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On the characterization of the controllability property for linear control systems on nonnilpotent, solvable three-dimensional Lie groups

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Abstract

In this paper we show that a complete characterization of the controllability property for linear control system on three-dimensional solvable nonnilpotent Lie groups is possible by the LARC and the knowledge of the eigenvalues of the derivation associated with the drift of the system.

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1. Introduction

Linear control systems on Euclidean spaces appear in several physical applications (see for instance [12,15,17]). A natural extension of a linear control system on Lie groups appears first in [13] for matrix groups and then in [4] for any Lie group. In the subsequent years, several works addressing the main problems in control theory for such systems, such as controllability, observability and optimization appeared (see [1–3,5,6,8,9]). In [9] P. Jouan shows that such gen-

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eralization is also important for the classification of general affine control systems on abstract connected manifolds. He shows that any affine control system on a connected manifold that generates a finite dimensional Lie algebra is diffeomorphic to a linear control system on a Lie group or on a homogeneous space.

Concerning controllability of linear control system, in [1] and [5] a more geometric approach was proposed by considering the eigenvalues of a derivation associated with the drift of the system. In particular, it was shown that a linear system is controllable if its reachable set from the identity is open and the associated derivation has only eigenvalues with zero real part. For restricted linear control systems on nilpotent Lie groups such condition is also necessary for controllability. In the same direction, Dath and Jouan show in [6] that linear control systems (restricted or not) on a two-dimensional solvable Lie group present the same behavior, they are controllable if and only if they satisfy the Lie algebra rank condition and the associated derivation has only zero eigenvalues. Here, the LARC is equivalent to the ad-rank condition which implies, in particular, the openness of the reachable set (see [4]).

In the present paper, we show that the behavior of nonrestricted linear control systems on three-dimensional solvable nonnilpotent Lie groups differ significantly from the two-dimensional case. By using a beautiful classification of three-dimensional solvable Lie groups (see Chapter 7 of [14]) we show that the geometry of the group strongly interferes in the controllability of the system and, although such systems do not behave the same as in the two-dimensional case, a complete characterization of their controllability is possible only by the knowledge of the eigenvalues of the associated derivation and the Lie algebra rank condition.

The paper is structured as follows: Section 2 is used to introduce the main properties and results concerning linear vector fields, linear control systems and decompositions of Lie algebras and Lie group induced by derivations. Section 3 is devoted to the study of nonnilpotent, solvable three-dimensional Lie groups. By using the classification in [14], we algebraically characterize the main elements needed in the proofs concerning controllability such as derivations, linear and invariant vector fields and so on. At the end of the section, we present some particular homogeneous spaces which will be of great importance when considering projections of linear control systems. In Section 4 we completely characterize the controllability of linear control systems on such groups. The work is divided into two cases, depending if the dimension of the Lie subalgebra generated by the control vectors is one or two, and then analyzed group by group using the classification presented in Section 3.

2. Preliminaries

2.1. Notations

In the whole paper, the Lie groups and subgroups considered are assumed to be connected unless we say the contrary. Their Lie algebras are identified with the set of left-invariant vector fields. If M, N are smooth manifolds and $f : M \to N$ is a differentiable map, we denote by $(df)_x$ the differential of f at the point $x \in M$ and by f_* the differential of f at any given point.

For any element $g \in G$ we denote by L_g and R_g the left and right translations of G and $e \in G$ stands for the identity element of G. If a Lie algebra is given by the semi-direct product $\mathfrak{g} = \mathfrak{h} \times_\theta \mathfrak{k}$ we will use the identification $\mathfrak{k} = \{0\} \times \mathfrak{k}$ and $\mathfrak{h} = \mathfrak{h} \times \{0\}$ and the same holds for Lie groups that are given as semi-direct product.

2.2. Linear vector fields and decompositions

In this section, we define linear vector fields and state their main properties. For the proof of the assertions in this section the reader can consult [4], [8] and [9].

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. A vector field $\mathcal{X}$ on G is said to be *linear* if its flow $(\varphi_t)_{t\in\mathbb{R}}$ is a 1-parameter subgroup of $\mathrm{Aut}(G)$. Associate to any linear vector field $\mathcal{X}$ there is a derivation $\mathcal{D}$ of $\mathfrak{g}$ defined by the formula

$$\mathcal{D}Y = -[\mathcal{X}, Y](e), \text{ for all } Y \in \mathfrak{g}. \tag{1}$$

The relation between φ_t and $\mathcal{D}$ is given by the formula

$$(d\varphi_t)_e = \mathrm{e}^{t\mathcal{D}} \quad \text{for all} \quad t \in \mathbb{R}. \tag{2}$$

In particular, it holds that

$$\varphi_t(\exp Y) = \exp(\mathrm{e}^{t\mathcal{D}}Y), \text{ for all } t \in \mathbb{R}, Y \in \mathfrak{g}.$$

The above equation implies that if $\mathcal{D} \equiv 0$ we necessarily have $\mathcal{X} \equiv 0$. Since we are interested in linear systems with nontrivial drift, we will always assume $\mathcal{D} \not\equiv 0$.

Let G be a Lie group and $\widetilde{G}$ its simply connected covering. Let $\mathcal{X}$ be a linear vector field on G and $\mathcal{D}$ its associated derivation. By Theorem 2.2 of [4], there exists a unique linear vector field $\widetilde{\mathcal{X}}$ on $\widetilde{G}$ whose associated derivation is $\mathcal{D}$. If we denote, respectively by, $\{\widetilde{\varphi}_t\}_{t\in\mathbb{R}}$ and $\{\varphi_t\}_{t\in\mathbb{R}}$ the flows of $\widetilde{\mathcal{X}}$ and $\mathcal{X}$ we have

$$\pi(\widetilde{\varphi}_t(\exp_{\widetilde{G}} X)) = \pi(\exp_{\widetilde{G}}(\mathrm{e}^{t\mathcal{D}}X)) = \exp_G(\mathrm{e}^{t\mathcal{D}}X) = \varphi_t(\exp_G X), \quad \text{for any} \quad X \in \mathfrak{g}$$

where $\pi : \widetilde{G} \to G$ is the canonical projection. By connectedness it holds that

$$\pi \circ \widetilde{\varphi}_t = \varphi_t \circ \pi \quad \text{for any} \quad t \in \mathbb{R}$$

implying that $\widetilde{\mathcal{X}}$ and $\mathcal{X}$ are π-related.

Next, we explicitly some decompositions of the Lie algebra $\mathfrak{g}$ induced by any given derivation $\mathcal{D}$. To do that, let us consider, for any eigenvalue α of $\mathcal{D}$, the real generalized eigenspaces of $\mathcal{D}$

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : (\mathcal{D} - \alpha I)^n X = 0 \text{ for some } n \geq 1\}, \quad \text{if} \quad \alpha \in \mathbb{R} \quad \text{and}$$

$$\mathfrak{g}_\alpha = \mathrm{span}\{\mathrm{Re}(v), \mathrm{Im}(v); \quad v \in \bar{\mathfrak{g}}_\alpha\}, \quad \text{if} \quad \alpha \in \mathbb{C}$$

where $\bar{\mathfrak{g}} = \mathfrak{g} + i\mathfrak{g}$ is the complexification of $\mathfrak{g}$ and $\bar{\mathfrak{g}}_\alpha$ the generalized eigenspace of $\bar{\mathcal{D}} = \mathcal{D} + i\mathcal{D}$, the extension of $\mathcal{D}$ to $\bar{\mathfrak{g}}$. By Proposition 3.1 of [16] it holds that $[\bar{\mathfrak{g}}_\alpha, \bar{\mathfrak{g}}_\beta] \subset \bar{\mathfrak{g}}_{\alpha+\beta}$ when $\alpha + \beta$ is an eigenvalue of $\mathcal{D}$ and zero otherwise. By considering in $\mathfrak{g}$ the subspaces $\mathfrak{g}_\lambda := \bigoplus_{\alpha;\mathrm{Re}(\alpha)=\lambda} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\lambda = \{0\}$ if $\lambda \in \mathbb{R}$ is not the real part of any eigenvalue of $\mathcal{D}$, we get

$$[\mathfrak{g}_{\lambda_1}, \mathfrak{g}_{\lambda_1}] \subset \mathfrak{g}_{\lambda_1+\lambda_2} \quad \text{when } \lambda_1 + \lambda_2 = \mathrm{Re}(\alpha) \text{ for some eigenvalue } \alpha \text{ of } \mathcal{D} \text{ and zero otherwise.}$$

This fact allow us to decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$$

where

$$\mathfrak{g}^+ = \bigoplus_{\alpha:\,\mathrm{Re}(\alpha)>0} \mathfrak{g}_\alpha, \qquad \mathfrak{g}^0 = \bigoplus_{\alpha:\,\mathrm{Re}(\alpha)=0} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{g}^- = \bigoplus_{\alpha:\,\mathrm{Re}(\alpha)<0} \mathfrak{g}_\alpha.$$

It is easy to see that $\mathfrak{g}^+, \mathfrak{g}^0, \mathfrak{g}^-$ are $\mathcal{D}$-invariant Lie algebras and $\mathfrak{g}^+, \mathfrak{g}^-$ are nilpotent.

At the Lie group level we will denote by G^+, G^-, G^0, $G^{+,0}$, and $G^{-,0}$ the connected Lie subgroups of G with Lie algebras $\mathfrak{g}^+$, $\mathfrak{g}^-$, $\mathfrak{g}^0$, $\mathfrak{g}^{+,0} := \mathfrak{g}^+ \oplus \mathfrak{g}^0$ and $\mathfrak{g}^{-,0} := \mathfrak{g}^- \oplus \mathfrak{g}^0$ respectively. The above subgroups play a fundamental role in the understand of the dynamics of linear control system as showed in [1], [2] and [5]. By Proposition 2.9 of [5], all the above subgroups are φ-invariant and closed. Moreover, if G is a solvable Lie group then $G = G^{+,0}G^- = G^{-,0}G^+$.

The next lemma shows that, for solvable Lie groups, the nilradical contains all the generalized eigenspaces associated with nonzero eigenvalues.

2.1 Lemma. *Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathfrak{n}$ its nilradical. If $\mathcal{D}$ is a derivation of $\mathfrak{g}$ then, for any nonzero eigenvalue α of $\mathcal{D}$, it holds that $\mathfrak{g}_\alpha \subset \mathfrak{n}$.*

Proof. If $\alpha \in \mathbb{R}^*$ then $\mathfrak{g}_\alpha = \bigcup_{n\in\mathbb{N}} \ker(\mathcal{D} - \alpha I)^n$. If $n = 1$ we have that $X \in \ker(\mathcal{D} - \alpha I)$ is such that $\mathcal{D}X = \alpha X$. Since $\mathcal{D}\mathfrak{g} \subset \mathfrak{n}$ and $\alpha \neq 0$ we get $X \in \mathfrak{n}$ implying that $\ker(\mathcal{D} - \alpha I) \subset \mathfrak{n}$. Inductively, if $\ker(\mathcal{D} - \alpha I)^n \subset \mathfrak{n}$ then, for any $X \in \ker(\mathcal{D} - \alpha I)^{n+1}$ it holds that

$$0 = (\mathcal{D} - \alpha I)^{n+1}X = (\mathcal{D} - \alpha I)^n(\mathcal{D} - \alpha I)X \implies (\mathcal{D} - \alpha I)X \in \ker(\mathcal{D} - \alpha I)^n \subset \mathfrak{n}.$$

Using again that $\alpha \neq 0$ and $\mathcal{D}\mathfrak{g} \subset \mathfrak{n}$ gives us $(\mathcal{D} - \alpha I)X \in \mathfrak{n}$ which implies that $X \in \mathfrak{n}$. Therefore, $\mathfrak{g}_\alpha \subset \mathfrak{n}$ as stated.

If $\alpha \in \mathbb{C}^*$ we have as in the real case that $\bar{\mathfrak{g}}_\alpha \subset \bar{\mathfrak{n}}$, where $\bar{\mathfrak{n}}$ is the nilradical of $\bar{\mathfrak{g}}$. Since the conjugation in $\bar{\mathfrak{g}}$ is an automorphism we have that $\bar{\mathfrak{n}}$ is invariant by conjugation and hence $\bar{\mathfrak{n}} = \mathfrak{n}^* + i\mathfrak{n}^*$ for some subspace $\mathfrak{n}^* \subset \mathfrak{g}$. A simple calculation shows that $\mathfrak{n}^*$ is in fact, a nilpotent ideal of $\mathfrak{g}$ and consequently $\mathfrak{n}^* \subset \mathfrak{n}$. Since $\mathrm{Re}(v), \mathrm{Im}(v) \in \mathfrak{n}^*$ for any $v \in \bar{\mathfrak{g}}_\alpha$ we get that $\mathfrak{g}_\alpha \subset \mathfrak{n}$ which concludes the proof. □

By Lemma 2.3 of [5], the above subalgebras and subgroups are preserved by homomorphisms in the following sense: If $\psi : G_1 \to G_2$ is a surjective homomorphism between Lie groups such that $(d\psi)_e \circ \mathcal{D}_1 = \mathcal{D}_2 \circ (d\psi)_e$, where $\mathcal{D}_i$ is a derivation in the Lie algebra $\mathfrak{g}_i$ of G_i, $i = 1, 2$ then

$$(d\psi)_e\mathfrak{g}_1^* = \mathfrak{g}_2^* \quad \text{and} \quad \psi(G_1^*) = G_2^*, \quad \text{where} \quad * = +, 0, -. \tag{3}$$

2.3. Linear control systems

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra identified with the vector space of all left-invariant vector fields. A *linear control system* on G is given a family of ordinary differential equations

$$\dot{g} = \mathcal{X}(g) + \sum_{j=1}^{m} u_i Y^j(g), \tag{4}$$

where the *drift* $\mathcal{X}$ is a linear vector field and the *control vectors* $Y^1, \ldots, Y^m$ are, left-invariant vector fields. The *control functions* $u = (u_1, \ldots, u_m)$ belongs to $\mathcal{U} \subset L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$, a subset that contains the piecewise constant functions and is stable by concatenations, that is, if $u_1, u_2 \in \mathcal{U}$ then the function u defined by

$$u(t) = \begin{cases} u_1(t), & t \in (-\infty, T) \\ u_2(t-T) & t \in [T, \infty) \end{cases}$$

belongs to $\mathcal{U}$.

If $\phi(t, g, u)$ denotes the solution of (4) associated with $u \in \mathcal{U}$ and starting at $g \in G$ then

$$\phi(t, g, u) = \varphi_t(g)\phi(t, e, u) = R_{\phi(t,e,u)}\left(\varphi_t(g)\right).$$

The *reachable set from g at time $t > 0$* and the *reachable set from g* are given, respectively, by

$$\mathcal{A}_t(g) := \{\phi(t, g, u),\ \ u \in \mathcal{U}\} \quad \text{and} \quad \mathcal{A}(g) := \bigcup_{t>0} \mathcal{A}_t(g)$$

Analogously, the *controllable set to g at time $t > 0$* and the *controllable set to g* are given, respectively, by

$$\mathcal{A}^*_t(g) := \{h \in G;\ \exists u \in \mathcal{U};\ \ \phi(t, h, u) = g\} \quad \text{and} \quad \mathcal{A}^*(g) := \bigcup_{t>0} \mathcal{A}^*_t(g)$$

For the particular case where $g = e$ is the identity element of G we denote the above sets only by $\mathcal{A}_t$, $\mathcal{A}$, $\mathcal{A}^*_t$ and $\mathcal{A}^*$, respectively.

We will say that the linear control system (4) is *controllable* if for any $g, h \in G$ it holds that $h \in \mathcal{A}(g)$. It is not hard to see that the controllability of (4) is equivalent to the equality $G = \mathcal{A} \cap \mathcal{A}^*$.

Let us denote by Δ the Lie subalgebra of $\mathfrak{g}$ generated by $Y^1, \ldots, Y^m$ and by G_Δ its associated connected Lie subgroup. The linear control system (4) is said to satisfy the *ad-rank condition* if $\mathfrak{g}$ is the smallest $\mathcal{D}$-invariant subspace containing Δ. It is said to satisfy the *Lie algebra rank condition* (LARC) if $\mathfrak{g}$ is the smallest $\mathcal{D}$-invariant subalgebra containing Δ.

Since we are interested in the controllability of linear control systems and the control functions are taking values in the whole $\mathbb{R}^m$, Theorem 3.5 in [10] implies that $G_\Delta \subset \operatorname{cl}(\mathcal{A}) \cap \operatorname{cl}(\mathcal{A}^*)$ and also that the closures $\operatorname{cl}(\mathcal{A})$ and $\operatorname{cl}(\mathcal{A}^*)$ remain the same if we change $Y_1, \ldots, Y^m$ for any basis of Δ. Furthermore, under the LARC it holds that $G = \operatorname{cl}(\mathcal{A}^{(*)})$ iff $G = \mathcal{A}^{(*)}$ and therefore, if $\dim\Delta = \dim\mathfrak{g}$ the system is trivially controllable. Since G is an analytic manifold and the linear and invariant vector fields are complete, Theorem 3.1 of [19] implies that the LARC is a necessary condition for controllability.

By the previous discussion, under the LARC the controllability of (4) only depends on $\mathcal{X}$ and on Δ. Therefore, we will use $\Sigma(\mathcal{X}, \Delta)$ to denote the linear system with drift $\mathcal{X}$ and control vectors given by any basis of Δ, where Δ is a proper, nontrivial subalgebra of $\mathfrak{g}$.

The next results relate the subgroups associated with the derivation induced by $\mathcal{X}$ with the reachable and controllable sets.

2.2 Lemma. *It holds:*

1. *Let $C \in \{\mathcal{A}, \mathcal{A}^*, \mathrm{cl}(\mathcal{A}), \mathrm{cl}(\mathcal{A}^*)\}$ and $g \in G$. If $\{\varphi_t(g), t \in \mathbb{R}\} \subset C$ then $L_g(C) \subset C$;*
2. *If $\varphi_t(g) = g$ for all $t \in \mathbb{R}$ then $g \in \mathcal{A}$ if and only if $g^{-1} \in \mathcal{A}^*$.*

Proof. Item 1. is an slight modification of Lemma 3.1 of [5] and hence we will omit its proof. For item 2., if $g \in \mathcal{A}$ there exists $t > 0, u \in \mathcal{U}$ with $g = \phi(t, e, u)$. Hence,

$$\phi(t, g^{-1}, u) = \varphi_t(g^{-1})\phi(t, e, u) = \varphi_t(g)^{-1}g = g^{-1}g = e$$

implying that $g^{-1} \in \mathcal{A}^*$. Reciprocally, if $g^{-1} \in \mathcal{A}^*$ then $e = \phi(t, g^{-1}, u)$ for some $t > 0$, $u \in \mathcal{U}$ and analogously

$$e = \phi(t, g^{-1}, u) = \varphi_t(g^{-1})\phi(t, e, u) = g^{-1}\phi(t, e, u) \implies g = \phi(t, e, u) \in \mathcal{A}$$

concluding the proof. $\square$

Concerning the controllability of linear control systems we have the following results from [5] (see Theorem 3.7).

2.3 Theorem. *If $\Sigma(\mathcal{X}, \Delta)$ is a linear system on a solvable Lie group G and assume that $\mathcal{A}$ is open, then*

$$G^{+,0} \subset \mathcal{A} \quad \textit{and} \quad G^{-,0} \subset \mathcal{A}^*.$$

In particular, if $\mathcal{D}$ has only eigenvalues with zero real part and $\mathcal{A}$ is open, then $\Sigma(\mathcal{X}, \Delta)$ is controllable.

2.4 Remark. We should notice that the condition on the openness of $\mathcal{A}$ is guaranteed, for instance, if $\Sigma(\mathcal{X}, \Delta)$ satisfies the ad-rank condition (see Theorem 3.5 of [4]). In particular, if Δ has codimension one in $\mathfrak{g}$ then the LARC is equivalent to the ad-rank condition.

2.5 Remark. An extension of Theorem 2.3 for a much larger class of Lie groups was proved in [1] under the assumption that $\mathcal{A}_\tau$ is a neighborhood of the identity element for some $\tau > 0$.

Let G and H be Lie groups and $\psi : G \to H$ a surjective Lie group homomorphism. If $\mathcal{X}$ is a linear vector field on G then $\mathcal{X}_\psi := \psi_*\mathcal{X}$ is a linear vector field on H. Therefore, if $\Sigma(\mathcal{X}, \Delta)$ is a linear control system on G, by considering $\Delta_\psi := (d\psi)_e\Delta$ we have that $\Sigma = \Sigma(\mathcal{X}_\psi, \Delta_\psi)$ is a linear control system on H that is ψ conjugated to $\Sigma(\psi, \Delta)$. As a particular case, if $\widetilde{G}$ is the simply connected covering of G and $\Sigma(\mathcal{X}, \Delta)$ is a linear control system on G, the control system $\Sigma(\widetilde{\mathcal{X}}, \Delta)$ is π-conjugated to $\Sigma(\mathcal{X}, \Delta)$. The next proposition states the main relationships between conjugated control systems.

2.6 Proposition. *Let $\Sigma(\mathcal{X}, \Delta)$ be a linear control system on G and $\psi : G \to H$ a surjective homomorphism. It holds:*

1. *If $\Sigma(\mathcal{X}, \Delta)$ is controllable, then $\Sigma(\mathcal{X}_\psi, \Delta_\psi)$ is controllable;*
2. *If $\ker\psi \subset \mathrm{cl}(\mathcal{A}) \cap \mathrm{cl}(\mathcal{A}^*)$ and $\Sigma(\mathcal{X}_\psi, \Delta_\psi)$ is controllable, then $\Sigma(\mathcal{X}, \Delta)$ is controllable;*
3. *If $\Sigma(\mathcal{X}, \Delta)$ satisfies the ad-rank condition then $\Sigma(\mathcal{X}, \Delta)$ is controllable if and only if $\Sigma(\widetilde{\mathcal{X}}, \Delta)$ is controllable.*

Proof. 1. It follows directly from the fact that $\psi(\mathcal{A}) = \mathcal{A}_\psi$ and $\psi(\mathcal{A}^*) = \mathcal{A}^*_\psi$.

2. It holds that $\psi^{-1}(\mathcal{A}_\psi) = \ker\psi \cdot \mathcal{A}$ and $\psi^{-1}(\mathcal{A}^*_\psi) = \ker\psi \cdot \mathcal{A}^*$. We know that $\ker\psi$ is invariant by the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ of $\mathcal{X}$, so, if $\ker\psi \subset \mathrm{cl}(\mathcal{A}) \cap \mathrm{cl}(\mathcal{A}^*)$ then $\{\varphi_t(g), t \in \mathbb{R}\} \subset \mathcal{A} \cap \mathcal{A}^*$ for all $g \in \ker\psi$. By Lemma 2.2 we get that $\ker\psi \cdot \mathcal{A} \subset \mathcal{A}$ and $\ker\psi \cdot \mathcal{A}^* \subset \mathcal{A}^*$ and therefore, if $\Sigma(\mathcal{X}_\psi, \Delta_\psi)$ is controllable we have

$$G = \psi^{-1}(H) = \psi^{-1}(\mathcal{A}_\psi \cap \mathcal{A}^*_\psi) = (\ker\psi \cdot \mathcal{A}) \cap (\ker\psi \cdot \mathcal{A}^*) \subset \mathcal{A} \cap \mathcal{A}^*$$

implying that $\Sigma(\mathcal{X}, \Delta)$ is controllable.

3. If $\Sigma(\widetilde{\mathcal{X}}, \Delta)$ is controllable, then by item 1. $\Sigma(\mathcal{X}, \Delta)$ is controllable. Reciprocally, since $\ker\pi$ is invariant by the flow of $\widetilde{\mathcal{X}}$ and it is a discrete subgroup we must have that $\widetilde{\mathcal{X}}(\ker\pi) = 0$ implying that $\ker\pi \subset \widetilde{G}^0$. If $\Sigma(\mathcal{X}, \Delta)$ is controllable and the ad-rank condition is satisfied, then necessarily $\widetilde{\mathcal{A}}$ is open which by Theorem 2.3 implies

$$\ker\pi \subset \widetilde{G}^0 \subset \widetilde{\mathcal{A}} \cap \widetilde{\mathcal{A}}^*$$

and by item 2. we have the controllability of $\Sigma(\widetilde{\mathcal{X}}, \Delta)$. □

We end this section with a result of Dath and Jouan characterizing the controllability of linear control systems on the two-dimensional solvable Lie group (see Theorem 3 of [6]), that it will be useful ahead.

2.7 Theorem. *Let G be the two-dimensional solvable Lie group and consider a linear control system $\Sigma(\mathcal{X}, \Delta)$ on G with $\dim\Delta = 1$. Then, $\Sigma(\mathcal{X}, \Delta)$ is controllable if and only if it satisfies the LARC and $\mathfrak{g} = \mathfrak{g}^0$.*

3. Three-dimensional solvable Lie groups

This section is devoted to analyze the main ingredients of nonnilpotent, solvable three-dimensional Lie groups and its corresponding Lie algebras such as derivations, linear vector fields, invariant vector fields and so on.

Following Chapter 7 of [14], any real three-dimensional nonnilpotent solvable Lie algebra is isomorphic to one (and only one) of the following Lie algebras:

(i) the semi-direct product $\mathfrak{r}_2 = \mathbb{R} \times_\theta \mathbb{R}^2$ where $\theta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$;

(ii) the semi-direct product $\mathfrak{r}_3 = \mathbb{R} \times_\theta \mathbb{R}^2$ where $\theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$;

(iii) the semi-direct product $\mathfrak{r}_{3,\lambda} = \mathbb{R} \times_\theta \mathbb{R}^2$ where $(\lambda \in \mathbb{R}, 0 < |\lambda| \leq 1)$ and $\theta = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$;

(iv) the semi-direct product $\mathfrak{r}'_{3,\lambda} = \mathbb{R} \times_\theta \mathbb{R}^2$ where $(\lambda \in \mathbb{R}, \lambda \neq 0)$ and $\theta = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$;

(v) the semi-direct product $\mathfrak{e} = \mathbb{R} \times_\theta \mathbb{R}^2$ where $\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The simply connected Lie groups R_3, $R_{3,\lambda}$, $R'_{3,\lambda}$, $\widetilde{E}$ and $\widetilde{R_2}$ with Lie algebras $\mathfrak{r}_3$, $\mathfrak{r}_{3,\lambda}$, $\mathfrak{r}'_{3,\lambda}$, $\mathfrak{e}$ and $\mathfrak{r}_2$, respectively, are given as the semi-direct product $\mathbb{R} \times_\rho \mathbb{R}^2$, where $\rho_t = \mathrm{e}^{t\theta}$.

Associated with $\mathfrak{e}$ we have also the groups $E_n := \widetilde{E}/D_n$ where $D_n := \{(2nk\pi, 0),\ k \in \mathbb{Z}\}$, $n \in \mathbb{N}$. The group E_1 is the group of proper motions of $\mathbb{R}^2$ (connected component of the whole group of motions of $\mathbb{R}^2$) and E_n its n-fold covering. Also, if we denote by $v_k = (2k\pi, 0) \in \mathbb{R}^2$ and consider the discrete central subgroup of R_2 given by $D = \{(0, v_k), k \in \mathbb{Z}\}$ we have the connected Lie group $R_2 = \widetilde{R_2}/D$.

In the above cases, the canonical projections are given by

$$\pi_n : \widetilde{E} \to E_n,\ \ \pi_n(t, v) = (\mathrm{e}^{i\frac{t}{n}}, v) \ \text{ and } \ \pi : \widetilde{R_2} \to R_2,\ \pi(t, (x, y)) = (t, (\mathrm{e}^{ix}, y))$$

and consequently

$$(d\pi_n)_{(t,v)}(a, w) = \left(\frac{a}{n}, w\right) \ \text{ and } \pi_* = \mathrm{id}. \tag{5}$$

Moreover, if G is a three-dimensional nonnilpotent, solvable, connected Lie group and $\widetilde{G}$ its simply connected covering it holds that:

(a) If $\widetilde{G} = \widetilde{R_2}$ then $G = \widetilde{G}$ or $G = R_2$;
(b) If $\widetilde{G} = R_3$, $R_{3,\lambda}$ or $R'_{3,\lambda}$ then $G = \widetilde{G}$;
(c) If $\widetilde{G} = \widetilde{E}$ then $G = E_n$ for some $n \in \mathbb{N}$.

With exception of the Lie group $\widetilde{E}$, all the three-dimensional nonnilpotent solvable Lie groups are exponential.

3.1 Remark. We denote by $\mathfrak{aff}(\mathbb{R})$ the only two-dimensional solvable Lie algebra. The associated connected Lie group is $\mathrm{Aff}_0(\mathbb{R})$, the connected component of the affine transformations in $\mathbb{R}$. For the Lie algebra $\mathfrak{r}_2$ it holds that $\mathfrak{r}_2 = \mathbb{R} \times \mathfrak{aff}(\mathbb{R})$ and consequently $\widetilde{R_2} = \mathbb{R} \times \mathrm{Aff}_0(\mathbb{R})$ and $R_2 = \mathbb{T} \times \mathrm{Aff}_0(\mathbb{R})$.

In what follows, we analyze the main properties of the above groups. Since we did not find the next results anywhere we present here their proofs in order to make the paper self-contained.

For any $s \in \mathbb{R}$ let us define Λ_s by

$$\Lambda_s := \begin{cases} (\rho_s - 1)\theta^{-1} & \text{if } \det\theta \neq 0 \\ \begin{pmatrix} s & 0 \\ 0 & \mathrm{e}^s - 1 \end{pmatrix} & \text{if } \det\theta = 0 \end{cases}.$$

A simple calculation shows that for any $t, s \in \mathbb{R}$ it holds that

$$\Lambda_0 = 0, \quad \frac{d}{ds}\Lambda_s = \rho_s, \quad \rho_s - \theta\Lambda_s = 1, \quad \rho_s\Lambda_t = \Lambda_t\rho_s \quad \text{and} \quad \Lambda_t + \rho_t\Lambda_s = \Lambda_{t+s}.$$

The above map will be extensively used in the next results.

3.2 Proposition. *If $G = \mathbb{R} \times_\rho \mathbb{R}^2$ then*

$$(dL_{(\tau_1,v_1)})_{(\tau_2,v_2)}(s, w) = (s, \rho_{\tau_1} w) \quad \textit{and} \quad (dR_{(\tau_1,v_1)})_{(\tau_2,v_2)}(s, w) = (s, w + s\theta\rho_{\tau_2} v_1)$$

and consequently

$$\exp(s, w) = \begin{cases} (0, w) & \textit{if } a = 0 \\ \left(s, \frac{1}{s}\Lambda_s w\right), & \textit{if } s \neq 0 \end{cases}$$

Proof. We show the expressions for the left translation since for the right translation are analogous. The curve $\gamma(t) = (\tau_2, v_2) + t(s, w) \in G$ satisfies that $\gamma(0) = (\tau_2, v_2)$ and $\gamma'(0) = (s, w)$ and therefore

$$\begin{aligned}(dL_{(\tau_1,v_1)})_{(\tau_2,v_2)}(s, w) &= \frac{d}{dt}_{|t=0} L_{(\tau_1,v_1)}(\gamma(t)) = (\tau_1, v_1)(\tau_2 + ts, v_2 + tw) \\ &= \frac{d}{dt}_{|t=0}(\tau_1 + \tau_2 + ts, v_1 + \rho_{\tau_1}(v_2 + tw)) = (s, \rho_{\tau_1} w).\end{aligned}$$

To prove the assertion on the exponential, let us consider $(s, w) \in \mathfrak{g}$ and define the curve

$$\zeta(t) := \begin{cases} (0, tw) & \text{if } s = 0 \\ \left(ts, \frac{1}{s}\Lambda_{ts} w\right), & \text{if } s \neq 0 \end{cases}.$$

Since in both cases $\zeta(0) = 0$ and

$$\zeta'(t) = \begin{cases} \frac{d}{dt}(0, tw) = (0, w) = (dL_{\zeta(t)})_{(0,0)}(0, w) & \text{if } s \neq 0 \\ \frac{d}{dt}\left(ts, \frac{1}{s}\Lambda_{ts} w\right) = (s, \rho_{ts} w) = (dL_{\zeta(t)})_{(0,0)}(s, w), & \text{if } s \neq 0 \end{cases},$$

by unicity we obtain that $\zeta(t) = \exp t(s, w)$ concluding the proof. □

3.3 Remark. The above result and equation (5) imply that the right and left invariant vector fields on any connected solvable nonnilpotent Lie group G are given respectively by

$$Y^L(\pi(t, v)) = (a, \rho_t w) \quad \text{and} \quad Y^R(\pi(t, v)) = (a, w + a\theta v)$$

where $Y = (a, w) \in \mathfrak{g}$ and $\pi : \widetilde{G} \to G$ is the canonical projection.

Let $\mathcal{X}$ be a linear vector field on $G = \mathbb{R} \times_\rho \mathbb{R}^2$ and denote by $\mathcal{D}$ its associated derivation. Since $\mathcal{D}(\mathbb{R} \times_\theta \mathbb{R}^2) \subset \mathbb{R}^2$ we have a well defined linear map $\mathcal{D}^* : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $\mathcal{D}(0, v) = (0, \mathcal{D}^* v)$ for $v \in \mathbb{R}^2$. The map $\mathcal{D}^*$ satisfies $\mathcal{D}^* \circ \theta = \theta \circ \mathcal{D}^*$. In fact, for any $v \in \mathbb{R}^2$ it holds that

$$(0, \mathcal{D}^*\theta v) = \mathcal{D}(0, \theta v) = \mathcal{D}[(1,0),(0,v)] = \underbrace{[\mathcal{D}(1,0),(0,v)]}_{=0} + [(1,0), \mathcal{D}(0,v)] = (0, \theta\mathcal{D}^* v)$$

and therefore $\mathcal{D}^* \circ \theta = \theta \circ \mathcal{D}^*$. As a consequence, $\mathcal{D}^* \rho_t = \rho_t \mathcal{D}^*$ for any $t \in \mathbb{R}$.

3.4 Proposition. *Let G be a three-dimensional nonnilpotent, solvable, connected Lie group and denote by $\widetilde{G}$ its simply connected covering. If $\mathcal{X}$ is a linear vector field on G with associated derivation $\mathcal{D}$ then*

$$\mathcal{X}(\pi(t,v)) = \left(0, \mathcal{D}^* v + \Lambda_t \xi\right), \quad \textit{where } (0,\xi) = \mathcal{D}(1,0)$$

and $\pi : \widetilde{G} \to G$ is the canonical projection.

Proof. Let us first consider the case where $G = \mathbb{R} \times_\rho \mathbb{R}^2$. Since

$$\mathcal{X}(t,v) = \mathcal{X}((t,0)(0,\rho_{-t}v)) = (dL_{(t,0)})_{(0,\rho_{-t}v)}\mathcal{X}(0,\rho_{-t}v) + (dR_{(0,\rho_{-t}v)})_{(t,0)}\mathcal{X}(t,0)$$

it is enough to compute the values of $\mathcal{X}(t,0)$ and $\mathcal{X}(0,\rho_{-t}v)$. Moreover, the fact that $(t,0) = \exp(t,0)$ and $(0,w) = \exp(0,w)$ implies that

$$\varphi_s(0,\rho_{-t}v) = \varphi_s(\exp(0,w)) = \exp\left(\mathrm{e}^{s\mathcal{D}}(0,\rho_{-t}v)\right) = \exp(0, \mathrm{e}^{s\mathcal{D}^*}\rho_{-t}v) = (0, \mathrm{e}^{s\mathcal{D}^*}\rho_{-t}v)$$

and

$$\varphi_s(t,0) = \varphi_s(\exp(t,0)) = \exp\left(\mathrm{e}^{s\mathcal{D}}(t,0)\right) = \exp t\left(1, \sum_{j\geq 0}\frac{s^{j+1}}{(j+1)!}(\mathcal{D}^*)^j\xi\right)$$
$$= \left(t, \sum_{j\geq 0}\frac{s^{j+1}}{(j+1)!}\Lambda_t\left((\mathcal{D}^*)^j\xi\right)\right).$$

Therefore,

$$\mathcal{X}(0,\rho_{-t}v) = \frac{d}{ds}_{|s=0}\varphi_s(0,\rho_{-t}v) = (0, \mathcal{D}^*\rho_{-t}v), \quad \text{and} \quad \mathcal{X}(t,0) = \frac{d}{ds}_{|s=0}\varphi_s(t,0) = (0, \Lambda_t\xi).$$

By using the formulas in Proposition 3.2 we get that

$$\mathcal{X}(t,v) = \left(0, \mathcal{D}^* v + \Lambda_t\xi\right)$$

proving the assertion for the simply connected case.

If G is not simply connected, we can consider the linear vector field $\widetilde{\mathcal{X}}$ on $\widetilde{G}$ that is π-related to $\mathcal{X}$. Since $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ have the same associated derivation $\mathcal{D}$ we have by equation (5) and the above calculations that

$$\mathcal{X}(\pi(t,v)) = (d\pi)_{(t,v)}\widetilde{\mathcal{X}}(t,v) = (0, \mathcal{D}^* v + \Lambda_t \xi)$$

concluding the proof. □

3.5 Remark. With the notation of the above proposition, it holds that the flow associated to $\mathcal{X}$ on $\widetilde{G}$ is given by

$$\varphi_s(t,v) = (t, \mathrm{e}^{s\mathcal{D}^*} v + F_s \Lambda_t \xi), \quad \text{where} \quad F_s = \sum_{j\geq 1} \frac{s^j (\mathcal{D}^*)^{j-1}}{j!}. \tag{6}$$

In fact, since $\mathcal{D}^* F_s = F_s \mathcal{D}^* = \mathrm{e}^{s\mathcal{D}^*} - 1$ and $F_s' = \mathrm{e}^{s\mathcal{D}^*}$ we get that

$$\frac{d}{ds}\varphi_s(t,v) = \left(0, \mathcal{D}^* \mathrm{e}^{s\mathcal{D}^*} v + \mathrm{e}^{s\mathcal{D}^*}\Lambda_t\xi\right) = \left(0, \mathcal{D}^*\Big(\underbrace{\mathrm{e}^{s\mathcal{D}^*} v + F_s \Lambda_t \xi}_{=\varphi_s(t,v)}\Big) + \Lambda_t \xi\right) = \mathcal{X}(\varphi_s(t,v)).$$

In particular, if $\mathcal{D}^*$ is invertible we obtain $F_s = (\mathrm{e}^{s\mathcal{D}^*} - 1)(\mathcal{D}^*)^{-1}$.

The next technical lemma will be useful in the proof of the main results.

3.6 Lemma. *Let G be a three-dimensional solvable nonnilpotent connected Lie group. For any $v_0 \in \mathbb{R}^2$ there exists $\psi \in \mathrm{Aut}(G)$ satisfying $(d\psi)_e(1, v_0) = (1, 0)$.*

Proof. Let us first consider the simply connected case $\widetilde{G} = \mathbb{R} \times_\rho \mathbb{R}^2$. The map $\psi : \widetilde{G} \to \widetilde{G}$ given by

$$\psi(t,v) = (t, v - \Lambda_t v_0) \quad \text{has inverse} \quad \psi^{-1}(t,v) = (t, v + \Lambda_t v_0)$$

and satisfies

$$\begin{aligned}
\psi(t_1,v_1)\psi(t_2,v_2) &= \left(t_1, v_1 - \Lambda_{t_1} v_0\right)\left(t_2, v_2 - \Lambda_{t_2} v_0\right) \\
&= \left(t_1 + t_2, v_1 - \Lambda_{t_1} v_0 + \rho_{t_1}\left(v_2 - \Lambda_{t_2} v_0\right)\right) \\
&= \left(t_1 + t_2, v_1 + \rho_{t_1} v_2 - \Lambda_{t_1+t_2} v_0\right) = \psi(t_1 + t_2, v_1 + \rho_{t_1} v_2) \\
&= \psi\left((t_1,v_1)(t_2,v_2)\right)
\end{aligned}$$

implying that $\psi \in \mathrm{Aut}(\widetilde{G})$. Moreover,

$$(d\psi)_{\tilde{e}}(1, v_0) = \frac{d}{ds}_{|s=0}\psi(s, sv_0) = \frac{d}{ds}_{|s=0}(s, sv_0 - \Lambda_s v_0) = (1, v_0 - \rho_s v_0)|_{s=0} = (1, 0),$$

proving the result for any $\widetilde{G}$ is simply connected.

If G is not simply connected, one easily shows that the above automorphism satisfies $\psi(D_n) = D_n$, $n \in \mathbb{N}$ and $\psi(D) = D$, where $D_n, n \in \mathbb{N}$ and D are the discrete central subgroups satisfying $E_n = \widetilde{E}/D_n$ and $R_2 = \widetilde{R_2}/D$, respectively. Therefore, ψ factors to an element in $\mathrm{Aut}(G)$ whose differential coincides with $(d\psi)_e$, which proves the result. □

The next lemma states the main properties of derivations in the three-dimensional Lie algebras under consideration.

3.7 Lemma. *Let $\mathfrak{g} = \mathbb{R} \times_\theta \mathbb{R}^2$ and let $\mathcal{D} : \mathfrak{g} \to \mathfrak{g}$ be a derivation. It holds:*

1. $\dim \mathfrak{g}^0 = 1$ *if and only if $\mathcal{D}^*$ is invertible;*
2. $\mathbb{R}^2 \subset \ker \mathcal{D}$ *then $\mathcal{D}$ is nilpotent;*
3. *Any derivation on $\mathfrak{r}_3$ or on $\mathfrak{r}_{3,\lambda}$ with $0 < |\lambda| < 1$ has only real eigenvalues. Therefore, if $\mathcal{D}^*$ admits a pair of complex eigenvalues we must have $\lambda = 1$;*
4. *Any derivation $\mathcal{D}$ of $\mathfrak{r}'_{3,\lambda}$ or of $\mathfrak{e}$ satisfies* $\mathcal{D}^* = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.

Proof. 1. Since $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$, if $\dim \mathfrak{g}^0 = 1$ we have that $\dim(\mathfrak{g}^+ \oplus \mathfrak{g}^-) = 2$. By Lemma 2.1 it holds that $\mathfrak{g}^+ \oplus \mathfrak{g}^- \subset \mathbb{R}^2$ and consequently $\mathbb{R}^2 = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ implying that $\mathcal{D}^*$ is invertible. Reciprocally, since $\mathcal{D}$ invertible implies $\mathfrak{g}$ nilpotent we must have necessarily that $\dim \mathfrak{g}^0 \geq 1$. The fact that the eigenvalues of $\mathcal{D}^*$ are also eigenvalues of $\mathcal{D}$ implies then that $\dim \mathfrak{g}^0 = 1$ if $\mathcal{D}^*$ is invertible.

2. In fact, if $\mathbb{R}^2 \subset \ker \mathcal{D}$ then necessarily $\mathcal{D}^* \equiv 0$ and consequently $\mathcal{D}^2(\mathfrak{g}) \subset \mathcal{D}^*\mathbb{R}^2 = \{0\}$ showing that $\mathcal{D}$ is nilpotent.

3. Since all the nonzero eigenvalues of $\mathcal{D}$ are also eigenvalues of $\mathcal{D}^*$ and $\mathcal{D}^*$ commutes with θ it holds that

$$\mathcal{D}^* = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad \mathcal{D}^* = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

when $\mathcal{D}$ is a derivation of $\mathfrak{r}_3$ and $\mathfrak{r}_{3,\lambda}$, $|\lambda| \in (0, 1)$, respectively.

4. It follows directly from item 3.

5. It follows from the fact that $\mathcal{D}^*$ and θ commutes. □

We have also the following.

3.8 Lemma. *The only two-dimensional Lie subalgebra of $\mathfrak{e}$ or $\mathfrak{r}'_{3,\lambda}$ is the nilradical $\mathbb{R}^2$.*

Proof. In fact, if $\mathfrak{h}$ is a two-dimensional Lie subalgebra of $\mathfrak{g}$, where $\mathfrak{g} \in \{\mathfrak{e}, \mathfrak{r}'_{3,\lambda}\}$ then necessarily $\dim(\mathfrak{h} \cap \mathbb{R}^2) \geq 1$ and consequently $(0, w) \in \mathfrak{h}$. If $(t, w') \in \mathfrak{h}$ is such that $\mathfrak{h} = \mathrm{span}\{(t, w'), (0, w)\}$ then

$$[(t, w'), (0, w)] = (0, t\theta w) \in \mathfrak{h}$$

and hence $t\theta w = t'w$. Since θ does not admit any invariant one-dimensional subspace we must have that $t = 0$ and consequently $\mathfrak{h} = \mathbb{R}^2$ as desired. □

Concerning linear systems on nonnilpotent, solvable three-dimensional Lie groups, the next result states that we can concentrate our studies to two specific kind of systems.

3.9 Proposition. *Any linear control system $\Sigma(\mathcal{X}, \Delta)$ on a three-dimensional, solvable, connected, nonnilpotent Lie group G that satisfies the LARC is equivalent to one of the following linear systems:*

$$\dot{g} = \mathcal{X}(g) + uY^1(g) \quad \textit{or} \quad \dot{g} = \mathcal{X}(g) + u_1Y^1(g) + u_2Y^2(g),$$

where $Y_1 = (1, 0)$ and $Y_2 = (0, w)$, for some $w \in \mathbb{R}^2$.

Proof. We only have to analyze the cases where $\dim \mathfrak{h} = 1$ or 2. For both cases, the fact that $\Sigma(\mathcal{X}, \Delta)$ satisfies the LARC implies that $\Delta \not\subset \mathbb{R}^2$ and hence:

1. If $\dim \Delta = 1$ then $\Delta = \text{span}\{(1, v_0)\}$ for some $v_0 \in \mathbb{R}^2$.
2. If $\dim \Delta = 2$ then $\Delta = \text{span}\{(1, v_0), (0, w)\}$ for some $v_0, w \in \mathbb{R}^2$.

By considering the isomorphism ψ given by Proposition 3.6 we get that $\Sigma(\mathcal{X}, \Delta)$ is equivalent to the system $\Sigma(\mathcal{X}_\psi, \Delta_\psi)$ that has necessarily the form

$$\dot{g} = \mathcal{X}(g) + uY^1(g) \quad \text{or} \quad \dot{g} = \mathcal{X}(g) + u_1Y^1(g) + u_2Y^2(g),$$

for $Y_1 = (1, 0)$ and $Y_2 = (0, w)$, for some $w \in \mathbb{R}^2$, concluding the proof. □

3.1. Homogeneous space

In this section we analyze homogeneous space of the three-dimensional nonnilpotent solvable Lie groups which will be used in the sections ahead. Our particular interest are the projections of linear and invariant vector fields to these homogeneous spaces.

• *$G = \mathbb{R} \times_\rho \mathbb{R}^2$ and $\mathcal{D}^*$ is invertible*

For this case, we will consider the group of the singularities of $\mathcal{X}$ given by $F = \{(t, v) \in G;\ \mathcal{X}(t, v) = 0\}$. Since $\mathcal{D}^*$ is invertible, it holds that F is a one-dimensional closed Lie subgroup of G. Moreover, by Proposition 3.4 it holds that

$$(t, v) \in F \iff \mathcal{D}^*v = -\Lambda_t\xi, \quad \text{where } (0, \xi) = \mathcal{D}(1, 0).$$

Hence,

$$F(t_1, v_1) = F(t_2, v_2) \iff \rho_{-t_1}\mathcal{D}^*v_1 - \Lambda_{-t_1}\xi = \rho_{-t_2}\mathcal{D}^*v_2 - \Lambda_{-t_2}\xi.$$

Consequently, we can identify the homogeneous space $F \setminus G$ with $\mathbb{R}^2$ by using the map

$$F \cdot (t, v) \in F \setminus G \mapsto \rho_{-t}\mathcal{D}^*v - \Lambda_{-t}\xi \in \mathbb{R}^2.$$

Under this identification, the projection $\pi : G \to F \setminus G$ is given by

$$\pi(t,v)=\rho_{-t}\mathcal{D}^*v-\Lambda_{-t}\xi \quad \text{and its differential by}$$
$$(d\pi)_{(t,v)}(a,w)=-a\rho_{-t}(\mathcal{D}^*\theta v-\xi)+\rho_{-t}\mathcal{D}^*w.$$

Therefore, we have that

$$(d\pi)_{(t,v)}\mathcal{X}(t,v)=\mathcal{D}^*\pi(t,v) \quad \text{and} \quad (d\pi)_{(t,v)}Y^L(t,v)=-a(\theta\pi(t,v)-\xi)+\mathcal{D}^*w \tag{7}$$

where $Y=(a,w)\in\mathfrak{g}$.

3.10 Remark. It is not hard to see that if we consider the above setup on E_n, $n\in\mathbb{N}$, both, the homogeneous space $F\setminus G$ and the projection $\pi: G\to F\setminus G$ have the same expression.

- *$G=\mathbb{R}\times_\rho\mathbb{R}^2$ and $\mathcal{D}^*$ is identically zero*
  Let $w_0\in\mathbb{R}^2$ and consider the one-parameter subgroup of $(0,w_0)$ given by

$$H_{w_0}=\{\exp s(0,w_0),\ s\in\mathbb{R}\}=\{(0,sw_0),\ \ s\in\mathbb{R}\}.$$

It holds that

$$H_{w_0}(t_1,v_1)=H_{w_0}(t_2,v_2) \iff t_1=t_2 \ \text{ and } v_2-v_1\in\operatorname{span}\{w_0\}.$$

If we consider $v_0\in\mathbb{R}^2$ such that $\langle w_0,v_0\rangle=0$ then $v_2-v_1\in\operatorname{span}\{w_0\}\iff\langle v_1,v_0\rangle=\langle v_2,v_0\rangle$ and consequently we can identify the homogeneous space $H_{w_0}\setminus G$ with $\mathbb{R}^2$ using the map

$$H_{w_0}\cdot(t,v)\in H_{w_0}\setminus G\mapsto(t,\langle v,v_0\rangle)\in\mathbb{R}^2.$$

Under this identification, the projection $\pi: G\to H_{w_0}\setminus G$ is given by

$$\pi(t,v)=(t,\langle v,v_0\rangle) \quad \text{and since it is linear} \quad (d\pi)_{(t,v)}=\pi.$$

We obtain,

$$(d\pi)_{(t,v)}\mathcal{X}(t,v)=(0,\langle\Lambda_t\xi,v_0\rangle) \quad \text{and} \quad (d\pi)_{(t,v)}Y^L(t,v)=(a,\langle\rho_t w,v_0\rangle), \tag{8}$$

where $Y=(a,w)\in\mathfrak{g}$.

- *$G=R_2$ and $\mathcal{D}^*$ is identically zero*
  Let $w_0\in\mathbb{R}^2$ and assume that $w_0=(\alpha,\beta)$ with $\alpha,\beta\in\mathbb{R}^*$. The one-parameter subgroup of $(0,w_0)$ is given by

$$H_{w_0}=\{\exp s(0,w_0),\ s\in\mathbb{R}\}=\{(0,(\mathrm{e}^{is\alpha},s\beta)),\ \ s\in\mathbb{R}\}.$$

Then

$$\begin{aligned}H_{w_0}(t_1,v_1)=H_{w_0}(t_2,v_2) &\iff t_1=t_2\ ,\mathrm{e}^{i(x_1+s\alpha)}=\mathrm{e}^{ix_2} \ \text{ and } \ y_1+s\beta=y_2\\ &\iff t_1=t_2 \ \text{ and } \ \mathrm{e}^{i(x_1-\frac{\alpha}{\beta}y_1)}=\mathrm{e}^{i(x_2-\frac{\alpha}{\beta}y_2)}\end{aligned}$$

If we consider $v_0 \in \mathbb{R}^2$ such that $\langle w_0, v_0\rangle = 0$ then $v_0 = (\beta, -\alpha)$. Consequently, we can identify the homogeneous space $H_{w_0} \setminus G$ with $\mathbb{R} \times \mathbb{T}$ using the map

$$H_{w_0} \cdot (t, v) \in H_{w_0} \setminus G \mapsto \left(t, e^{i\beta^{-1}\langle v, v_0\rangle}\right) \in \mathbb{R} \times \mathbb{T}.$$

Under this identification, the projection $\pi : G \to H_{w_0} \setminus G$ is given by

$$\pi(t, v) = \left(t, e^{i\beta^{-1}\langle v, v_0\rangle}\right) \quad \text{and its differential by} \quad (d\pi)_{(t,v)}(a, w) = (a, \beta^{-1}\langle v, v_0\rangle).$$

In particular, we get

$$(d\pi)_{(t,v)}\mathcal{X}(t, v) = (0, \beta^{-1}\langle \Lambda_t \xi, v_0\rangle) \quad \text{and} \quad (d\pi)_{(t,v)}Y^L(t, v) = (a, \beta^{-1}\langle \rho_t w, v_0\rangle), \tag{9}$$

where $Y = (a, w) \in \mathfrak{g}$.

4. Controllability

In this section we analyze the controllability property of linear control systems on three-dimensional nonnilpotent solvable Lie groups. Since the LARC is a necessary condition for controllability our work is reduced to the analysis of linear systems on $\Sigma(\mathcal{X}, \Delta)$, where $\dim \Delta = 1$ or 2.

4.1. The one-dimensional case

In this section we analyze the case where $\dim \Delta = 1$. In this context, the next theorem summarizes the controllability of linear control systems on the different classes of three-dimensional nonnilpotent, solvable Lie groups. Its proof will be divided in several propositions.

4.1 Theorem. *Let $\Sigma(\mathcal{X}, \Delta)$ be a linear system on a three-dimensional nonnilpotent solvable Lie group G that satisfies the LARC and* $\dim \Delta = 1$. *It holds:*

1. *If $G = R_2$: $\Sigma(\mathcal{X}, \Delta)$ is controllable if and only if $\mathfrak{g}^0 \simeq \mathfrak{aff}(\mathbb{R})$ or $\mathfrak{g} = \mathfrak{g}^0$;*
2. *If $G = \widetilde{R_2}$: $\Sigma(\mathcal{X}, \Delta)$ is controllable if and only if $\mathfrak{g}^0 = \mathfrak{aff}(\mathbb{R})$;*
3. *If $G = E_n, \widetilde{E}$ or R_3: $\Sigma(\mathcal{X}, \Delta)$ is controllable if and only if $\mathfrak{g} = \mathfrak{g}^0$ and $\mathcal{D}^* \not\equiv 0$;*
4. *If $G = R_{3,\lambda}$: $\Sigma(\mathcal{X}, \Delta)$ is controllable if and only if $\lambda = 1$ and $\mathcal{D}^*$ has a pair of complex eigenvalues;*
5. *If $G = R'_{3,\lambda}$: $\Sigma(\mathcal{X}, \Delta)$ is controllable.*

In the sequel, we prove the above theorem analyzing case by case.

4.1.1. The case $G = R_2$ or $G = \widetilde{R_2}$

4.2 Proposition. *Let $\Sigma(\mathcal{X}, \Delta)$ be a linear control system on G. Then, $\Sigma(\mathcal{X}, \Delta)$ is controllable if and only if it satisfies the LARC and*

(i) *$G = \widetilde{R_2}$ and $\mathfrak{g}^0 \simeq \mathfrak{aff}(\mathbb{R})$;*
(ii) *$G = R_2$ and $\mathfrak{g}^0 \simeq \mathfrak{aff}(\mathbb{R})$ or $\mathfrak{g} = \mathfrak{g}^0$.*

Proof. Let us start by proving the following facts:

a) *If* $\Sigma(\widetilde{\mathcal{X}}, \Delta)$ *is a linear control system on* $\widetilde{R_2}$ *which satisfies the LARC and* $\mathfrak{g}^0 \simeq \mathfrak{aff}(\mathbb{R})$ *then, it is controllable.*

In fact, if $\mathfrak{g}^0 \simeq \mathfrak{aff}(\mathbb{R})$ then necessarily $\mathfrak{g} = \mathbb{R}e_1 \oplus \mathfrak{g}^0$ implying that $\mathfrak{g}^0$ is an ideal of $\mathfrak{g}$. Hence, $\mathcal{D}e_1 = \lambda e_1$ with $\lambda \in \mathbb{R}^*$ and $\mathcal{D}e_2 = 0$. Moreover, if $\Sigma(\widetilde{\mathcal{X}}, \Delta)$ satisfies the LARC and $(0, \xi) = \mathcal{D}(1, 0)$ we must have $\xi = ae_1 + be_2$ with $a, b \in \mathbb{R}^*$. Thus, $\mathcal{D}^2(1, 0) = (0, a\lambda e_1)$ showing that $\{(1, 0), \mathcal{D}(1, 0), \mathcal{D}^2(1, 0)\}$ is a basis for $\mathfrak{g}$ and therefore that $\Sigma(\widetilde{\mathcal{X}}, \Delta)$ satisfies the ad-rank condition.

Since $\mathfrak{g}^0$ is an ideal of $\mathfrak{g}$ we can consider the induced linear system on $G/\widetilde{G}^0 \simeq \mathbb{R}$ that is controllable by the ad-rank condition (see Theorem 3 of [18]). Moreover, the ad-rank condition implies also by Theorem 2.3 that $G^0 \subset \mathcal{A} \cap \mathcal{A}^*$ which by Lemma 2.2 gives us the controllability $\Sigma(\widetilde{\mathcal{X}}, \Delta)$.

b) *If* $\Sigma(\mathcal{X}, \Delta)$ *is a linear system on* R_2 *which satisfies the LARC and* $\mathfrak{g}^0 \simeq \mathfrak{aff}(\mathbb{R})$ *then* Σ *cannot be controllable.*

In fact, for such system we have the following possibilities:

- $\dim \mathfrak{g}^0 = 1$: By Lemma 3.7 the linear map $\mathcal{D}^*$ is invertible and consequently, the induced system on $G/\mathbb{T} \simeq \mathrm{Aff}_0(\mathbb{R})$ cannot be controllable by Theorem 2.7.
- $\dim \mathfrak{g}^0 = 2$: In this case $\mathfrak{g}^0$ is Abelian and necessarily $e_1 \in \mathfrak{g}^0$ since otherwise the whole Lie algebra $\mathfrak{g}$ would be Abelian. Moreover, since $\mathcal{D}e_3 = 0$ would imply $\mathfrak{g} = \mathfrak{g}^0$ we must have $\mathcal{D}e_3 = \mu e_3$ for some $\mu \in \mathbb{R}^*$. As in the previous item, we do not have controllability of the induced system on $G/\mathbb{T} \simeq \mathrm{Aff}_0(\mathbb{R})$.

Let us now consider π-related linear control systems $\Sigma(\widetilde{\mathcal{X}}, \Delta)$ and $\Sigma(\mathcal{X}, \Delta)$ on $\widetilde{R_2}$ and R_2 respectively, where $\pi : \widetilde{R_2} \to R_2$ is the canonical projection. Note that $\Sigma(\widetilde{\mathcal{X}}, \Delta)$ satisfies the LARC if and only if $\Sigma(\mathcal{X}, \Delta)$ also satisfies it.

(i) By item a) above, if $\mathfrak{g}^0 \simeq \mathfrak{aff}(\mathbb{R})$ and $\Sigma(\widetilde{\mathcal{X}}, \Delta)$ satisfies the LARC then it is controllable. Reciprocally, if $\dim \mathfrak{g}^0 = 1$ or $\dim \mathfrak{g}^0 = 2$ and $\mathfrak{g}^0$ is Abelian, $\Sigma(\mathcal{X}, \Delta)$ is not controllable by item b) and consequently $\Sigma(\widetilde{\mathcal{X}}, \Delta)$ cannot be controllable. The only remaining possibility is $\mathfrak{g} = \mathfrak{g}^0$. Since in this case we necessarily have $\mathcal{D}^* \equiv 0$, for any $w_0 \in \mathbb{R}^2$, we can consider the induced system on $\widetilde{H}_{w_0} \setminus \widetilde{G}$ for $w_0 \in \mathbb{R}^2$ given by

$$\begin{cases} \dot{t} = u \\ \dot{z} = \langle \Lambda_t \xi, v_0 \rangle \end{cases}, \quad \text{where} \quad \langle w_0, v_0 \rangle = 0. \tag{10}$$

By assuming that $\Sigma(\widetilde{\mathcal{X}}, \Delta)$ satisfies the LARC it holds that $\xi = (\xi_1, \xi_2)$ with $\xi_1, \xi_2 \in \mathbb{R}^*$. By considering $w_0 = (-\xi_2^{-1}, \xi_1^{-1})$ we get that

$$\dot{z} = \langle \Lambda_t \xi, v_0 \rangle = 1 + t - \mathrm{e}^t \leq 0$$

implying that (10) cannot be controllable since the region $\mathcal{C} = \{(z, t);\ z \leq 0\}$ is invariant by its solutions. Consequently, $\Sigma(\widetilde{\mathcal{X}}, \Delta)$ cannot be controllable concluding the proof of case (i).

(ii) By item a) if $\mathfrak{g}^0 \simeq \mathfrak{aff}_0(\mathbb{R})$ and $\Sigma(\mathcal{X}, \Delta)$ satisfies the LARC it follows that $\Sigma(\widetilde{\mathcal{X}}, \Delta)$ is controllable and consequently $\Sigma(\mathcal{X}, \Delta)$ is controllable. By item b) $\Sigma(\mathcal{X}, \Delta)$ cannot be controllable when $\dim \mathfrak{g}^0 = 1$ or when $\dim \mathfrak{g}^0 = 2$ and $\mathfrak{g}^0$ is Abelian. Therefore, we only have to show that $\mathfrak{g} = \mathfrak{g}^0$ together with the LARC implies the controllability of $\Sigma(\mathcal{X}, \Delta)$.

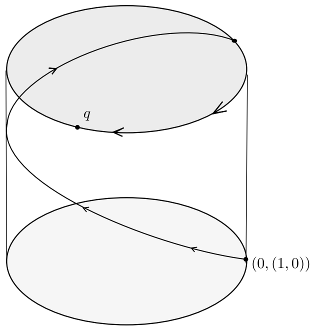

Fig. 1. Solutions connecting $(0, (1, 0))$ to $q \in \mathbb{R} \times \mathbb{T}$.

In this case, for any $w_0 = (\alpha, \beta) \in \mathbb{R}^2$ with $\alpha, \beta \in \mathbb{R}^*$ we have the induced system on $H_{w_0} \setminus G \simeq \mathbb{R} \times \mathbb{T}$ given by

$$\begin{cases} \dot{t} = u \\ \dot{z} = \beta^{-1} \langle \Lambda_t \xi, v_0 \rangle \end{cases}, \quad \text{where} \quad \langle w_0, v_0 \rangle = 0. \tag{11}$$

Since $\Sigma(\mathcal{X}, \Delta)$ satisfies the LARC we must have that $\mathcal{D}(1, 0) = (0, \xi)$ with $\xi = (\xi_1, \xi_2)$ with $\xi_1, \xi_2 \in \mathbb{R}^*$ and hence, by considering $w_0 = \xi$ we get that $v_0 = (-\xi_2, \xi)$ and

$$\dot{z} = \xi_2^{-1} \langle \Lambda_t \xi, v_0 \rangle = \xi_1 (\mathrm{e}^t - t - 1).$$

For such system we have the following equalities

$$\begin{array}{ll} \pi_1 (\varphi(s, (t, z), u)) = t + us, & \text{if } u \equiv \text{cte} \\ \varphi(s, (t, z), u) = (t, z \cdot \mathrm{e}^{is\xi_1(\mathrm{e}^t - t - 1)}) & \text{if } u \equiv 0 \end{array}$$

where $\pi_1 : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ is the projection onto the first coordinate. For any given $q = (a, z) \in \mathbb{R} \times \mathbb{T}$ with $a \neq 0$ we construct a trajectory from $(0, (1, 0))$ to q as follows:

1. We go from $(0, (1, 0))$ to a point $q' = (a, z')$ by using a constant control function;
2. By "switching off" the control we can go from $q' = (a, z')$ to $q = (a, z)$, since $z' \cdot \mathrm{e}^{is\xi_1(\mathrm{e}^a - a - 1)}$ covers the whole $\mathbb{T}$ if $a \neq 0$ (Fig. 1).

Since $\Sigma(\mathcal{X}, \Delta)$ projects to the system (11), the projection of $\mathcal{A}$ and $\mathcal{A}^*$ are dense in $H_\xi \setminus G$ and consequently $H_\xi \cdot \mathcal{A}$ and $H_\xi \cdot \mathcal{A}^*$ are dense in G.

On the other hand, for any $t, s \in \mathbb{R}$ it holds that

$$\varphi_t(s(1, 0)) = \varphi_t(\exp s(1, 0)) = \exp s(\mathrm{e}^{\mathcal{D}}(1, 0)) = \exp s(1, t\xi) = (s, t\Lambda_s \xi).$$

In particular, for any $r > 0$ we can consider $t = r/|s|$ and so $\left(s, \frac{rs}{|s|} \frac{1}{s} \Lambda_s v_0\right) = \varphi_t(s(1, 0)) \in \mathcal{A}$ for any $s \in \mathbb{R}$. By considering $s \to 0$ from both sides we get that $(0, \pm r\xi) \in \overline{\mathcal{A}}$ and since $r > 0$

was arbitrary we conclude that $H_\xi \subset \overline{\mathcal{A}}$ which by Lemma 2.2 implies also that $H_\xi \subset \overline{\mathcal{A}^*}$ and consequently

$$G = \overline{H_\xi \cdot \mathcal{A}} \subset \overline{\mathcal{A}} \quad \text{and } G = \overline{H_\xi \cdot \mathcal{A}^*} \subset \overline{\mathcal{A}^*}.$$

Since $\Sigma(\mathcal{X}, \Delta)$ satisfies the LARC $G = \overline{\mathcal{A}} \cap \overline{\mathcal{A}^*}$ implies $G = \mathcal{A} \cap \mathcal{A}^*$ concluding the proof. □

4.1.2. The case $G = E_n$ or $G = \widetilde{E_2}$

4.3 Proposition. *A linear control system $\Sigma(\mathcal{X}, \Delta)$ on G is controllable if and only if it satisfies the LARC and $\mathcal{D}$ has a pair of purely imaginary eigenvalues.*

Proof. If $\Sigma(\mathcal{X}, \Delta)$ satisfies the LARC then $\mathcal{D}(1, 0) = (0, \xi) \neq 0$. If we also assume that $\mathcal{D}$ has a pair of purely imaginary eigenvalues, then $\mathfrak{g} = \mathfrak{g}^0$ and $\{\xi, \mathcal{D}^*\xi\}$ is linearly independent implying that $\Sigma(\mathcal{X}, \Delta)$ satisfies the ad-rank condition, which by Theorem 2.3 implies its controllability.

Reciprocally, let us then assume that $\mathcal{D}$ does not admit a pair of purely imaginary eigenvalues. By Proposition 3.7 the eigenvalues of $\mathcal{D}^*$ are of the form $\alpha \pm i\beta$. If $\Sigma(\mathcal{X}, \Delta)$ satisfies the LARC then we can assume w.l.o.g. that $(1, 0) \in \Delta$ and we have the following possibilities:

• If $\alpha \neq 0$ we have that $\mathcal{D}^*$ is invertible and so, the induced system on the homogeneous space $F \setminus G$ is given by $\dot{v} = \mathcal{D}^* v - u(\theta v - \xi)$ which in coordinates reads as

$$\begin{cases} \dot{x} = \alpha x - \beta y + u(y + \xi_1) \\ \dot{y} = \beta x + \alpha y - u(x - \xi_2) \end{cases} \quad \text{where} \quad \xi = (\xi_1, \xi_2). \tag{12}$$

Using the fact that $\alpha \neq 0$, a simple calculation shows that $(x - \xi_2)\dot{x} + (y + \xi_1)\dot{y} = \alpha K(x, y)$ where

$$K(x, y) = \left[\left(x + \frac{\beta\xi_1 - \alpha\xi_2}{2\alpha} \right)^2 + \left(y + \frac{\beta\xi_2 + \alpha\xi_1}{2\alpha} \right)^2 - \frac{(\beta^2 + \alpha^2)|\xi|^2}{4\alpha^2} \right].$$

Therefore, if $\alpha > 0$ the solutions of (12) let the exterior $\mathcal{C}$ of any circumference with center at ξ and radius $R > \frac{((\alpha+\beta)^2+2\alpha^2)|\xi|^2}{2\alpha^2}$ invariant (Fig. 2). Analogously, if $\alpha < 0$ the interior of any such circumference is invariant by the solutions of (12). Therefore, (12) cannot be controllable and consequently $\Sigma(\mathcal{X}, \Delta)$ cannot be controllable.

• If $\alpha = \beta = 0$ we can consider the induced system on $H_\xi \setminus G$. By (8) such system is given by

$$\begin{cases} \dot{t} = u \\ \dot{x} = \langle \Lambda_t \xi, \theta\xi \rangle \end{cases}. \tag{13}$$

However, since $\theta^{-1} = -\theta$ and $\|\rho_t\| = 1$ we have that

$$\langle \Lambda_t \xi, \theta\xi \rangle = \langle (1 - \rho_t)\theta\xi, \theta\xi \rangle = |\xi|^2 - \langle \rho_t \xi, \xi \rangle \geq |\xi|^2 - \|\rho_t\| |\xi|^2 \geq 0$$

implying that $\dot{x} \geq 0$ and hence that (13) cannot be controllable.

Therefore, the condition on $\mathcal{D}$ admitting a pair of purely imaginary eigenvalues is a necessary condition for the controllability of Σ concluding the proof. □

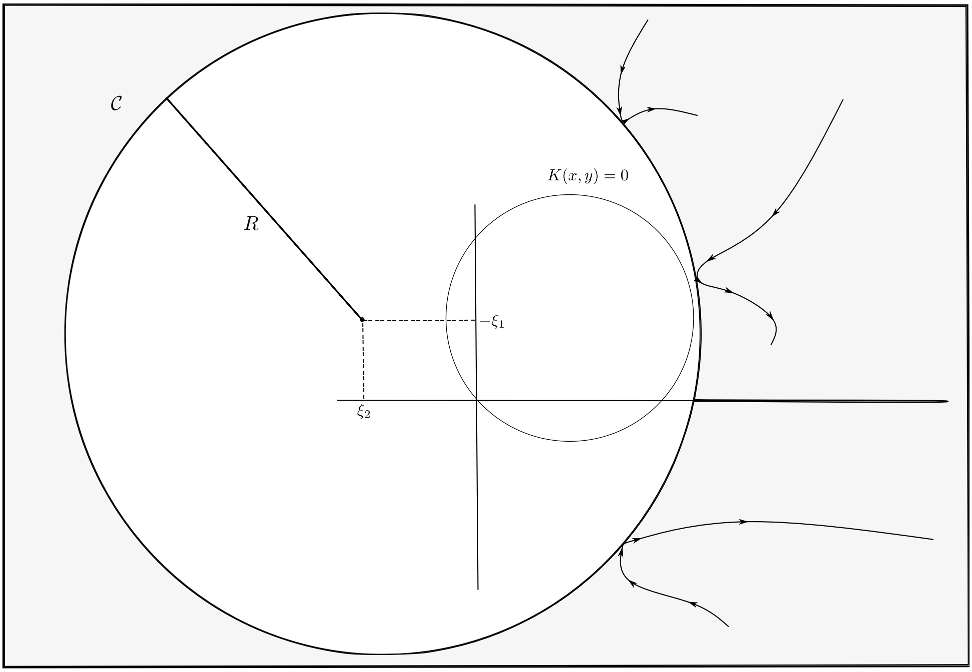

Fig. 2. Invariant region $\mathcal{C}$ for $\alpha > 0$.

4.1.3. The case $G = R_3,\ R_{3,\lambda}$ *or* $R'_{3,\lambda}$

By using Lemma 3.7 we can divide the analysis of linear control systems on R_3, $R_{3,\lambda}$ and $R'_{3,\lambda}$ as follows:

- $G = R_3$ **or** $G = R_{3,\lambda}$ **and** $\mathcal{D}$ **has only real eigenvalues.**

4.4 Proposition. *Let* $\Sigma(\mathcal{X}, \Delta)$ *be a linear control system on* $G = R_{3,\lambda}$ *or* R_3. *Then,*

1. *If* $G = R_{3,\lambda}$ *the linear system cannot be controllable;*
2. *If* $G = R_3$ *the linear system is controllable if and only if it satisfies the LARC and* $\mathfrak{g} = \mathfrak{g}^0$ *with* $\mathcal{D}^* \not\equiv 0$.

Proof. Let us analyze the possibilities for $\mathcal{D}^*$.

- $\dim\ker\mathcal{D}^* = 0$. In this case $\mathcal{D}^*$ is invertible and we can consider the induced system on $F \setminus G$ given by $\dot{v} = \mathcal{D}^* v - u\theta(v - \xi)$ which in coordinates reads as

$$\begin{cases} \dot{x} = \alpha x + by - u\left((x - \xi_1) + \delta(y - \xi_2)\right) \\ \dot{y} = \beta y - u\lambda(y - \xi_2) \end{cases}, \tag{14}$$

where $\xi = (\xi_1, \xi_2)$, $\alpha, \beta \in \mathbb{R}^*$, $|\lambda| \in (0, 1]$ and $\delta \in \{0, 1\}$. Such system is not controllable since the line $y = \xi_2$ works as a barrier for its solutions. In fact, if for instance $\beta\xi_2 \geq 0$, we have that on points of the form (x, ξ_2) it holds that $\dot{y} \geq 0$ showing that the solutions starting on the upper half-plane $\mathcal{C}^+ = \{(x, y) \in \mathbb{R}^2; y \geq \xi_2\}$ will remain there (Fig. 3). Hence $\Sigma(\mathcal{X}, \Delta)$ cannot be controllable.

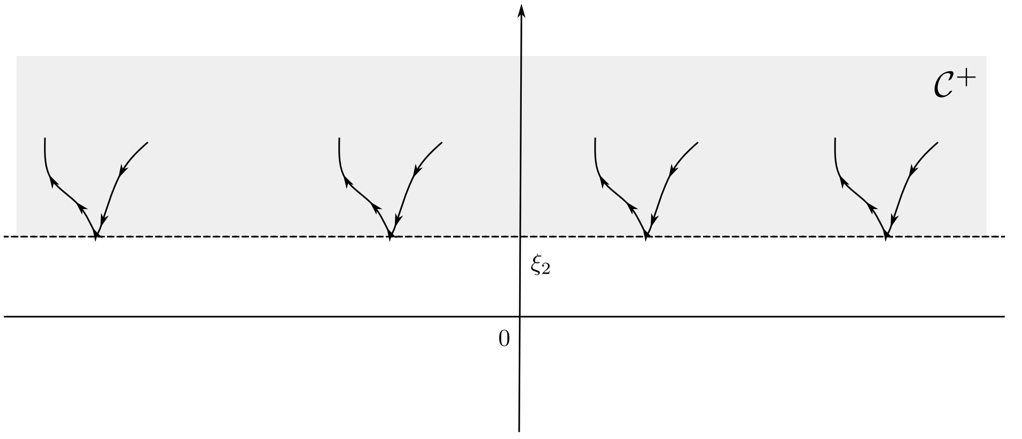

Fig. 3. Invariant region $\mathcal{C}^+$ for $\beta > 0$.

• $\dim\ker\mathcal{D}^* = 1$ and $G = R_{3,\lambda}$. In this case $\mathcal{D}^*$ admits two distinct eigenvalues. Moreover, if $w_0 \in \ker\mathcal{D}^*$ is nonzero, the quotient $H_{w_0} \setminus G$ is isomorphic to $\mathrm{Aff}_0(\mathbb{R})$ and the induced linear system admits a nonzero eigenvalue. By Theorem 2.7 such system cannot be controllable and consequently $\Sigma(\mathcal{X}, \Delta)$ is not controllable.

• $\dim\ker\mathcal{D}^* = 1$ and $G = R_3$. Since $\mathcal{D}^*$ and θ commutes $\ker\mathcal{D}^*$ is θ-invariant and consequently $\ker\mathcal{D}^* = \mathrm{span}\{e_1\}$. By Proposition 3.7 the eigenvalue zero is of multiplicity two for $\mathcal{D}^*$ implying that $\mathfrak{g} = \mathfrak{g}^0$. On the other hand, since $\Sigma(\mathcal{X}, \Delta)$ satisfies the LARC we must have that $\xi = (\xi_1, \xi_2)$ with $\xi_2 \neq 0$ and therefore $\{\xi, \mathcal{D}^*\xi\}$ is a linear independent set. Thus, $\Sigma(\mathcal{X}, \Delta)$ satisfies the ad-rank condition. By Theorem 2.3 we have the controllability of $\Sigma(\mathcal{X}, \Delta)$.

• $\dim\ker\mathcal{D}^* = 2$. Let us assume w.l.o.g. that $\Sigma(\mathcal{X}, \Delta)$ satisfies the LARC. By (8), for any $w_0 \in \mathbb{R}^2$ the induced system on the homogeneous space $H_{w_0} \setminus G$ is given in coordinates by

$$\begin{cases} \dot{t} = u \\ \dot{x} = \langle \Lambda_t \xi, v_0 \rangle \end{cases} \quad \text{where} \quad \langle w_0, v_0 \rangle = 0 \tag{15}$$

and we have the following possibilities:

1. In R_3 it holds that $\xi = (\xi_1, \xi_2)$ with $\xi_2 \in \mathbb{R}^*$. By considering $w_0 = \theta^{-1}\xi$ the induced system becomes

$$\begin{cases} \dot{t} = u \\ \dot{x} = \xi_2^2 t \mathrm{e}^t \end{cases} \tag{16}$$

which is certainly noncontrollable.

2. In $R_{3,\lambda}$ it holds that $\xi = (\xi_1, \xi_2)$ with $\xi_1, \xi_2 \in \mathbb{R}^*$. By considering $w_0 = \lambda(\xi_2^{-1}, \xi_1^{-1})$ the induced system becomes

$$\begin{cases} \dot{t} = u \\ \dot{x} = \lambda(\mathrm{e}^t - 1) - (\mathrm{e}^{\lambda t} - 1) \end{cases} \tag{17}$$

which is certainly noncontrollable since $\dot{x} \le 0$ if $\lambda \in [-1, 0)$ and $\dot{x} \ge 0$ if $\lambda \in (0, 1]$.

In both cases, $\mathcal{D}^* \equiv 0$ implies that $\Sigma(\mathcal{X}, \Delta)$ cannot be controllable, which concludes the proof. □

- **$\mathcal{D}$ has a pair of complex eigenvalues and $G = R_{3,1}$ or $G = R'_{3,\lambda}$**

4.5 Proposition. *A linear control system $\Sigma(\mathcal{X}, \Delta)$ on G is controllable if and only if it satisfies the LARC.*

Proof. Let us assume that Σ satisfies the LARC and by Proposition 3.9 that $(1, 0) \in \Delta$. We have three possibilities to consider:

- $\{\mathcal{D}^*, \theta\}$ is linearly independent. In this case, it holds that $\mathcal{D}^*$ is invertible and we can consider the induced system on the homogeneous space $F \setminus G \simeq \mathbb{R}^2$ given by

$$\dot{v} = \mathcal{D}^* v + u(\theta v - \xi), \quad \text{where} \quad (0, \xi) = \mathcal{D}(1, 0) \neq 0. \tag{18}$$

Since $\{\mathcal{D}^*, \theta\}$ is linear independent, it holds that the associate bilinear system $\dot{w} = (\mathcal{D}^* + u\theta)w$ satisfies

(i) There exists $u \in \mathbb{R}$ such that $\mathcal{D}^* + u\theta$ is skew-symmetric,
(ii) It satisfies the LARC,

and hence it is controllable in $\mathbb{R}^2 \setminus \{0\}$ (see Theorem 3.3 of [7]). Moreover, the fact that $\dot{v} \neq 0$ for any $u \in \mathcal{U}$, implies by Theorem 2 of [11] that (18) is controllable and so $G = F \cdot \mathcal{A} = F \cdot \mathcal{A}^*$.

If $\mathcal{D}$ admits a pair of complex eigenvalues then Σ satisfies the ad-rank condition and by Lemma 2.2 and Theorem 2.3 it holds that $\Sigma(\mathcal{X}, \Delta)$ is controllable. On the other hand, if $\mathcal{D}^* = \alpha\, \mathrm{id}$ then equation (6) gives us that

$$\varphi_s(t(1,0)) = \left(t, (\mathrm{e}^{s\alpha} - 1)\alpha^{-1}\Lambda_t \xi\right), t, s \in \mathbb{R}.$$

If $s > 0$ and $\alpha < 0$ we obtain

$$\mathcal{A} \ni \varphi_s(t(1,0)) = \left(t, (\mathrm{e}^{s\alpha} - 1)\alpha^{-1}\Lambda_t \xi\right) \to (t, -\alpha^{-1}\Lambda_t \xi) \quad \text{as} \quad s \to +\infty.$$

Analogously, if $\alpha > 0$ we get

$$\mathcal{A}^* \ni \varphi_{-s}(t(1,0)) = \left(t, (\mathrm{e}^{-s\alpha} - 1)\alpha^{-1}\Lambda_t \xi\right) \to (t, -\alpha^{-1}\Lambda_t \xi) \quad \text{as} \quad s \to +\infty.$$

By Lemma 2.2, in any case $(t, -\alpha^{-1}\Lambda_t \xi) \in \mathrm{cl}(\mathcal{A}) \cap \mathrm{cl}(\mathcal{A}^*)$ for any $t \in \mathbb{R}$. Since $F = \{(t, -\alpha^{-1}\Lambda_t \xi), t \in \mathbb{R}\}$ we have that $F \subset \overline{\mathcal{A}} \cap \overline{\mathcal{A}^*}$ implying by Lemma 2.2 and the LARC that $\Sigma(\mathcal{X}, \Delta)$ is controllable.

- $\mathcal{D}^* = a\theta$ with $a \neq 0$. In this case, the induced system on $F \setminus G$ has the form

$$\dot{v} = (a - u)\theta v + u\xi$$

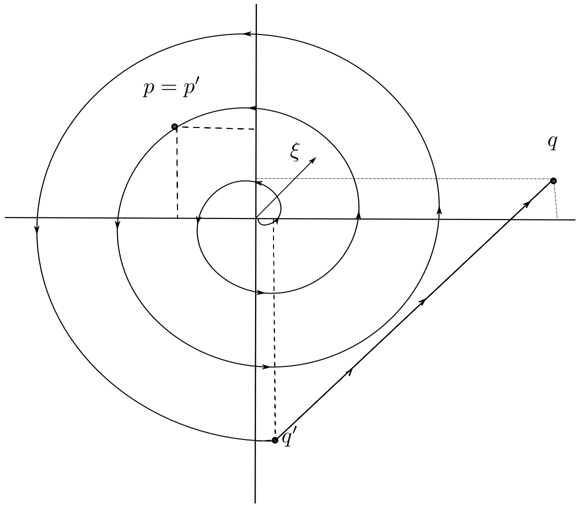

Fig. 4. Solution connecting $p, q \in \mathbb{R}^2$.

and we get that

$$\varphi(t, v, u) = \begin{cases} v + ta\xi, & \text{if } u \equiv a \\ \rho_{at} v & \text{if } u \equiv 0 \end{cases} . \tag{19}$$

Let us assume $a, a\lambda \in \mathbb{R}^+$, since the other possibilities are analogous. For any given $p, q \in \mathbb{R}^2$ we construct a trajectory from p to q as follows (see Fig. 4):

1. If $p = 0$ consider $p' = 0 + t_0 a\xi$ where t_0 is any positive real number. If $p \neq 0$ consider $p' = p$;
2. Go through the spiral $\rho_{at} p'$ from p' to a point q' of the form $q + t_1 a\xi$ where $t_1 \leq 0$;
3. Go from q' to q through the line $q' + ta\xi$.

• $\mathcal{D}^*$ is identically zero. By using the fact that $\mathcal{D}^* \equiv 0$ a simple calculation gives us that $\mathcal{D} = \mathrm{ad}(0, \zeta)$ where $\zeta = -\theta^{-1}\xi$. Moreover, since $H \subset \mathcal{A}_s$ for any $s > 0$ we have

$$(t, 0)\varphi_s(-t, 0) = (t, 0)(-t, (1 - \rho_{-t})s\zeta) = (0, (\rho_t - 1)s\zeta) = (0, s\Lambda_t\zeta) \in \mathcal{A}, \quad \text{for all } \; t \in \mathbb{R},\ s > 0. \tag{20}$$

However, the fact that $\{\Lambda_t\zeta, t \in \mathbb{R}\}$ is a spiral implies that $\{s\Lambda_t\zeta,\ t \in \mathbb{R}, s > 0\} = \mathbb{R}^2$ and consequently that $\mathbb{R}^2 \subset \mathcal{A}$. By Lemma 2.2 we have also that $\mathbb{R}^2 \subset \mathcal{A}^*$ and therefore,

$$G = \mathbb{R}^2 \cdot H \subset \mathbb{R}^2 \cdot \mathcal{A} \subset \mathcal{A} \quad \text{and} \quad G = \mathbb{R}^2 \cdot H \subset \mathbb{R}^2 \cdot \mathcal{A}^* \subset \mathcal{A}^*$$

concluding the proof. □

4.2. The two-dimensional case

For the two-dimensional case it holds that

4.6 Theorem. *Let $\Sigma(\mathcal{X}, \Delta)$ be a linear control system on a three-dimensional nonnilpotent solvable Lie group G that satisfies the LARC and $\dim \Delta = 2$. It holds:*

1. *If $G = R_2$ or $\widetilde{R_2}$: $\Sigma(\mathcal{X}, \Delta)$ is controllable if and only if $\dim \mathfrak{g}^0 > 1$ or $\dim \mathfrak{g}^0 = 1$ and $\Delta \simeq \mathfrak{aff}(\mathbb{R})$;*
2. *If $G = E_n, \widetilde{E}$ or $R'_{3,\lambda}$: $\Sigma(\mathcal{X}, \Delta)$ is controllable;*
3. *If $G = R_3$: $\Sigma(\mathcal{X}, \Delta)$ is controllable if and only if $\mathfrak{g} = \mathfrak{g}^0$;*
4. *If $G = R_{3,\lambda}$: $\Sigma(\mathcal{X}, \Delta)$ is controllable if and only if $\ker \mathcal{D}^* \not\subset \Delta$ or $\mathcal{D}$ has a pair of complex eigenvalues.*

4.2.1. The case $G = R_2$ or $G = \widetilde{R_2}$

4.7 Theorem. *Let $\Sigma(\mathcal{X}, \Delta)$ be a linear control system on G. Then $\Sigma(\mathcal{X}, \Delta)$ is controllable if and only if it satisfies the LARC and $\dim \mathfrak{g}^0 > 1$ or $\dim \mathfrak{g}^0 = 1$ and $\Delta \simeq \mathfrak{aff}(\mathbb{R})$.*

Proof. Let us assume that $\Sigma(\mathcal{X}, \Delta)$ satisfies the LARC. By Propositions 2.6 and 3.9 we only have to consider a linear control system $\Sigma(\mathcal{X}, \Delta)$ on $\widetilde{R_2}$ such that $\Delta = \operatorname{span}\{(1, 0), (0, w)\}$. Moreover,

$$\Delta \ni [(1, 0), (0, w)] = (0, \theta w) \implies \theta w \in \operatorname{span}\{w\}$$

and therefore $\Delta = \operatorname{span}\{(1, 0), (0, e_2)\} \simeq \mathrm{Aff}_0(\mathbb{R})$ or $\Delta = \operatorname{span}\{(1, 0), (0, e_1)\}$ is Abelian.

1. If $\dim \mathfrak{g}^0 = 3$ then $\mathfrak{g} = \mathfrak{g}^0$ and since in this case the LARC is equivalent to the ad-rank condition Theorem 2.3 implies the controllability of $\Sigma(\mathcal{X}, \Delta)$.
2. If $\dim \mathfrak{g}_0 = 2$ we have that $\mathfrak{g} = \mathfrak{g}^+\Delta$ and consequently $G = G^0 \cdot G_\Delta$. Since $\Sigma(\mathcal{X}, \Delta)$ satisfies the ad-rank condition Theorem 2.3 implies $G^0 \subset \mathcal{A} \cap \mathcal{A}^*$ and by Lemma 2.2

$$G = G^0 \cdot G_\Delta \subset G^0 \cdot \mathcal{A} \subset \mathcal{A} \quad \text{and} \quad G = G^0 \cdot G_\Delta \subset G^0 \cdot \mathcal{A}^* \subset \mathcal{A}^*$$

   implying the controllability of $\Sigma(\mathcal{X}, \Delta)$.
3. If $\dim \mathfrak{g}^0 = 1$ then necessarily $\mathcal{D}^*$ is invertible.
   - If $\Delta = \operatorname{span}\{(1, 0), (0, e_2)\}$ we can consider the induced linear system on $H_{e_2} \setminus G \simeq \mathbb{R}^2$. Since such system satisfies the ad-rank condition it is controllable implying that $G = H_{e_2} \cdot \mathcal{A} = H_{e_2} \cdot \mathcal{A}^*$. On the other hand, the fact that H_{e_2} is φ-invariant and $H_{e_2} \subset G_\Delta \subset \mathcal{A} \cap \mathcal{A}^*$ implies by Lemma 2.2 the controllability of $\Sigma(\mathcal{X}, \Delta)$.
   - If $\Delta = \operatorname{span}\{(1, 0), (0, e_1)\}$, the induced system on the two-dimensional solvable Lie group $H_{e_1} \setminus G \simeq \mathrm{Aff}_0(\mathbb{R})$ cannot be controllable. In fact, the induced derivation admits a nonzero eigenvalue and consequently $\Sigma(\mathcal{X}, \Delta)$ cannot be controllable, concluding the proof. □

4.2.2. The case $G = R_3,\ R_{3,\lambda},\ R'_{3,\lambda},\ E_n$ or $\widetilde{E_2}$

Let us separate the cases as follows:

• $G = R_3$ **or** $G = R_{3,\lambda}$.

4.8 Proposition. *Let $\Sigma(\mathcal{X}, \Delta)$ be a linear control system on $G = R_{3,\lambda}$ or R_3. Then,*

1. *If $G = R_{3,\lambda}$ the linear system is controllable if and only if it satisfies the LARC and* $\ker \mathcal{D}^* \not\subset \Delta$ *or $\mathcal{D}^*$ has a pair of complex eigenvalues;*
2. *If $G = R_3$ the linear control system is controllable if and only if it satisfies the LARC and $\mathfrak{g} = \mathfrak{g}^0$.*

Proof. We can as before assume that $\Delta = \text{span}\{(1, 0), (0, w)\}$ where $w \in \mathbb{R}^2$ is a common eigenvector of θ and $\mathcal{D}^*$.

1. If $\mathcal{D}^*$ has a pair of complex eigenvalues $G = R_{3,1}$ and the linear control system $\Sigma(\mathcal{X}, \Delta)$ is controllable if and only if it satisfies the LARC by the one-dimensional case. Let us then assume that $\mathcal{D}^*$ has a pair of real eigenvalues and analyze the dimension of $\ker \mathcal{D}^*$.

• $\dim \ker \mathcal{D}^* = 0$. In this case $\ker \mathcal{D}^* = \{0\} \subset \Delta$. By Theorem 2.7 the induced linear control system on $H_w \setminus G \simeq \text{Aff}_0(\mathbb{R})$ cannot be controllable, since the associated derivation has a nonzero eigenvalue. Therefore, $\Sigma(\mathcal{X}, \Delta)$ can not be controllable.

• $\dim \ker \mathcal{D}^* = 1$. If $\ker \mathcal{D}^* = \text{span}\{w\} \subset \Delta$ we have as before, that the derivation of the induced linear control system on $H_w \setminus G \simeq \text{Aff}_0(\mathbb{R})$ has a nonzero eigenvalue and therefore cannot be controllable implying that $\Sigma(\mathcal{X}, \Delta)$ is not controllable.

On the other hand, if $\ker \mathcal{D}^* \not\subset \Delta$ it follows that $\mathbb{R}^2 = \ker \mathcal{D}^* \oplus \mathbb{R}w \subset \mathcal{A} \cap \mathcal{A}^*$ and consequently

$$G = \mathbb{R}^2 \cdot H \subset \mathbb{R}^2 \cdot \mathcal{A} \subset \mathcal{A} \quad \text{and} \quad G = \mathbb{R}^2 \cdot H \subset \mathbb{R}^2 \cdot \mathcal{A}^* \subset \mathcal{A}^*$$

implying the controllability of $\Sigma(\mathcal{X}, \Delta)$.

• $\dim \ker \mathcal{D}^* = 2$. In this case we have that $\mathfrak{g} = \mathfrak{g}^0$ which by the Theorem 2.3, and the fact that $\Sigma(\mathcal{X}, \Delta)$ satisfies the ad-rank condition implies the controllability.

2. Since $G = R_3$ we necessarily have $w = e_1$. Moreover, if $\dim \ker \mathcal{D}^* \geq 1$ then $\mathfrak{g} = \mathfrak{g}^0$ which by Theorem 2.3 implies the controllability of $\Sigma(\mathcal{X}, \Delta)$. Therefore, we only have to show that when $\mathcal{D}^*$ is invertible, $\Sigma(\mathcal{X}, \Delta)$ cannot be controllable. In order to do that let us consider the induced system on $F \setminus G$ given by $\dot{v} = \mathcal{D}^* v - u_1 \theta(v - \xi) + u_2 \mathcal{D}^* e_1$. In coordinates we have that

$$\begin{cases} \dot{x} = \alpha x + by - u_1\left((x - \xi_1) + (y - \xi_2)\right) + \alpha u_2 \\ \dot{y} = \beta y - u(y - \xi_2) \end{cases}, \tag{21}$$

where $\xi = (\xi_1, \xi_2)$ and $\alpha, \beta \in \mathbb{R}^*$. As for the one-dimensional case, such system is not controllable since the line $y = \xi_2$ works as a barrier for its solutions (see Fig. 2), concluding the proof. □

• $G = E_n, \widetilde{E}$ **or** $G = R'_{3,\lambda}$.

When $G = E_n, \widetilde{E}$ or $G = R'_{3,\lambda}$ the only two-dimensional subalgebra of their associated Lie algebra is $\mathbb{R}^2$ as follows from Lemma 3.8. Therefore, any linear system $\Sigma(\mathcal{X}, \Delta)$ with $\dim \Delta > 1$ is trivially controllable if it satisfies the LARC, since in this case we necessarily have that $\dim \Delta = 3$.

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