$ and $\delta = (b-a)-(d-c)$), and to the second class if $b-a < d-c$ (with $x = a-c$ and $\delta = (d-c)-(b-a)$). For the canonical representation of an equivalence class we will use the notation (x,δ), which implies that

$$(x,\delta) = \begin{cases} \langle [x, x+\delta], 0\rangle, & \text{if } \delta \geqslant 0, \\ \langle 0, \ [-x, -x-\delta]\rangle, & \text{if } \delta < 0. \end{cases}$$

Consequently, to each $(x,\delta) \in \mathcal{B}$, we can associate an interval $A \in \mathcal{K}_C$ as follows:

$$A = \begin{cases} [x, x+\delta], & \text{if } \delta \geqslant 0, \\ [x+\delta, x], & \text{if } \delta < 0. \end{cases} \tag{4}$$

The following result is proven in [37].

Theorem 3 ([37]). *The following results hold*

(a) $(x,\delta) = (y,\beta) \Leftrightarrow x = y,\ \delta = \beta$,
(b) $(x,\delta) + (y,\beta) = (x+y, \delta+\beta)$,
(c) $a(x,\delta) = (ax, a\delta)$ *for each* $a \in \mathbb{R}$.

Now, we are in a position to give the expression of the π-difference of two intervals:

$$[a,b] -_{\pi} [c,d] = \pi([a,b]) - \pi([c,d]) = \langle [a,b],[c,d]\rangle = (a-c, (b-a)-(d-c)).$$

We can associate with $[a,b] -_{\pi}[c,d]$ the interval (see (4))

$$C = \begin{cases} [a-c, b-d] \text{ if } (b-a)-(d-c) \geqslant 0 \\ [b-d, a-c] \text{ if } (b-a)-(d-c) < 0 \end{cases} = [\min\{a-c, b-d\}, \max\{a-c, b-d\}]. \tag{5}$$

Now, if we define the π-difference $[a,b] -_{\pi}[c,d]$ to be the interval (5) then, for $n = 1$, that is, for compact intervals, the *gH*-difference, the difference defined in Markov [30] and the π-difference of two intervals are the same concept.

Based on the *gH*-difference, the following definition of differentiability for interval-valued functions was introduced by [43].

Definition 5 ([43]). The *gH*-derivative of an interval-valued function $F : T \to \mathcal{K}_C$ at $t_0 \in T$ is defined as

$$F'(t_0) = \lim_{h \to 0} \frac{F(t_0+h) \ominus_g F(t_0)}{h}. \tag{6}$$

If $F'(t_0) \in \mathcal{K}_C$ satisfying (6) exists, we say that F is generalized Hukuhara differentiable (*gH*-differentiable) at t_0. We say that F is *gH*-differentiable on T if F is *gH*-differentiable at each point $t_0 \in T$.

In view of the equivalence of the *gH*-difference, difference defined by [30] and π-difference, the *gH*-differentiability, the differentiability concept introduced in [30] and the π-differentiability for interval-valued functions will also coincide.

The next result expresses the π-derivative (or *gH*-derivative or the derivative defined in [30]) in terms of the endpoints of the interval-valued function.

Theorem 4 ([30]). *Let* $F : T \to \mathcal{K}_C$ *be a set-valued function such that* $F(t) = [f(t), g(t)]$. *If f and g are differentiable functions on T, then F is* π*-differentiable on T and*

$$F'(t) = [\min\{f'(t), g'(t)\}, \max\{f'(t), g'(t)\}]. \tag{7}$$

The next result expresses the π-derivative in terms of the canonical representation of the equivalence classes which is very useful in some practical applications. It will also appear in Section 5.

Theorem 5 ([37]). *Let* $F : T \to \mathcal{K}_C$ *be a set-valued function such that* $F(t) = [f(t), g(t)]$. *If f and g are differentiable functions on T, then F is* π*-differentiable on T and*

$$D_{\pi}(x(t), \delta(t)) = (x'(t), \delta'(t)),$$

where $D_{\pi}(x(t), \delta(t))$ *is the* π*-derivative ([8]) of the pair* $(x(t), \delta(t))$.

It is interesting to observe that the converse of Theorem 4 (and thus Theorem 5) does not hold, that is, the π-differentiability of $F(t) = [f(t), g(t)]$ does not imply the differentiability of f and g. Next example shows this fact.

Example 5. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = |t|$. We have g is not differentiable at $t_0 = 0$. Now, consider the interval $C = [-1, 1]$ and define the interval-valued function F by $F(t) = C \cdot g(t)$. Then

$$\frac{F(0+h)-_{\pi}F(0)}{h}=\frac{1}{h}\{[-|h|,|h|]-_{\pi}[0,0]\}=\frac{1}{h}[-|h|,|h|]=\frac{|h|}{h}[-1,1]=[-1,1].$$

So, $F'(0) = [-1,1]$.

In general, we have the following result.

Theorem 6. *Let $F : T \to \mathcal{K}_C$ be a set-valued function such that $F(t) = [f(t), g(t)]$. Then, F is π-differentiable at $t_0 \in T$ if and only if one of the following cases holds*

(*a*) *f and g are differentiable at t_0;*
(*b*) $f'_-(t_0),\ f'_+(t_0),\ g'_-(t_0)$ *and* $g'_+(t_0)$ *exist and satisfy* $f'_-(t_0)=g'_+(t_0)$ *and* $f'_+(t_0)=g'_-(t_0)$.

Proof. That the π-differentiability of F at t_0 implies (a) or (b) has been proved by [30]. Next we prove the converse. If f and g are differentiable at t_0, from Theorem 4 F is π-differentiable at $t_0 \in T$. Suppose that (b) is valid. Then we consider three cases: (i) $f'_+(t_0) < g'_+(t_0)$, (ii) $f'_+(t_0) = g'_+(t_0)$ and (iii) $f'_+(t_0) > g'_+(t_0)$ and study each case separately.

Case (i): since $f'_+(t_0) < g'_+(t_0)$ then

$$\lim_{h\to 0^+}\frac{F(t_0+h)-_{\pi}F(t_0)}{h}=\lim_{h\to 0^+}\left[\min\left\{\frac{f(t_0+h)-f(t_0)}{h},\frac{g(t_0+h)-g(t_0)}{h}\right\},\right.$$

$$\left.\max\left\{\frac{f(t_0+h)-f(t_0)}{h},\frac{g(t_0+h)-g(t_0)}{h}\right\}\right]=\lim_{h\to 0^+}\left[\frac{f(t_0+h)-f(t_0)}{h},\frac{g(t_0+h)-g(t_0)}{h}\right]=\left[f'_+(t_0),g'_+(t_0)\right].$$

Since $f'_-(t_0)=g'_+(t_0)$ and $f'_+(t_0)=g'_-(t_0)$, then $g'_-(t_0)<f'_-(t_0)$. So,

$$\lim_{h\to 0^-}\frac{F(t_0+h)-_{\pi}F(t_0)}{h}=\lim_{h\to 0^-}\left[\frac{g(t_0+h)-g(t_0)}{h},\frac{f(t_0+h)-f(t_0)}{h}\right]=\left[g'_-(t_0),f'_-(t_0)\right].$$

Moreover, since $f'_-(t_0)=g'_+(t_0)$ and $f'_+(t_0)=g'_-(t_0)$, we have

$$\lim_{h\to 0^+}\frac{F(t_0+h)-_{\pi}F(t_0)}{h}=\lim_{h\to 0^-}\frac{F(t_0+h)-_{\pi}F(t_0)}{h}.$$

Therefore, F is π-differentiable at t_0.

Case (ii): Since $f'_+(t_0)=g'_+(t_0),\ f'_-(t_0)=g'_+(t_0)$ and $f'_+(t_0)=g'_-(t_0)$, we have that f and g are differentiable at t_0, then by Theorem 4, F is π-differentiable at $t_0 \in T$.

Case (iii): For this case, we proceed analogously to Case (i). □

The following result relates G-differentiability and π-differentiability.

Theorem 7. *Let $F : T \to \mathcal{K}_C$ be a set-valued function. If F is G-differentiable at $t_0 \in T$ then F is also π-differentiable at t_0.*

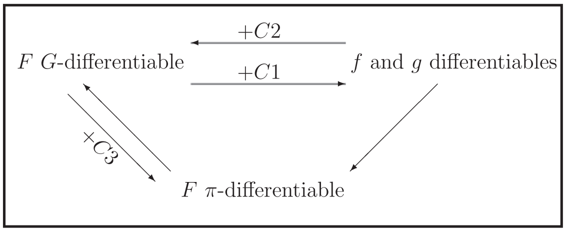
Proof. If F is G-differentiable at t_0 in the first form (i) or in the second form (ii), from Theorem 1, f and g are differentiable at t_0. Thus, from Theorem 4, F is π-differentiable at t_0. If F is G-differentiable in the third form (iii) or in the fourth form (iv), from Theorem 1, the functions f and g satisfy the case (b) of the Theorem 6. Therefore, F is π-differentiable at t_0 and the proof is completed. □

Example 4 shows that the converse of Theorem 7 is not valid. The equivalence between the G-differentiability and the gH-differentiability has been proved in [43] by assuming the following additional condition.

Condition C3: The set of switching points is finite.

Theorem 8 ([43]). *Let $F : T \to \mathcal{K}_C$ be a set-valued function. Then F is G-differentiable on T if and only if F is π-differentiable on T and condition C3 holds.*

The following diagram summarizes the relationships between G-differentiability, π-differentiability and the differentiability of the extreme functions.



5. An application to interval differential equations

We consider the initial value problem

$$\begin{cases} X'(t) = F(t, X(t)), \\ X(0) = X_0, \end{cases} \tag{8}$$

where $F : [0,T] \times \mathcal{K}_C \to \mathcal{K}_C$ is a continuous function and X_0 is an interval. Note that there are different interpretations of the problem (8). For example, we can consider the H-derivative ([19,40–42]), the generalized derivative ([9,10,15–17]), or the gH-derivative ([43]). The problem (8) can also be seen as a family of differential inclusions [20,21,25].

This section considers the π-derivative for the problem (8). For this, we denote by $X'_\pi(t)$ the π-derivative of X at t. Then, we will study the following problem

$$\begin{cases} X'_\pi(t) = F(t, X(t)), \\ X(0) = X_0, \end{cases} \tag{9}$$

where $F : [0,T] \times \mathcal{K}_C \to \mathcal{K}_C$ is a continuous function and X_0 is an interval. A solution to (9) is a continuous function $X : T \to \mathcal{K}_C$ which verifies the Eq. (9) for each $t \in [0,T]$.

Taking into account Theorems 3 and 5, we obtain a useful procedure to solve the interval differential Eq. (9). In fact, denoting

$$X(t) = [f(t), g(t)], \quad X_0 = [x_0^L, x_0^U],$$

we obtain

$$\pi(X(t)) = (x(t), \delta(t)), \quad \pi(X_0) = (x^0, \delta^0),$$

where $x(t) = f(t),\ \delta(t) = g(t) - f(t),\ x^0 = x_0^L$ and $\delta_0 = x_0^U - x_0^L$. Let

$$F(t, X(t)) = [U(t, x(t), \delta(t)), V(t, x(t), \delta(t))].$$

Then, associated with the interval differential Eq. (9), we have the following two ordinary (systems of) differential equations:

$$\text{Case(I)} \begin{cases} x'(t) = U(t, x(t), \delta(t)), \\ \delta'(t) = V(t, x(t), \delta(t)) - U(t, x(t), \delta(t)), \\ x(0) = x^0, \quad \delta(0) = \delta^0, \end{cases}$$

and

$$\text{Case(II)} \begin{cases} x'(t) = V(t, x(t), \delta(t)), \\ \delta'(t) = U(t, x(t), \delta(t)) - V(t, x(t), \delta(t)), \\ x(0) = x^0, \quad \delta(0) = \delta^0. \end{cases}$$

whose solutions give the two solutions to our interval differential Eq. (9).

Example 6. Let us consider the Malthusian problem,

$$\begin{cases} X'_\pi(t) = -\lambda X(t), \\ X(0) = X_0, \end{cases} \tag{10}$$

where $\lambda > 0$ and the interval initial condition $X_0 = [-a, a]$, $a > 0$.

Case (I) The first solution to (10) is obtained by solving the differential system

$$\begin{cases} x'(t) = -\lambda x(t) - \lambda\delta(t), & x(0) = -a, \\ \delta'(t) = \lambda\delta(t), & \delta(0) = 2a. \end{cases}$$

The solution to the above system is

$$x(t) = -ae^{\lambda t}, \quad \delta(t) = 2ae^{\lambda t}.$$

Note that $\delta(t) \geqslant 0$, for all $t > 0$. Thus, there exists a solution $X(t)$ to problem (10) for all $t > 0$, such that

$$X(t) = [x(t), x(t) + \delta(t)] = \left[-ae^{\lambda t}, ae^{\lambda t}\right] = X(0)e^{\lambda t}.$$

Case (II) The second solution to (10) is obtained by solving the differential system

$$\begin{cases} x'(t) = -\lambda x(t), & x(0) = -a, \\ \delta'(t) = -\lambda \delta(t), & \delta(0) = 2a. \end{cases}$$

The solution to this system is

$$x(t) = -ae^{-\lambda t}, \quad \delta(t) = 2ae^{-\lambda t}.$$

Note that $\delta(t) \geqslant 0$ for all $t > 0$. Thus, there exists a solution $X(t)$ to (10) for all $t > 0$, such that

$$X(t) = [x(t), x(t) + \delta(t)] = [-ae^{-\lambda t}, ae^{-\lambda t}] = X(0)e^{-\lambda t}.$$

5.1. π-derivative for fuzzy functions

Recently in [18] the authors have introduced a concept of π-derivative for fuzzy functions as a generalization of π-derivative for interval valued functions, similar to that introduced by Seikkala [45]. As an application, they have also studied fuzzy differential equation with π-derivative. Next we sketch the ideas introduced in [18].

Let $\mathcal{F}$ be the set of all fuzzy intervals with bounded α-level intervals. This means that if $u \in \mathcal{F}$ then the α-level set is a closed bounded interval which we denote by $[u]^\alpha$, for all $\alpha \in [0,1]$. A mapping $X : T \to \mathcal{F}$ is called a fuzzy function. Let us denote

$$[X(t)]^\alpha = X_\alpha(t) = [f_\alpha(t), g_\alpha(t)], \quad t \in T, \ 0 \leqslant \alpha \leqslant 1.$$

Note that X_α is a set-valued function for each $\alpha \in [0,1]$. The π-derivative $X'_\pi(t)$ of a fuzzy function X is defined by

$$[X'_\pi(t)]^\alpha = X'_{\alpha\pi}(t), \quad 0 \leqslant \alpha \leqslant 1, \tag{11}$$

provided that is equation defines a fuzzy interval $X'_\pi(t) \in \mathcal{F}$. Note that, if the family $\{X'_{\alpha\pi}(t)\}$ satisfies the conditions of the Representation Theorem [19], then there exists the π-derivative $X'_\pi(t)$ of the fuzzy function X.

Example 7. Consider the fuzzy function $X : (0, +\infty) \to \mathcal{F}_C$ defined by

$$X(t)(s) = \begin{cases} \frac{s}{t} + 1, & \text{if } -t \leqslant s \leqslant 0, \\ -\frac{s}{t^2} + 1, & \text{if } 0 \leqslant s \leqslant t^2, \\ 0, & \text{if } s \notin [-t, t^2]. \end{cases}$$

Then, for all $\alpha \in [0,1]$ we have

$$[X(t)]^\alpha = [f_\alpha(t), g_\alpha(t)] = \left[(\alpha - 1)t, (1 - \alpha)t^2\right],$$

and

$$X'_{\alpha\pi}(t) = [(\alpha - 1), 2(1 - \alpha)t].$$

Now, the family $\{X'_{\alpha\pi}(t)\}_{\alpha \in [0,1]}$ satisfies the conditions in the Representation Theorem [19] for each $t > 0$. Therefore, there exists the π-derivative $X'_\pi(t)$ of the fuzzy function X for each $t > 0$ and

$$\left[X'_\pi(t)\right]^\alpha = [(\alpha - 1), 2(1 - \alpha)t].$$

6. Conclusions and future research

This article has studied relationships between the concepts of G-derivative, gH-derivative, Markov-derivative and π-derivative for interval-valued functions. For the interval case, some of the results have been obtained separately by [9,43]. Our study can be applied directly to intervals and to the extensions of these derivatives to the fuzzy case which was developed in Sub Section 5.1, (also see [18]).

An interesting line of work is the study of fuzzy differential equations with π-derivative and the analysis of the relationships between this interpretation and those already existing in the literature. For instance, developing relationships between what we developed and fuzzy differential equations with H-derivative [11,35,40,42], fuzzy differential inclusions [20,21,25] and fuzzy differential equations using Zadeh's Extension Principle [16,22,27,34], among others.

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