Some remarks on fuzzy differential equations via differential inclusions

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Abstract

In this paper we discuss the formulation and procedure for solving fuzzy differential equations via differential inclusions. We give several examples showing the correct and incorrect procedure for solving fuzzy differential equations. We show the connection between fuzzy differential equations and fuzzy differential inclusions. Finally, we give some remarks on numerical algorithms for solving fuzzy differential equations via differential inclusions.

Keywords: Fuzzy differential equations; Fuzzy differential inclusions; Differential inclusions

1. Introduction

We consider the following problem

\[ X'(t) = F(t, X(t)), \quad X(0) = X_0, \]  

(1)

where \( F : [0, T] \times F^n_C \to F^n_C \) is a continuous function, \( F^n_C \) is the space of all convex and compact fuzzy sets on \( \mathbb{R}^n \) and \( X_0 \in F^n_C \).

Some questions appear naturally associated with problem (1). What is an interpretation of the problem (1)? What does it mean to be a solution of (1)?

A first approach is to consider the derivative \( X' \) of the fuzzy function \( X : [0, T] \to F^n_C \), for instance Hukuhara derivative [36,37,56,62,64] or generalized Hukuhara derivative [9,10,13,17,69,70]. Thus, a solution of (1) is a differentiable fuzzy function \( X \) that satisfies (1).

Let us consider the following dynamical control system:

\[ x' = f(t, x, w), \quad x(0) = x_0, \quad w \in U \subset \mathbb{R}^m \]  

(2)

where \( f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is a continuous function. It is well-known that (2) is an important model for a wide class of real problems. However, in some cases, these equations are restrictive in their ability to describe phenomena.

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For example, in mathematical models that describe biological phenomena, the parameters and initial conditions usually are inherently uncertain and consequently the variables will also possess uncertainty, see for example [6,7,30–32,45,57]. How can we translate (2) to fuzzy context if the parameters w, initial conditions $x_0$ and variables are fuzzy intervals? If we have a function whose parameters are fuzzy intervals as the right-hand side of (2), what is the appropriate extension within fuzzy theory? To obtain a fuzzy differential equation from such a problem (2), we consider

$$X'(t) = \hat{f}(t, X, W), \quad X(0) = X_0$$

where $\hat{f} : [0, T] \times F_C^R \times F_C^m \rightarrow F_C^m$ was obtained from a continuous function $f : [0, T] \times R^n \times R^m \rightarrow R^n$ by applying Zadeh’s extension principle, $X_0 \in F_C^R$ and $W \in F_C(U)$.

Note that, in some cases, there is another possible extension to transform problem (2) into a fuzzy context. It is

$$X'(t) = f_S(t, X, W), \quad X(0) = X_0$$

where $f_S : [0, T] \times F_C^m \times F_C^m \rightarrow F_C^m$ was obtained from a continuous function $f : [0, T] \times R^n \times R^m \rightarrow R^n$ by applying fuzzy standard interval arithmetic (see Section 2).

We consider that the most adequate extension of (2) to a fuzzy context is problem (3) since Zadeh’s extension principle preserves the main properties and characteristics of the original function whereas fuzzy standard interval arithmetic generates overestimation, so that it does not preserve the main properties and characteristic of the original function (for more details see [18,31,32]). However, given that the calculus of $f$ is very complex, in some cases suitable approximations by $f_S$ are most often used [10,18,31]. In addition, there are several approaches to avoid overestimation of fuzzy standard interval arithmetic with respect to arithmetic via Zadeh’s extension principle [44,49,53,60]. Consequently, problems (3) and (4) are particularly important subclasses of (1) considering $F(t, X) = \hat{f}(t, X, W)$ and $F(t, X) = f_S(t, X, W)$, respectively.

Now, for the problem (3), there is another interpretation besides the first approach using derivative of a fuzzy function. We obtain a fuzzy solution $X : [0, T] \rightarrow F_C^n$ of (3) by applying, for each $t \in [0, T]$ fixed, Zadeh’s extension principle to deterministic solution $\phi_t : \Omega \times U \rightarrow \mathbb{R}^n$ of (2), with $[X_0]^0 \subset \Omega \subset \mathbb{R}^n$, i.e. $X(t) = \hat{\phi}_t(X_0, W)$, for each $t \in [0, T]$. Note that in this approach, $X'$ in (3) does not denote any derivative (it is symbolic) and the fuzzy solution $X$ is obtained directly by applying Zadeh’s extension principle to the deterministic solution. For more details see [14,35,43,51,57].

Another approach to the interpretation of (1) is based on a family of differential inclusions [3–5,8,21–23,30,33,34,50,67]. Fuzzy differential equations via differential inclusions, differently from fuzzy differential equations involving the generalized Hukuhara derivative, allow us to characterize the main properties of ordinary differential equations in a natural way, such as periodicity, stability, bifurcation, among others (15,21–23,46). But, there is some criticism of this approach since one does not have the derivative of a fuzzy function. Fuzzy differential equations are directly interpreted with the help of differential inclusions without having a derivative of fuzzy functions. However, the connection between these three approaches with some conditions on $F$ has been shown in [14]. Also, the connection between fuzzy differential equations via fuzzy derivative and via differential inclusions has been shown in [37] for a particular class of fuzzy functions $F$.

This paper presents some remarks about fuzzy differential equations via differential inclusions. We show that this approach cannot be used for an arbitrary continuous fuzzy function $F$ in (1), as was used in [3,21,23,50]. That is, not all fuzzy differential equations where $F$ is a continuous function can be written as a family of differential inclusions. This approach is valid when we consider that $F$ is obtained from a continuous function $f : [0, T] \times R^n \rightarrow R^n$ by applying Zadeh’s extension principle, i.e. this approach is valid for (3).

We give some remarks about the correct procedure for solving (3) via differential inclusions. In particular, we show that with some conditions on $f$ solving (3) is equivalent to solving a family of dynamical systems controlled by parameters. We give several examples for illustrating this procedure and we correct some examples which were previously given in [4,21,22,46]. Also, we show the connection between fuzzy differential equations via differential inclusions and fuzzy differential inclusions introduced by H"{u}llermeier in [33,34].

One could expect that a solution algorithm for obtaining a fuzzy solution of (3) via differential inclusions might be obtained as some extension of known algorithms for ordinary differential equations. Unfortunately this is not the case [58,65,66]. We present some remarks and comments on numerical methods for solving fuzzy differential equations via differential inclusions.
2. Notation, the space of fuzzy sets and fuzzy arithmetic

We denote by $K^n$ the family of all nonempty compact subsets of $n$-dimensional space $\mathbb{R}^n$ and by $K^n_C$ the family of all nonempty compact and convex subsets of $\mathbb{R}^n$, i.e. $K^n = \{ A \subseteq \mathbb{R}^n | A$ is nonempty and compact $\}$, $K^n_C = \{ A \in K^n | A$ is convex $\}$.

If $A, B \in K^n$ and $\lambda \in \mathbb{R}$, then the operations of sum and multiplication by a scalar are defined as

$$A + B = \{a + b | a \in A, b \in B\}, \quad \lambda A = \{\lambda a | a \in A\}. \tag{5}$$

On the space of all compact and convex intervals $K_C$, we have the standard interval arithmetic which is due Moore [52]. He states that, given two intervals $A$ and $B$,

$$A \ast B = \{c | c = a \ast b, a \in A, b \in B, \ast \in \{+,-,\times,\div\}\}, \tag{6}$$

and the multiplication of a real number $\lambda$ by an interval $A$ is given by (5). Note that the sum given by (5) and the sum given by (6) on the space of intervals $K_C$ are equivalent.

Recall that a fuzzy set $u$ on a universe set $X$ is a mapping $u : X \to [0, 1]$. We think of $u$ as assigning to each element $x \in X$ a degree of membership, $0 \leq u(x) \leq 1$. If $u$ is a fuzzy set on $\mathbb{R}^n$, we define $[u]^x = \{x \in \mathbb{R}^n | u(x) \geq x\}$ the $x$-level of $u$, with $0 < x \leq 1$. For $x = 0$ the support of $u$ is defined as $[u]^0 = \text{supp}(u) = \{x \in \mathbb{R}^n | u(x) > 0\}$, where $A$ denotes the closure of $A \subseteq \mathbb{R}^n$.

A fuzzy set $u$ on $\mathbb{R}^n$ is called compact if $[u]^x \in K^n, \forall x \in [0, 1]$. Also, $u$ is called convex if $[u]^x$ is a convex set for all $x \in [0, 1]$. We denote by $F^n$ and $F^n_C$ the spaces of all compact fuzzy sets on $\mathbb{R}^n$ and all convex and compact fuzzy sets on $\mathbb{R}^n$, respectively, i.e. $F^n = \{ u : \mathbb{R} \to [0, 1] | [u]^x \in K^n, \forall x \in [0, 1]\}, F^n_C = \{ u : \mathbb{R} \to [0, 1] | [u]^x \in K^n_C, \forall x \in [0, 1]\}$.

Given a subset $\Omega \subseteq \mathbb{R}^n$ we denote by $F^n_C(\Omega)$ the family of all compact and convex fuzzy sets $u$ on $\mathbb{R}^n$ such that $\Omega$ contains the support of $u$, i.e. $F^n_C(\Omega) = \{ u \in F^n | [u]^0 \subset \Omega \}$.

The space $F^n_C(\Omega) \subseteq F^n_C$ is called the space of non-interacting fuzzy sets.

The following characterization of elements of $F^n$ is important in fuzzy theory. It is called theorem of representation.

**Theorem 1** (Negoita and Ralescu [54]). Let $Y_2 \subseteq \mathbb{R}^n | 0 \leq x \leq 1$ be a family of compact subsets satisfying the following:

- $Y_2 \in K^n$ for all $0 \leq x \leq 1$;
- $Y_\beta \subset Y_x$ for $0 \leq x \leq \beta \leq 1$;
- $Y_2 = \bigcap_{i=1} \{ Y_{x_i} \}$ for any nondecreasing sequence $x_i \to x$ in $[0, 1]$.

Then there is a unique fuzzy set $u \in F^n$ such that $[u]^x = Y_2$. In particular, if the $Y_2$ are also convex, then $u \in F^n_C$. Conversely, the level sets $[u]^x$ for any $u \in F^n$ satisfy these conditions.

If $u \in F^n_C$, then $u$ is called fuzzy interval and the $x$-level sets $[u]^x$ are nonempty compact intervals. We denote by $[u]^x = [u^x_-, u^x_+]$ the $x$-level of a fuzzy interval $u$. If the core of a fuzzy interval $u$ is a singleton, i.e. if $[u]^1 = \{a\}$, with $a \in \mathbb{R}$, then $u$ is called fuzzy number. A special fuzzy number is the triangular fuzzy number which is well-defined by three real numbers. So we denote a triangular fuzzy number by $u = (a, b, c)$ with $a, b, c \in \mathbb{R}$ and $a \leq b \leq c$ where $[a, c]$ is the support and $b$ is the core of $u$.

Given two fuzzy intervals $u_1$ and $u_2$ we can define the pair $(u_1, u_2)$, a fuzzy set on $\mathbb{R}^2$, such that

$$(u_1, u_2)(x, y) = u_1(x) \land u_2(y),$$

where $\land$ denotes the minimum. Then $[(u_1, u_2)]^x = [u_1]^x \times [u_2]^x$, for all $x \in [0, 1]$, where $\times$ denotes the usual Cartesian product [20]. So, we can define a fuzzy vector as being $(u_1, \ldots, u_n)$ such that $u_i \in F^n_C$, for $i = 1, \ldots, n$. We denote by $(F^n)^n$ the space of all fuzzy vectors, that is,

$$(F^n)^n = \{(u_1, \ldots, u_n) | u_i \in F^n, i = 1, \ldots, n\}.$$
We have the following result for the subspaces $\mathcal{F}_c^n(A)$.

**Proposition 1.** Let $A$ be a subset of $\mathbb{R}^n$ and let $A_i$ be a subset of $\mathbb{R}$, for each $i = 1, \ldots, n$. Then

(i) If $A$ is a nonempty open subset of $\mathbb{R}^n$, then $\mathcal{F}_c^n(A)$ is a nonempty open subset of $\mathcal{F}_c^n$.

(ii) If $A$ is a nonempty open subset of $\mathbb{R}^n$, then $\mathcal{F}_c^n(A)$ is a nonempty open subset of $\mathcal{F}_c^n$.

(iii) If $A_i$ is a nonempty open subset of $\mathbb{R}$, for $i = 1, \ldots, n$, then the product $\Pi_{i=1}^n\mathcal{F}_c(A_i)$ is a nonempty open subset of $(\mathcal{F}_c)^n$.

**Proof.** For a proof of (i) and (ii) see Proposition 1 in [61]. For part (iii) we have from (ii) that each $\mathcal{F}_c(A_i)$ is a nonempty open subset of $\mathcal{F}_c$. Consequently, considering the product topology, $\Pi_{i=1}^n\mathcal{F}_c(A_i)$ is a nonempty open subset of $(\mathcal{F}_c)^n$. □

Zadeh proposed an extension principle in [73], which has become an important tool in fuzzy theory and its applications. The idea is that each function $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ induces a unique fuzzy function $\hat{f} : (\mathcal{F}_c)^m \rightarrow (\mathcal{F}_c)^n$, defined for each fuzzy vector $u = (u_1, \ldots, u_m)$ by

$$\hat{f}(u)(y) = \begin{cases} \sup_{x \in \mathbb{R}^m : f(x) = y} u(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

(7)

We say that $\hat{f}$ is obtained from $f$ by applying the Zadeh’s extension principle (ZPE).

Arithmetic operations are continuous real-valued functions, excluding division by zero. Therefore, the extension principle can be used for obtaining a fuzzy arithmetic [20,31,48,49]. For instance, if we consider the function $f_+ : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f_+(x, y) = x + y$, applying ZPE we obtain a unique fuzzy function $\hat{f}_+ : (\mathcal{F}_c)^2 \rightarrow \mathcal{F}_c$ defined by

$$\hat{f}_+(u, v)(x) = \sup_{x_1 + x_2 = x} u(x_1) \land v(x_2).$$

(8)

Thus, we define the sum between two fuzzy intervals $u, v \in \mathcal{F}_c$ by $u + v = \hat{f}_+(u, v)$. Also, considering the function $f_m$ defined by $f_m(x) = \lambda \cdot x$, with $\lambda \in \mathbb{R}$, and applying ZPE we obtain $\lambda \cdot u = \hat{f}_m(u)$, the multiplication of a scalar $\lambda$ by a fuzzy interval $u$.

Any continuous function $f$ can be extended to a unique fuzzy function by applying the extension principle. For example, if we consider the function $f_1 : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by $f_1(x) = \sqrt{x}$, applying ZPE we obtain the fuzzy function $\hat{f}_1 : \mathcal{F}_c(\mathbb{R}^+) \rightarrow \mathcal{F}_c$ and we define $\sqrt{u}$ by

$$\sqrt{u} = \hat{f}_1(u),$$

(9)

for all fuzzy interval $u$ such that $[u]^0 \subset \mathbb{R}^+$.

In general, the computation of the fuzzy function $\hat{f}$ is a complex problem [15,16,31,32]. But there exists a relation between the $\alpha$-levels of $\hat{f}(u)$ and the image of $\alpha$-level of $u$ by $f$, which helps to obtain $\hat{f}$.

**Theorem 2 (Román-Flores and Rojas-Medar [62]).** The function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous if and only if its extension $\hat{f} : (\mathcal{F}_c)^n \rightarrow (\mathcal{F}_c)^m$ is a well-defined function and it is continuous with respect to the Haussdorff–Pompieu metric. Moreover

$$[\hat{f}(u_1, \ldots, u_n)]^z = f([u_1]^z, \ldots, [u_n]^z),$$

(10)

for all $z \in [0, 1]$ and for all $u_i \in \mathcal{F}_c$, where $f(A) = \{f(a)/a \in A\}$.

Theorem 2 and Eq. (10) states that we can obtain the sum of two fuzzy intervals $u + v$ applying ZPE (8) so that the following property holds

$$[u + v]^z = [u]^z + [v]^z, \quad \forall z \in [0, 1],$$
where $[u]^2 + [v]^2$ is the sum of two compact intervals defined by (6). Multiplication of a fuzzy set by a scalar $\lambda \cdot u$ has the following property:

$$[\lambda \cdot u]^2 = \lambda [u]^2, \quad \forall \lambda \in [0, 1],$$

where $\lambda [u]^2$ is the multiplication of a scalar by a compact interval (5).

The fuzzy interval space $\mathcal{F}_C$, in addition to the sum and multiplication by a scalar, has subtraction obtained by applying ZPE, $u - v$, multiplication $u \times v$, and division $u \div v$ ($0 \notin [v]^0$) between two fuzzy intervals $u$, $v$.

We can define other fuzzy arithmetics on the fuzzy interval space $\mathcal{F}_C$: (see [49,52,53]). Given two fuzzy intervals $u$, $v \in \mathcal{F}_C$ fuzzy standard interval arithmetic (FSIA) $u \ast v$, with $\ast \in \{+,-,\times,\div\}$, is defined via its $x$-levels by

$$[u \ast v]^2 = [u]^2 [v]^2 \quad \text{for all} \quad x \in [0, 1],$$  \hspace{1cm} (11)

where $[u]^2 [v]^2$ is obtained from standard interval arithmetic (6).

Note that the sum of two different fuzzy intervals $u$ and $v$ using ZPE and using FSIA are equivalent. In the case of multiplication of a scalar by a fuzzy interval the two concepts are also equivalent. However, in case of the multiplication of the same fuzzy intervals using ZPE and using FSIA the results are not equivalent. For example, if $u = (-1, 0, 1)$ then $u \times u = (-1, 0, 1)$ using FSIA while $u \times u = (0, 0, 1)$ using ZPE.

Every fuzzy algebraic expression obtained via ZPE has a unique fuzzy interval expression when extended from a continuous real function of real variables. For example, when $u^3 - u^2$ is obtained by applying ZPE to function $f(x) = x^3 - x^2$ we have from (7) that

$$u^3 - u^2 = u \times (u^2 - u) = u^2 \times (u - 1),$$

for all $u \in \mathcal{F}_C$. Now, if we apply the FSIA to function $f(x) = x^3 - x^2$ then, in general, we have that

$$u^3 - u^2 \neq u \times (u^2 - u) \neq u^2 \times (u - 1).$$ \hspace{1cm} (12)

On the other hand, if we denote by $f_S$ the fuzzy function obtained from a function $f$ by applying FSIA, then

$$\hat{f}(u) \subseteq f_S(u)$$ \hspace{1cm} (13)

for all $u \in \mathcal{F}_C^n$.

Zadeh’s extension principle preserves the main properties and characteristics of the original function whereas fuzzy standard interval arithmetic overestimates (13) and ambiguity of representation (12). However, there are several approaches to avoid overestimation and ambiguity of representation of fuzzy functions obtained via FSIA that respect ZPE [44,49,53,60].

3. Fuzzy differential equations via differential inclusions

We consider the following class of fuzzy differential equations:

$$X'(t) = \hat{f}(t, X, W), \quad X(0) = X_0 \hspace{1cm} (14)$$

where $\hat{f} : [0, T] \times (\mathcal{F}_C)^n(\Omega) \times (\mathcal{F}_C)^m \to \mathcal{F}_C^n$ is obtained from a real-valued continuous function $f : [0, T] \times \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ by applying the Zadeh’s extension principle, $X_0 \in (\mathcal{F}_C)^n$.

As we have said previously, for problem (14) there are at least three possibilities for representing a fuzzy solution: the first involves the derivative of fuzzy functions; the second is obtained through of Zadeh’s extension principle applied to deterministic solution and the last one is based on a family of differential inclusions.

This section devotes attention to the last approach, obtaining a fuzzy solution of (14) via differential inclusions.

3.1. Solving fuzzy differential equations via differential inclusions

Note that in (14) $\hat{f}$ is continuous since $f$ is continuous and by (10) we have

$$[\hat{f}(t, X, W)]^2 = f(t, [X]^2, [W]^2),$$

where $f(t, A, B) = \{ f(t, a, b)/a \in A, b \in B \}$.  

The fuzzy initial value problem (14) can be rewritten as a family of differential inclusions
\[ x'_2(t) \in \overline{f}_x(t, x_2(t)), \quad x_2(0) \in [X_0]^2, \quad 0 \leq x \leq 1, \] (15)
where \( \overline{f}_x : [0, T] \times \Omega \rightarrow \mathcal{K}^m_{\Omega} \) is a set-valued function defined by
\[ \overline{f}_x(t, x_2) = f(t, x_2, [W]^2) = \{ f(t, x_2, w)/w \in [W]^2 \}, \]
which is continuous. The subscript \( x \) indicates the \( x \)-level of the problem, i.e. we have a differential inclusion at each \( x \)-level.

We denote by \( \mathcal{A}_x(t, X^x_0, [W]^2) \) the attainable sets associated with problem (15) and it is defined, for each \( x \in [0, 1] \), by
\[ \mathcal{A}_x(t, X^x_0, [W]^2) = \{ x_2(t)/x_2(\cdot) \text{ is a solution of (15)} \text{ in the interval } [0, T] \}. \]

If for each \( t \in [0, T] \) the family \( \{\mathcal{A}_x(t, X^x_0, [W]^2)\}_{x \in [0, 1]} \) are \( x \)-levels of a compact fuzzy set \( \mathcal{A}(t, X_0, W) \), then we say that there exists a weak fuzzy solution \( X_{Iw}(t) = \mathcal{A}(t, X_0, W) \) of (14) via differential inclusion (IW-fuzzy solution, for short).

If for each \( t \in [0, T] \) the family \( \{\mathcal{A}_x(t, X^x_0, [W]^2)\}_{x \in [0, 1]} \) is \( x \)-levels of a compact and convex fuzzy set \( \mathcal{A}(t, X_0, W) \), then we say that there exists a fuzzy solution \( X_I(t) = \mathcal{A}(t, X_0, W) \) of (14) via differential inclusions (I-fuzzy solution, for short) and
\[ [\mathcal{A}(t, X_0, W)]^2 = \mathcal{A}_x(t, X^x_0, [W]^2) \quad \text{for all } x \in [0, 1]. \]

From above, we can see that obtaining a solution of the fuzzy differential equation (14) is equivalent to finding the attainable sets of the family of differential inclusions (15).

How do we obtain \( \mathcal{A}_x(t, X^x_0, [W]^2) \) for each \( x \in [0, 1] \)? It is well-known that under some suitable conditions on \( f \) the family of differential inclusions (15) is equivalent to a family of dynamical systems controlled by parameters \( c_2(t) \in [W]^2 \) (for more details see [1,2,11,66])
\[ x'_2(t) = f(t, x_2(t), c_2(t)), \quad x_2(0) = x^0_2 \in [X_0]^2. \] (16)

Let us denote by \( \mathcal{M}([0, T], [W]^2) \) the family of all the measurable functions \( c_2 : [0, T] \rightarrow \mathbb{R}^m \) such that \( c_2(t) \in [W]^2 \).

So, for each \( c_2 \in \mathcal{M}([0, T], [W]^2) \) and \( x^0_2 \in [X_0]^2 \) we have a unique solution \( x_2(t, c_2, x^0_2) \) of the problem (16) and it is also a solution of the differential inclusion (15) [1,2,11,65]. Thus, for obtaining \( \mathcal{A}_x(t, X^x_0, [W]^2) \), with \( x \in [0, 1] \), we need to consider all the measurable functions \( c_2 \in \mathcal{M}([0, T], [W]^2) \), all initial conditions \( x^0_2 \in [X_0]^2 \) and, for each one of these, obtain the solutions \( x_2(t, c_2, x^0_2) \) associated with \( c_2 \) and \( x^0_2 \). Thus,
\[ \mathcal{A}_x(t, X^x_0, [W]^2) = \bigcup_{c_2(t) \in \mathcal{M}([0, T], [W]^2), x^0_2 \in [X_0]^2} x_2(t, c_2, x^0_2), \] (17)
for each \( x \in [0, 1] \).

**Example 1.** We consider of following fuzzy differential equation:
\[ X'(t) = (-1) \cdot X + t + 1, \quad X(0) = X_0, \] (18)
where \( X_0 \) is the triangular fuzzy number \([0, 1, 2]\).

Note that in problem (18), the right hand side is independent of any fuzzy parameter \( W \), i.e. in this fuzzy differential equation the unique fuzzy interval (not a real number) is the variable \( X \), and it is obtained by applying the Zadeh’s extension principle to the function \( f(t, x) = -x + t + 1 \) in relation to the second argument.

The family of differential inclusions associated to problem (18) is given by
\[ x'_2(t) = -x_2 + t + 1, \quad x_2(0) \in [x, 2 - x], \quad x \in [0, 1]. \] (19)

For each \( x \in [0, 1] \), we need to obtain the attainable set of the differential inclusions (19). In this case
\[ \mathcal{A}_x(t, X^x_0) = \bigcup_{x_2(0) \in X^x_0} x_2(t, x_2(0)) = \bigcup_{x_2(0) \in X^x_0} (t + x_2(0)e'). \]
The family of attainable sets \( \{ A_x(t, X_0^x) \}_{x \in [0, 1]} \) verifies the conditions of the representation theorem (Theorem 1) and these are convex. Then there exists a unique \( I \)-fuzzy solution \( X_I \), defined for \( t \geq 0 \), of the problem (20) which is given by
\[ X(t) = t + X_0 \cdot e^{-t}. \]

In general, when a fuzzy differential equation is independent of a fuzzy parameter \( W \), i.e. if we consider the following problem:
\[ X'(t) = \tilde{f}(t, X), \quad X(0) = X_0, \]  
(20)
where only the variable \( X \) and the initial condition are fuzzy vectors, then the family of differential inclusions associated to problem (20) will be written as
\[ x'_x(t) = f(t, x_x(t)), \quad x_0^x \in [X_0]^x, \quad x \in [0, 1], \]  
(21)
and if there exists a \( I \)-fuzzy solution \( X_I : [0, T] \rightarrow \mathcal{F}_C \) of (20) then for each \( t \in [0, T] \)
\[ [X_I(t)]^x = A_x(t, X_0^x) = \bigcup_{x_0^x \in X_0^x} x_x(t, x_0^x), \]
where \( x_x(\cdot, x_0^x) \) is a solution of (21). In particular, if \( \tilde{f} \) is a function defined on fuzzy intervals, i.e. \( \tilde{f} : [0, T] \times \mathcal{F}_C \rightarrow \mathcal{F}_C \), and the initial condition is a fuzzy interval, then
\[ [X_I(t)]^x = \left[ \inf_{x_0^x \in X_0^x} x_x(t, x_0^x), \sup_{x_0^x \in X_0^x} x_x(t, x_0^x) \right]. \]

Next we present an example, which was given by Kaleva in [37], to illustrate the last equation.

**Example 2.** Consider the fuzzy differential equation
\[
\begin{align*}
X'(t) &= X^2(t), \\
X(0) &= (1, 2, 3),
\end{align*}
\]  
(22)
The family of differential inclusions associated with (22) is given by
\[ x'_x(t) = x^2_x(t), \quad x_0(t) = p_x \in [(1, 2, 3)]^x, \]
and the solutions to this problem in the interval \([0, 1/3]\) are
\[ x_x(t, p_x) = \frac{p_x}{1 - tp_x}, \quad p_x \in [(1, 2, 3)]^x = [1 + x, 3 - x]. \]
Therefore, there exists an \( I \)-fuzzy solution \( X_I \) of (22) and, for each \( x \in [0, 1] \) and \( t \in [0, 1/3] \), we have
\[ [X_I(t)]^x = \left[ \inf_{p_x \in [1+x,3-x]} x_x(t, p_x), \sup_{p_x \in [1+x,3-x]} x_x(t, p_x) \right] = \left[ \frac{1 + x}{1 - t - t^2}, \frac{3 - x}{1 - 3t + t^2} \right]. \]

Now, we present an example such that a fuzzy parameter is involved in the fuzzy differential equation.

**Example 3.** We consider the following problem:
\[
\begin{align*}
X'(t) &= Y(t) - W, \\
Y'(t) &= -X(t), \\
X(0) &= 0, \quad Y(0) = 0,
\end{align*}
\]  
(23)
where \( W \) is the triangular fuzzy number \((-1, 0, 1)\).
The problem (23) can be written as the following family of differential inclusions:

\[
\begin{cases}
  x'_2(t) \in y_2(t) - [W]^2 \\
  y'_2(t) = -x_2(t) \\
  x_2(0) = 0, \ y_2(0) = 0.
\end{cases}
\]  

(24)

The problem (24) is equivalent to a family of dynamical systems controlled by parameters \(c_2(t) \in [W]^2\)

\[
\begin{cases}
  x'_2(t) = y_2(t) - c_2(t) \\
  y'_2(t) = -x_2(t) \\
  x_2(0) = 0, \ y_2(0) = 0.
\end{cases}
\]  

(25)

For each measurable function \(c_2 : [0, T] \rightarrow \mathbb{R}\) such that \(c_2(t) \in [W]^2\) we have a unique solution \((x_2(t, c_2), y_2(t, c_2))\) of the problem (25) and it is also a solution of the differential inclusion (24). Thus, to obtain \(A_2(t, [W]^2)\), with \(\alpha \in [0, 1]\), we need to consider all the measurable functions \(c_2(t) \in [W]^2\) and then obtain the solution \((x_2(t, c_2), y_2(t, c_2))\) associated to \(c_2\). So,

\[A_2(t, [W]^2) = \bigcup_{c_2(t) \in [0,T], [W]^2} (x_2(t, c_2), y_2(t, c_2)),\]

for each \(\alpha \in [0, 1]\).

If we take only constant function \(c_2(t) = w_2 \in [W]^2\), then we have

\[\bigcup_{w_2 \in [W]^2} (x_2(t, w_2), y_2(t, w_2)) = (-[W]^2 \cdot \sin t, [W]^2 \cdot (1 - \cos t)).\]

But we can see that

\[A_2(t, [W]^2) \neq \bigcup_{w_2 \in [W]^2} (x_2(t, w_2), y_2(t, w_2)).\]

In fact, if we fix \(\alpha = 0\) and we take \(c_0(t) = \frac{1}{2} \sin t/2\), then we have a solution of (24) given by

\[(x_0(t, c_0), y_0(t, c_0)) = \left(\frac{1}{3} \cos t - \frac{1}{3} \cos \frac{t}{2}, -\frac{1}{3} \sin t + \frac{2}{3} \sin \frac{t}{2}\right).\]

Then for \(t = \pi\),

\[\bigcup_{w_2 \in [W]^2} (x_2(\pi, w_2), y_2(\pi, w_2)) = (0, [-2, 2]),\]

and \((x_0(\pi, c_0), y_0(\pi, c_0)) = (-\frac{1}{3}, \frac{2}{3})\). So

\[(x_0(\pi, c_0), y_0(\pi, c_0)) \notin \bigcup_{w_2 \in [W]^2} (x_2(t, w_2), y_2(t, w_2)).\]

This means that to obtain \(A_2(t, [W]^2)\) it is not sufficient to consider \(c_2\) as a constant function.

We denote by \(A_2(t, X_0^2, [W]^2)\) the set of all the solutions of (15) obtained by considering \(c_2\) as a constant function \(c_2(t) = w_2 \in [W]^2\), i.e.

\[A_2(t, X_0^2, [W]^2) = \bigcup_{w_2 \in [W]^2, x_0^2 \in [X_0]^2} x_2(t, w_2, x_0^2).
\]

(26)

Clearly, for all \(\alpha \in [0, 1],\)

\[A_2(t, X_0^2, [W]^2) \subset A_2(t, X_0^2, [W]^2).\]

In general the converse is not valid, as was shown in the previous example.
From previous example we can see that it is incorrect to consider $c_2$ as being a constant function. Next we present other examples where we can see the correct procedure to solving a fuzzy differential equation via differential inclusions.

**Example 4.** In [4, 21, 22, 46] the following example was considered:

\[
\begin{cases}
X'(t) = -X + W \cos(t) \\
X(0) = (-1, 0, 1),
\end{cases}
\]

where $W = (-1, 0, 1)$.

The family of differential inclusions associated to (27) is given by

\[
\begin{cases}
x'_2(t) \in -x_2 + [x - 1, 1 - x] \cos(t) \\
x_2(0) \in [x - 1, 1 - x].
\end{cases}
\]

All articles [4, 21, 22, 46] stated that the $x$-solutions set is given, for $t \geq 0$, by

\[
x_2(t) \in \frac{1}{2}(\sin t + \cos t)[W]^2 + ([X_0]^2 - \frac{1}{2}[W]^2)e^{-t},
\]

considering fuzzy standard interval arithmetic. However, since $[x - 1, 1 - x] \cos(t) = [x - 1, 1 - x] \cos(t)$ for all $t$, the family of differential inclusions (28) is equivalent to

\[
\begin{cases}
x'_2(t) \in -x_2 + [x - 1, 1 - x] \cos(t) \\
x_2(0) \in [x - 1, 1 - x].
\end{cases}
\]

Thus, for $x = 0$, $x^1_0(t) = e^{-t}(1 + \int_0^t e^s \cos s|ds|$ is a solution of (30), consequently of (28), and

\[
x^1_0(t) \notin \frac{1}{2}(\sin t + \cos t) [W]^0 + ([X_0]^0 f - \frac{1}{2}[W]^0)e^{-t}.
\]

Fig. 1 displays the upper and lower bound of the 0-solutions set given by (29) and the solution $x^1_0$. Clearly, $x^1_0$ does not satisfy (29).

By comparison, we have that the $I$-fuzzy solution $X_I(t)$ of the fuzzy differential equation (27) is such that

\[
[X_I(t)]^2 = A_x(t, X_0^2, [W]^2) = e^{-t} \left([X_0]^2 + \int_0^t e^s \cos s|[W]^2| ds \right)
\]
Fig. 2 displays the upper and lower bound of \([X_I(t)]^0\).

Fig. 3. The \(I\)-fuzzy solution \(X_I\).

\[
= e^{-t} \left( [x - 1, 1 - x] + \int_0^t e^s \cos s \, ds \right)
\]

\[
= e^{-t} \left[ (x - 1) \left( 1 + \int_0^t e^s \cos s \, ds \right) + (1 - x) \left( 1 + \int_0^t e^s \cos s \, ds \right) \right]
\]

Fig. 2 displays the upper and lower bound of support of the \(I\)-fuzzy solution \(X_I\) and Fig. 3 displays the dynamics of the fuzzy differential equation (27).

**Remark 1.** Note that fuzzy differential equations via differential inclusions are directly interpreted with the help of differential inclusions without having a derivative of fuzzy functions. However, the connection between fuzzy differential equation via differential inclusions and fuzzy differential equations considering \(X'\) as being the generalized Hukuhara derivative [9,13] has been shown for problem (14) in [14]. In particular \(X(t) = X_I(t)\) when \(f\) is monotone with respect to the second variable. Also, the connection between fuzzy differential equations via Hukuhara derivative and via differential inclusions has been shown in [37], for a particular class of fuzzy functions \(\hat{f}\).
3.2. Fuzzy differential equations via differential inclusions are not always applicable

Fuzzy differential equations via differential inclusions were initially studied in [21] (see also [3,22,23]). The existence of solutions for fuzzy differential equations via differential inclusions was studied in those papers. But, in [3,21,23,50] the authors suppose that any fuzzy differential equation of solutions for fuzzy differential equations via differential inclusions was studied in those papers. However, if we consider the fuzzy differential equation (31) with the fuzzy function \( F \) defined by (33), it can not be interpreted as a family of differential inclusions. However, the equation (31) not always can be written as a family of differential inclusions, as we can see in the following examples.

**Example 5.** Let \( X \) be a fuzzy interval and we denote it by \( X = (x^-(z), x^+(z)) \). We consider the fuzzy function \( F : [0, T] \times C \rightarrow C \) defined by
\[
[F(t, X)]^2 = [x^-(1), x^+(z)],
\]
that is, \( F \) associates to each fuzzy interval \( X \) the side right of it. For instance, if we take the fuzzy interval \( X = (x^-(z), x^+(z)) = (z - 1, 2 - z) \) then we have
\[
F(t, X) = (0, 2 - z).
\]
Then the fuzzy differential equation (31), considering the fuzzy function \( F \) defined by (32), can not be interpreted as a family of differential inclusions (15). There is no connection between the variable \( x_{20} \) to be considered and the extremities of the \( z \)-levels of \( F(t, X) \). In fact, all the \( z \)-levels of \( F(t, X) \) are dependent on the \( 1 \)-level.

Next we present a more interesting example.

**Example 6.** We denote by \( T_p(x_1, y_1) \) the diamond on the plane centered at \((x_1, y_1)\), i.e.
\[
T_p(x_1, y_1) = \{(x, y) \in \mathbb{R}^2 | |x - x_1| + |y - y_1| \leq p\}.
\]
We define the fuzzy function \( F : (C)^2 \rightarrow C \) via its \( x \)-levels by
\[
[F(X, Y)]^x = T_{1-x}(x^+(1), y^+(1)).
\]
In this case, \( F \) associates to each fuzzy vector \((X, Y)\) the fuzzy set \([F(X, Y)]^x\) which only dependent on the core \( x^+(1) \) of fuzzy interval \( X \) and on the core \( y^+(1) \) of fuzzy interval \( Y \).

If we consider the fuzzy differential equation (31) with the fuzzy function \( F \) defined by (33) clearly that it can not be interpreted as the family of differential inclusions (15) since we can not obtain a relation between the variable to be considered \( x_2 \) and the set \([F(X, Y)]^x\).

In same way, if we denote by \( S_p(x_1, y_1) \) the square on the plane centered in \((x_1, y_1)\), i.e.
\[
S_p(x_1, y_1) = \{(x, y) \in \mathbb{R}^2 | \max \{|x - x_1|, |y - y_1|\} \leq p\},
\]
and we define the fuzzy function \( F_s : (C)^2 \rightarrow C \) via its \( x \)-levels by
\[
[F_s(X, Y)]^x = S_{1-x}(x^+(1), y^+(1)),
\]
then \( F_s(X, Y) \) depends only on the core \( x^+(1) \) of fuzzy interval \( X \) and on the core \( y^+(1) \) of fuzzy interval \( Y \). Thus, the fuzzy differential equation (31) with the fuzzy function \( F \) defined by (34) can not be interpreted as a family of differential inclusions (15) since each \( z \)-level of \( F \) are not independent in relation to \( z \). All \( z \)-levels always are dependent on the \( 1 \)-level.

Consequently, when considering differential inclusions to solve fuzzy differential equations a condition is that each \( z \)-level of \( F(t, x) \) is required to be independent on \( z \). Thus, we cannot use this approach to solve the fuzzy differential equation (31) for an arbitrary fuzzy function \( F \).

A particular case, as we have previously seen, is when we consider \( F(t, X) = \hat{f}(t, X, W) \) as being a fuzzy function generated by applying the Zadeh’s extension principle. In this case, \([F(t, X)]^x\) depends only on \( z \).
3.3. Fuzzy differential equations and fuzzy differential inclusions

Fuzzy differential inclusions were introduced by Hüllermeier [33,34] as a natural extension of differential inclusions. Basically, it is considered the following problem:

\[ x'(t) \in H(t, x), \quad x(0) \in X_0 \]  

(35)

where \( H : [0, T] \times \mathbb{R}^n \to \mathcal{F}_C^n \) is a fuzzy-valued function and \( X_0 \in \mathcal{F}_C^n \). What does \( \hat{e} \) in (35) mean? This has a symbolic meaning and problem (35) is interpreted as a family of differential inclusions

\[ x'_2(t) \in [H(t, x_2)]^2, \quad x(0) \in [X_0]^2, \]  

(36)

where the subscript \( \alpha \) indicates that the \( \alpha \)-level of a fuzzy set is involved. For more details see [5,8,22,30,46].

Example 7. Let \( H : \mathbb{R} \to \mathcal{F}_C \) be a fuzzy-valued function defined by \( H(x) = \chi_{[2x, x^2+1]} \). We consider the fuzzy differential inclusion

\[ x'(t) \in \chi_{[2x, x^2+1]}, \quad x(0) \in (0, 0.1, 0.2). \]  

(37)

Then, the family of differential inclusions associated to (37) is given by

\[ x'_2(t) \in [2x_2, x_2^2 + 1], \quad x_2(0) \in [0.1x, 0.2 - 0.1x], \]  

(38)

with \( \alpha \in [0, 1] \). Thus, the solution \( X_t \) of the fuzzy differential inclusion (37) is given, for \( t \in [0, 2] \), by

\[ [X_t]^2 = \mathcal{A}_0(t, [0.1x, 0.2 - 0.1x]) = [0.1xe^{2t}, \tan(t + \arctan(0.2 - 0.1x))]. \]

Note that, in this case, the fuzzy-valued function \( H \) is not generated from any real-valued function.

Since this is the case, fuzzy differential inclusions (35) are more general than differential inclusions. Also, fuzzy differential inclusions (35) are more general than fuzzy differential equations via differential inclusions. In fact, given the fuzzy differential equation (14), we define the fuzzy-valued function \( H : [0, T] \times \Omega \to \mathcal{F}_C^n \) by \( H(t, x) = \hat{f}(t, x, W) \), where \( \hat{f}(t, x, W) \) is obtained from the continuous function \( f : [0, T] \times \Omega \times \mathbb{R}^m \to \mathbb{R}^n \) by applying the Zadeh’s extension principle only in relation to the third variable. If we consider the fuzzy differential inclusion

\[ x'(t) \in \hat{f}(t, x, W), \quad x(0) \in X_0, \]  

(39)

then the family of differential inclusions associated to fuzzy differential inclusion (39) is given by

\[ x'_2(t) \in f(t, x_2, [W]^2), \quad x_2(0) \in [X_0]^2, \quad 0 \leq \alpha \leq 1. \]  

(40)

Since the family of differential inclusions (15) and (40) are exactly equal, fuzzy differential equations via differential inclusions and fuzzy differential inclusions are equivalent for the particular class of fuzzy differential equations (14).

3.4. Some results on existence of solutions

Next we present some results with respect the existence of an \( I \)-fuzzy solution for the fuzzy differential equation (14). Since fuzzy differential inclusions are equivalent to fuzzy differential equations via differential inclusions, all the results presented in [5,22,30,46,67] for the existence of solutions of fuzzy differential inclusions are valid for existence of \( Iw \)-fuzzy solutions (\( I \)-fuzzy solutions). We are going to include the results in this paper for completeness.

Theorem 3. Let \( f : [0, T] \times \Omega \times \mathbb{R}^m \to \mathbb{R}^n \) be a continuous function in \( (t, x, w) \), let \( X_0 \in (\mathcal{F}_C)^n(\Omega), W \in (\mathcal{F}_C)^m \) with \( \Omega \) an open subset of \( \mathbb{R}^n \). Let the boundedness assumption with constants \( b, M, T \) (see [21]), hold for all \( x_0 \in [X_0]^0 \) and the inclusion

\[ x' \in \overline{F}_0(t, x), \quad x(0) \in [X_0]^0. \]  

(41)

Then the attainable sets \( A_2(t, X_0^2, [W]^2) \) associated to problem (15) are nonempty and compact subset of \( \mathbb{R}^n \), for each \( t \in [0, T] \).
Proposition 2. Let \( f \in \mathbb{R}^n \) with \( \mathcal{A}_f(t, [W]^2) \) be a continuous function in \( (t, x, w) \), let \( X_0 \in \mathcal{F}_C, W \in (\mathcal{F}_C)^m \). If the Lipschitz condition is satisfied, i.e.
\[
D(\hat{f}(t, X, W), \hat{f}(t, Y, W)) \leq LD(X, Y),
\]
for all \( X, Y \in (\mathcal{F}_C) \) with a Lipschitz constant \( L > 0 \), then the attainable sets \( \mathcal{A}_f(t, X_0^t, [W]^2) \) associated with problem (15) are nonempty, convex and compact subsets of \( \mathbb{R}^n \), for each \( t \in [0, T] \).

Proof. It is a consequence of Corollaries 7.1 and 7.2 in [19].

The regularity of the attainable set of a differential inclusion was studied in [23]. Basically, a quasi-concave type of regularity with respect to a parameter was considered in [23], which will be essential for our purpose. An application \( H : \mathcal{F}_C \times [0, 1] \rightarrow \mathcal{K}_n \), with \( I \) a real compact interval, is said to be regularly quasi-concave on \( I \) if

- for all \( (t, x, w) \in I \) and \( \alpha, \beta \in I \),
  \[
  H(t, x; \alpha) \leq H(t, x; \beta), \quad \text{whenever} \quad \alpha \leq \beta.
  \]

- If \( \{z_n\} \) is a nondecreasing sequence in \( I \) converging to \( z \), then for all \( (t, x) \in I \)
  \[
  \cap_n H(t, x; z_n) = H(t, x; z).
  \]

Theorem 4. Let \( f : [0, T] \times \mathcal{F}_C \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a continuous function in \( (t, x, w) \), \( W \in (\mathcal{F}_C)^m \) and \( X_0 \in (\mathcal{F}_C)^m(\Omega) \), with \( \Omega \) an open subset of \( \mathbb{R}^n \). Let the boundedness assumption with constants \( b, M, T \), hold for all \( X_0 \in [X_0]^0 \) and the inclusion
\[
x^t \in \mathcal{T}_f(t, x), \quad x(0) \in [X_0]^0.
\]

Then there exist a unique Iw-fuzzy solution (3) on the interval \([0, T]\).

Proof. From Theorem 3 we have that \( \mathcal{A}_f(t, X_0^t, [W]^2) \in \mathcal{K}_n \) for all \( \alpha \in [0, 1] \) and \( t \in [0, T] \). On the other hand, for each \( \alpha \in [0, 1] \), the set-valued function \( \mathcal{A}_f \) is continuous on \([0, T] \times \mathbb{R}^n \), it is regularly quasi-concave on \([0, 1] \) and the boundedness assumption holds. Then the application \( \alpha \rightarrow \mathcal{A}_f(\alpha, X_0^\alpha, [W]^2) \) is a regularly quasi-concave map from \([0, 1] \) to \( \mathcal{K}_n \) (see Theorem 1 [23]). Thus, from Theorem of Representation (Theorem 1) there exists a unique compact fuzzy set \( \mathcal{A}(t, X_0, W) \in \mathcal{F}_n \) such that \( [\mathcal{A}(t, X_0, W)]^2 = \mathcal{A}(t, X_0^t, [W]^2) \). Therefore there exists a unique Iw-fuzzy solution \( X_{1w}(t) = \mathcal{A}(t, X_0, W) \) of the problem (3) defined on the interval \([0, T] \).

Theorem 5. Let \( f : [0, T] \times \mathcal{F}_C \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) be a continuous function in \( (t, x, w) \), \( X_0 \in (\mathcal{F}_C)^m(\Omega) \), \( W \in (\mathcal{F}_C)^m \) with \( \Omega \) an open subset of \( \mathbb{R}^n \). Let the boundedness assumption with constants \( b, M, T \), hold for all \( x_0 \in [X_0]^0 \) and the inclusion
\[
x^t \in \mathcal{T}_f(t, x), \quad x(0) \in [X_0]^0.
\]

Additionally, suppose that \( \mathcal{A}_f(t, X_0^t, [W]^2) \) is convex for each \( \alpha \in [0, 1] \). Then there exist a unique Iw-fuzzy solution (3) on the interval \([0, T] \).
Proof. From Theorem 4, there exists a unique compact fuzzy set $\mathcal{A}(t, X_0, W) \in \mathcal{F}^n$ such that $[\mathcal{A}(t, X_0)]^x = \mathcal{A}^x(t, X^x_0, [W]^x)$. Since $\mathcal{A}^x(t, X^x_0, [W]^x)$ is convex for each $x \in [0, 1]$, there exists a unique convex and compact fuzzy set $X^x(t) = \mathcal{A}(t, X_0, W) \in \mathcal{F}^n_c$, defined on the interval $[0, T]$, which is a $I$-fuzzy solution of the problem (3).

Remark 2. The essential condition in the previous theorems is that the boundedness assumption holds for the differential inclusion involving $f$ (43). We can give some conditions on $f$ (Lipschitz, bounded) such that this boundedness assumption holds.

3.5. Approximation of fuzzy solutions

In the literature there are articles devoted to algorithms for approximating the fuzzy solution of problem (14). See for instance [3,4,34,50]. In some of these articles, there is a confusion when implementing the algorithm or giving examples of this. For instance, in [3], Abbasbandy et al. presented a numerical method for solving the fuzzy differential equation (20). On the other hand, in [4] the authors have presented the TRS algorithm for solving fuzzy differential equation (31). For this, they assumed that given a fuzzy function $F$ we can construct $n$ families of $\alpha$-parameterized interval-valued functions such that (see Eq. (8) in [3])

$$F(t, x_2(t); \alpha) \doteq \times_{k=1}^n [f^k_1(t, x_2(t); \alpha), f^k_2(t, x_2(t); \alpha)], \quad x_2(t) \in [X(t)]^x,$$

for all $\alpha \in [0, 1]$, where $\times$ denotes the usual Cartesian product.

A fuzzy set $u$ on $\mathbb{R}^n$ is called pyramidal if its $\alpha$-levels are $n$-dimensional rectangles for $0 \leq \alpha \leq 1$ (see [22,71]). So, if a fuzzy function $F$ verifies (44), then $F$ can be approximated by a pyramidal fuzzy set. But, the relation (44) is valid only if $F$ is generated from a function $f$ by applying the Zadeh’s extension principle [71]. In fact, if we consider the fuzzy function $F$ defined by (32), we have that $F(t, X)$ is a pyramid fuzzy interval, but it cannot be interpreted as a family of differential inclusions. In the same form, if we consider the fuzzy function $F$ defined by (33), then using (44) we have that the pyramidal fuzzy set associated to $F$ is $F_{\alpha}$ (see Eq. (34)) which has $\alpha$-levels being a square centered in $(x^+1, y^+1)$. However, the fuzzy differential equation $X' = F_{\alpha}(t, X)$ cannot be interpreted as a family of differential inclusions. Therefore, the Runge–Kutta method for fuzzy differential equations introduced in [3] is valid for solving the fuzzy differential equation (20).

On the other hand, in [4] the authors have presented the TRS algorithm for solving a fuzzy differential equation via differential inclusions. Apparently, this algorithm is not suitable. For example, if we consider the fuzzy differential equation (see Example 4)

$$\begin{align*}
X'(t) &= -X + (-1, 0, 1) \cos(t) \\
X(0) &= (-1, 0, 1),
\end{align*}$$

then $X_1(0.5)$ and $Y(0.5)$, where $Y(t)$ is the approximation of $I$-fuzzy solution $X_1(t)$ of (45) using TRS algorithm, are not close. Fig. 4 displays $X_1(0.5)$ and $Y(0.5)$. Note that $Y(0.5)$ was obtained in [4] (see Fig. 2 in [4]). On the other hand, the authors stated the fuzzy solution exactly as given by (29) and this algorithm is compared with this, being that (29) is not the correct fuzzy solution of (45). However, we can say that this algorithm is correct.

One could expect that an algorithm for constructing the attainable set of a differential inclusion might be obtained as some extension of known algorithms for ordinary differential equations. Unfortunately, this is not the case. First of all, the attainable set is a set in the time-state space, where the graphs of all possible trajectories of a differential inclusion are included. Finding the boundary of such set is not an easy task even in the case of linear dynamics. The estimation theory and related algorithms based on ideas of the construction of outer and inner set-valued estimates of attainable sets have been developed. For instances, using the advantages of ellipsoidal calculus an estimation for attainable sets of linear cases have been presented in [38,39–42]. Recently, this tool were extended for some class of nonlinear systems [25–27].

When the fuzzy differential inclusions (14) can be given in the form of an equivalent control system (15), like is our case, we can use optimal control theory to obtain the boundary of the family of attainable sets. In fact, one of the properties of the attainable sets is the fact that if a trajectory reaches a point on the boundary of the attainable set at the final time, then its entire graph must belong to the attainable boundary. This fact is well known and used in optimal
control theory [28,58,65,66]. Thus, to obtain the attainable set \( A_x(t, X_0, [W]^2) \) of the differential inclusion (15) we need to solve a family of problems of optimal control (see [58,65,66])

\[
\begin{align*}
\min & \quad -c^T x_T(T) \\
\text{s.t.} & \quad x'_T = f(t, x_T, u), \quad t \in [0, T] \\
& \quad u(t) \in [W]^2, \quad t \in [0, T] \\
& \quad x_T(0) \in [X_0]^2.
\end{align*}
\]

(46)

where \( c \in \mathbb{R}^n \) (assume \( \|c\| = 1 \)) is a fixed parameter. Clearly, \( x^*_T(T; c) \), optimal solution of (46), is a boundary point of \( A_x(T, X_0, [W]^2) \) and this is true for all \( c \in \mathbb{R}^n \). Thus, by varying this coefficient, different boundary point can be produced. The polytope with these points as vertices, constitutes an inner approximation of the attainable set \( A_x(T, X_0, [W]^2) \).

To illustrate the previous algorithm, let us consider the following linear fuzzy differential equation

\[
X' = a(t)X + b(t)W, \quad X(0) = X_0,
\]

(47)

where \( a, b : [0, T] \rightarrow \mathbb{R} \) are continuous functions and \( W, X_0 \in \mathcal{F}_C \). The problem (47) is rewritten as the family of differential inclusions

\[
x'_T \in a(t)x_T + b(t)[w_2, \overline{w}_2], \quad x_T(0) \in [x_0, \overline{x}_0].
\]

(48)

where \( [W]^2 = [w_2, \overline{w}_2] \) and \( [X_0]^2 = [x_0, \overline{x}_0] \).

If for each \( x \in [0, 1] \) we denote the attainable set of (48) by \( A_x(t, X_0, [W]^2) = [x^*_x(t), \overline{x}^*_x(t)] \), then the family of problems of optimal control (46) is equivalent to solving the following two problems of optimal control:

\[
\begin{align*}
\min & \quad x(T) \\
\text{s.t.} & \quad x'_T = a(t)x_T + b(t)u, \quad t \in [0, T] \\
& \quad u(t) \in [W]^2, \quad t \in [0, T] \\
& \quad x_T(0) \in [X_0]^2.
\end{align*}
\]

(49)
and
\[
\begin{align*}
x^*_p(T) &= \max x(T) \\
\text{s.t.} & \quad x'_p = a(t)x_p + b(t)u, \quad t \in [0, T] \\
& \quad u(t) \in [W]^2, \quad t \in [0, T] \\
& \quad x_p(0) \in [X_0]^2.
\end{align*}
\]
Applying the maximum principle [59] to problem (49), we obtain the optimal control \(u^*(t) = \overline{w}_x\) if \(b(t) \geq 0\) and \(u^*(t) = \underline{w}_x\) if \(b(t) < 0\). Thus, we have that the optimal solution \(x^*_p\) is solution of the ordinary differential equation
\[
x'(t) = a(t)x(t) + \left(\frac{\underline{w}_x + \overline{w}_x}{2}\right) b(t) - \left(\frac{\underline{w}_x - \overline{w}_x}{2}\right) |b(t)|, \quad x(0) = x_{0^x}.
\]
(51)
In the same way, applying the maximum principle to problem (50), we have that the optimal solution \(\overline{x}^*_x\) is solution of the ordinary differential equation
\[
x'(t) = a(t)x(t) + \left(\frac{\underline{w}_x + \overline{w}_x}{2}\right) b(t) + \left(\frac{\underline{w}_x - \overline{w}_x}{2}\right) |b(t)|, \quad x(0) = \overline{x}_{0^x}.
\]
(52)
Example 8. Consider the fuzzy differential equation (45). From (51), we have that \(x^*_p\) is solution of the ordinary differential equation
\[
x'(t) = -x(t) - (1 - \varepsilon)|\cos(t)|, \quad x(0) = \varepsilon - 1,
\]
whereas from (52), we have that \(\overline{x}^*_x\) is solution of the ordinary differential equation
\[
x'(t) = -x(t) + (1 - \varepsilon)|\cos(t)|, \quad x(0) = 1 - \varepsilon.
\]
Therefore, the \(X_I\) fuzzy solution of (45) is such that, for each \(\varepsilon \in [0, 1]\),
\[
[X_I(t)]^2 = \mathcal{A}_x(t, X_{0^x}, [W]^2)
\]
\[
= \left[(\varepsilon - 1)e^{-t} \left(1 + \int_0^t e^s |\cos s| ds\right), (1 - \varepsilon)e^{-t} \left(1 + \int_0^t e^s |\cos s| ds\right)\right] ds
\]
References


