Calculus for interval-valued functions using generalized Hukuhara derivative and applications

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Abstract

This paper is devoted to studying differential calculus for interval-valued functions by using the generalized Hukuhara differentiability, which is the most general concept of differentiability for interval-valued functions. Conditions, examples and counterexamples for limit, continuity, integrability and differentiability are given. Special emphasis is set to the class $F(t) = C \cdot g(t)$, where $C$ is an interval and $g$ is a real function of a real variable. Here, the emphasis is placed on the fact that $F$ and $g$ do not necessarily share their properties, underlying the extra care that must be taken into account when dealing with interval-valued functions. Two applications of the obtained results are presented. The first one determines a Delta method for interval valued random elements. In the second application a new procedure to obtain solutions to an interval differential equation is introduced. Our results are relevant to fuzzy set theory because the usual fuzzy arithmetic, extension functions and (mathematical) analysis are done on $\alpha$-cuts, which are intervals. © 2012 Elsevier B.V. All rights reserved.

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1. Introduction

The importance of the study of set-valued analysis from a theoretical point of view as well as from their applications is well known [2,3]. Many advances in set-valued analysis have been motivated by control theory and dynamical games [4]. Optimal control theory and mathematical programming were an engine driving these domains since the dawn of the sixties [4]. Interval analysis is a particular case and it was introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. The first monographs dealing with interval analysis were due to Moore [42,43].

Other issues of control theory, dynamic economy and biological evolution theory, such as the regulation of control systems subjected to viability constraints, triggered the discovery of differential calculus of set-valued maps [4]. In this direction, several notions of derivative of a set-valued map were introduced. For instance, see [2–7,11,13–18,21,27,28,51]. The paper of Hukuhara [27] was the starting point for the topic of Set Differential Equations and later on for...
Fuzzy Differential Equations. Set differential equations have recently attracted the attention of many researchers [1,12,22–24,38,39,44,51]. Fuzzy Differential Calculus and Fuzzy Differential Equations are also two very important related fields [7–9,15,16,21,25,36,45,48].

An important class of interval-valued functions is defined by

\[ F(t) = C \cdot g(t), \]  

where \( C = [a, b] \) is a closed interval and \( g \) is a real function of a real variable. The class (1) is extensive. It includes, for example, interval-valued objective functions in interval (fuzzy) linear or nonlinear programming. As a practical matter, it is usually difficult to determine the coefficients of an objective function as a real number since, very often, they possess inherent uncertainty and/or inaccuracy. Given that this is the usual state, we consider interval (or fuzzy) mathematical programming as one approach to tackle uncertainty and inaccuracies in the objective function coefficients of mathematical programming models. The articles [10,29,30–34] and more recently [19,20,35,46,57–59] give the current development of interval mathematical programming and point to the importance of the class of functions we are studying.

On the other hand, differential equations are an important class of mathematical models. However, in some cases, these equations are restrictive in their ability to describe phenomena. For example, in mathematical models that describe biological phenomena, the parameters are usually inherently uncertain [26,37]. Consequently, these parameters can be considered as being intervals (fuzzy intervals) and thus arise a class of interval differential equations involving interval-valued functions (or fuzzy functions) of type (1)

The calculus in the class of functions that is the focus of our study includes differentiation and integration. In order to develop differentiation, we need an appropriate metric space and a well-defined subtraction. In the development of the calculus for this special class of functions, we point out common misconceptions. Typically, if it is known that a real valued function \( g(t) \) has a variety of properties (continuity and differentiability, for example), then both \( f(t) = cg(t) \) and \( g(t) \) share their properties, where \( c \) is a real constant, in other words, \( g \) inherits the properties of \( f \) and, conversely, \( f \) inherits the properties of \( g \). However, this is not the case for (1), as it will be seen later.

Most results in the paper are given for the class (1), but the paper also contains new results for the more general class of interval-valued functions. For instance, we make a detailed study on differentiability for interval-valued functions including a study on the classification of switching points. After been stated, these new general results are specialized for the class (1).

The paper is organized as follows. In Section 2 some basic definitions and the notation that will be used along the paper are given. The following sections deal with the continuity, integrability, differentiability for the class of interval-valued functions in (1). In Section 6 an interval-valued version of the fundamental theorem of calculus is derived. Finally, Section 7 displays two applications of the reported results: to derive a Delta method for random intervals and to obtain new solutions for interval differential equations.

2. Basic concepts and motivation

Let \( \mathbb{R} \) be the one-dimensional Euclidean space. Following Diamond and Kloeden [21], let \( K_C \) denote the family of all non-empty compact convex subsets of \( \mathbb{R} \), that is,

\[ K_C = \{ [a, b] | a, b \in \mathbb{R} \text{ and } a \leq b \}. \]

The Hausdorff metric \( H \) on \( K_C \) is defined by

\[ H(A, B) = \max\{ d(A, B), d(B, A) \}, \]

where \( d(A, B) = \max_{a \in A} d(a, B) \) and \( d(a, B) = \min_{b \in B} d(a, b) = \min_{b \in B} |a - b| \). It is well known that \( (K_C, H) \) is a complete metric space (see [2,3]). The Minkowski sum and scalar multiplication are defined by

\[ A + B = \{ a + b | a \in A, b \in B \} \quad \text{and} \quad \lambda A = \{ \lambda a | a \in A \}. \]

The space \( K_C \) is not a linear space since it does not possess an additive inverse and therefore subtraction is not well defined (see [2,3,21]).
A crucial concept in obtaining a useful working definition of derivative for interval-valued functions is deriving a suitable difference between two intervals. Toward this end, one way is to use (2) by requiring

\[ A - B = A + (-1)B. \]

However, this definition of the difference has the drawback that

\[ A - A \neq [0] \] (3)
in general (the exception is when \( A \) is a zero width interval, \( A = [a, a] \), that is, a real number). One of the first attempts to overcome (3) was due to Hukuhara [27] who defined what has come to be known as the Hukuhara difference. If \( A = B + C \), then the Hukuhara difference (H-difference) of \( A \) and \( B \), denoted by \( A - H B \), is equal to \( C \). The H-difference of two intervals does not always exist for arbitrary pairs of intervals. It only exists for intervals \( A \) and \( B \) for which the widths are such that

\[ \text{len}(A) \geq \text{len}(B), \]

where for \( A = [a, \overline{a}] \),

\[ \text{len}(A) = \overline{a} - a, \]

that is to say, \( \text{len}(A) \) is the length or diameter of interval \( A \).

Recently, Stefanini et al. [51,53] introduced the concept of generalized Hukuhara difference of two sets \( A, B \in K_C \) (\( gH \)-difference for short) and it is defined as follows:

\[ A \ominus_{gH} B = \begin{cases} A = B + C \quad \text{or} \\ B = A + (-1)C. \end{cases} \] (4)

In case (a), the \( gH \)-difference coincides with the H-difference. Thus, the \( gH \)-difference is a generalization of the H-difference. The \( gH \)-difference exists for any two compact intervals. Chalco-Cano et al. [13] have shown that the \( gH \)-difference, the \( \pi \)-difference and the \( M \)-difference, which is the difference concept introduced by Markov [41], between two intervals \( A = [a, b], B = [c, d] \in K_C \) are the same concept. Specifically,

\[ A \ominus_{gH} B = A - \pi B = [\min\{a - c, b - d\}, \max\{a - c, b - d\}]. \] (5)

Henceforth \( T = (t_1, t_2) \) denotes an open interval. Let \( F : T \to K_C \) be an interval-valued function. We will denote \( F(t) = \{f(t), \overline{f}(t)\} \), where \( f(t) \subseteq \overline{f}(t) \), \( \forall t \in T \). The functions \( f \) and \( \overline{f} \) are called the lower and the upper (endpoint) functions of \( F \), respectively. For the special class of interval-valued functions (1),

\[ F(t) = C \cdot g(t), \]

where \( g : T \to \mathbb{R} \) is a real function and \( C = [a, b] \in K_C \) is a fixed closed interval, we have that

\[ f(t) = \begin{cases} ag(t) & \text{if } g(t) \geq 0, \\ bg(t) & \text{if } g(t) < 0, \end{cases} \quad \overline{f}(t) = \begin{cases} bg(t) & \text{if } g(t) \geq 0, \\ ag(t) & \text{if } g(t) < 0. \end{cases} \] (6)

An important point is that the properties of the functions \( f \) and \( \overline{f} \) in (6) are not necessarily inherited from \( g \). For instance, it can be easily seen that if the functions \( f \) and \( \overline{f} \) in (6) are both differentiable at \( t_0 \), then \( g \) is differentiable at \( t_0 \). The converse of this assertion is not true. Next example shows this fact.

**Example 1.** Let \( C = [-1, 2] \) and \( g(t) = t^3 + t \). Obviously, \( g \) is differentiable. The functions \( f \) and \( \overline{f} \) defined in (6) now become

\[ f(t) = \begin{cases} 2(t^3 + t) & \text{if } t \leq 0, \\ -(t^3 + t) & \text{if } t > 0, \end{cases} \quad \overline{f}(t) = \begin{cases} -(t^3 + t) & \text{if } t \leq 0, \\ 2(t^3 + t) & \text{if } t > 0. \end{cases} \]

Observe that \( f \) and \( \overline{f} \) are not differentiable at \( t = 0 \).
Thus the properties of the class \( F(t) = C \cdot g(t) \) must be derived separately since they cannot be obtained from those of \( g \). Next sections deal with the properties and calculus for this class of interval-valued functions. Specifically, the connections between the limit, continuity, integrability and differentiability of \( F \) and the limit, continuity, integrability and differentiability of \( g \), respectively, are studied.

3. Limits and continuity

**Definition 1** (Aubin and Cellina [2], Aubin and Franskowska [3]). Let \( F : T \to K_C \) be an interval-valued function and let \( t_0 \in T \). We say that \( L \in K_C \) is limit of \( F \) in \( t_0 \), which we denote by

\[
\lim_{t \to t_0} F(t) = L,
\]

if given \( \varepsilon > 0 \), exists \( \delta > 0 \) such that

\[
0 < |t - t_0| < \delta \Rightarrow H(F(t), L) < \varepsilon.
\]

The following result establishes the relationship between the limit of an interval-valued function and the limit of the lower and the upper (endpoint) functions.

**Theorem 1** (Aubin and Cellina [2]). Let \( F : T \to K_C \) be an interval-valued function such that \( F(t) = [f(t), \overline{f}(t)] \) and let \( t_0 \in T \). Then

\[
\lim_{t \to t_0} F(t) = \left[ \lim_{t \to t_0} f(t), \lim_{t \to t_0} \overline{f}(t) \right].
\]

Next we study the connection between the limits of \( F \) and \( g \).

**Theorem 2.** Let \( F : T \to K_C \) with \( F(t) = C \cdot g(t) \) and let \( t_0 \in T \). If \( \lim_{t \to t_0} g(t) \) exists then \( \lim_{t \to t_0} F(t) \) exists and

\[
\lim_{t \to t_0} F(t) = C \cdot \lim_{t \to t_0} g(t).
\]

**Proof.** Suppose that \( \lim_{t \to t_0} g(t) = g_0 \) and that (7) does not hold. Then there exists \( \varepsilon > 0 \) such that for all \( \delta > 0 \) and for all \( 0 < |t - t_0| < \delta \),

\[
H([a, b] \cdot g(t), [a, b] \cdot g_0) > \varepsilon.
\]

So

\[
\max_{e \in [a, b] \cdot g(t)} d(e, [a, b] \cdot g_0) > \varepsilon \tag{8}
\]

and

\[
\max_{h \in [a, b] \cdot g_0} d(h, [a, b] \cdot g(t)) > \varepsilon. \tag{9}
\]

From (8) there exists \( c_1 \in [a, b] \) such that

\[
|c_1 g(t) - w g_0| > \varepsilon, \quad \forall w \in [a, b]. \tag{10}
\]

Let us see that the constant \( c_1 \) in (10) cannot be equal to 0: if 0 \( \notin [a, b] \) then the assertion is trivial; if 0 \( \in [a, b] \), since the inequality in (10) hold \( \forall w \in [a, b] \), for \( w = 0 \) we get \( |c_1 g(t)| > \varepsilon \), which implies that \( c_1 \neq 0 \).

Taking \( w = c_1 \) in (10) we have

\[
|c_1 g(t) - c_1 g_0| = |c_1||g(t) - g_0| > \varepsilon,
\]

which implies that \( \lim_{t \to t_0} g(t) \neq g_0 \). Analogously, from (9) we have that \( \lim_{t \to t_0} g(t) \neq g_0 \). In both cases we have a contradiction to the hypothesis. Consequently, there exists \( \lim_{t \to t_0} F(t) \) and (7) holds. \( \square \)
The following example shows that the converse of Theorem 2 is not true.

**Example 2.** Consider the function \( g : \mathbb{R} \to \mathbb{R} \) defined by

\[
g(t) = \begin{cases} 
1 & \text{if } t \geq 0, \\
-1 & \text{if } t < 0.
\end{cases}
\]

It is clear that there does not exist \( \lim_{t \to 0} g(t) \). The interval-valued function \( F \) defined by \( F(t) = [-2, 2] \cdot g(t) \) is equivalent to \( F(t) = [-2, 2] \) for all \( t \in \mathbb{R} \). So, \( \lim_{t \to 0} F(t) = [-2, 2] \).

Next result establishes the converse of Theorem 2. It tells us that one must be careful when one is trying to deduce the properties of \( F \) from those of \( g \) and vice versa. The result is a consequence of the fact that we are dealing with interval-valued functions whose space is different from that of real-valued functions.

**Theorem 3.** Let \( F : T \to \mathcal{K}_C \) with \( F(t) = C \cdot g(t) \) and let \( t_0 \in T \). If \( \lim_{t \to t_0} F(t) \) exists then one of the following cases holds:

(a) there exists \( \lim_{t \to t_0} g(t) \) and

\[
\lim_{t \to t_0} F(t) = C \cdot \lim_{t \to t_0} g(t),
\]

(b) there exists \( \lim_{t \to t_0} |g(t)| \), \( a = -b \) and

\[
\lim_{t \to t_0} F(t) = C \cdot \lim_{t \to t_0} |g(t)|.
\]

**Proof.** From (6) the interval-valued function \( F \) can be rewritten as

\[
F(t) = [\min\{ag(t), bg(t)\}, \max\{ag(t), bg(t)\}] = [m_1(t), m_2(t)].
\]

Since \( \lim_{t \to t_0} F(t) \) exists, then, from Theorem 1, the limits \( \lim_{t \to t_0} m_i(t), i = 1, 2 \), exist. So \( \lim_{t \to t_0} \{m_1(t) + m_2(t)\} \) exists. But

\[
m_1(t) + m_2(t) = (a + b)g(t).
\]

If \( a + b \neq 0 \), then \( \lim_{t \to t_0} g(t) \) exists. Therefore (a) holds.

If \( a + b = 0 \), then \( b = -a, b > 0 \) and \( C = [-b, b] \). Then \( F(t) = [-b, b] \cdot g(t) = [-b|g(t)|, b|g(t)|] \). Consequently, from Theorem 1, there exists \( \lim_{t \to t_0} |g(t)| \) and (b) holds. □

From the definition of limit for interval-valued functions it is clear that \( F : T \to \mathcal{K}_C \) is continuous at \( t_0 \in T \) if and only if

\[
\lim_{t \to t_0} F(t) = F(t_0).
\]

Next we state two immediate consequences of Theorems 2 and 3.

**Theorem 4.** Let \( F : T \to \mathcal{K}_C \) with \( F(t) = C \cdot g(t) \) and let \( t_0 \in T \). If \( g \) is continuous at \( t_0 \) then \( F \) is continuous at \( t_0 \).

**Theorem 5.** Let \( F : T \to \mathcal{K}_C \) with \( F(t) = C \cdot g(t) \) and let \( t_0 \in T \). If \( F \) is continuous at \( t_0 \) then one of the following cases holds:

(a) \( g \) is continuous at \( t_0 \),

(b) \( |g| \) is continuous at \( t_0 \) and \( a = -b \).

**4. Integral**

Let \( F : T \to \mathcal{K}_C \) and let \( S(F) \) denote the set of integrable selectors of \( F \) over \( T \), that is,

\[
S(F) = \{ f : T \to \mathbb{R} | f \text{ is an integrable function and } f(t) \in F(t) \text{ for all } t \in T \}.
\]
Definition 2 (Aubin and Cellina [2]). Let \( F : T \to \mathcal{K}_C \) be an interval-valued function. The integral (Aumann integral) of \( F \) over \( T \) is defined as
\[
\int_{t_1}^{t_2} F(t) \, dt = \left\{ \int_{t_1}^{t_2} f(t) \, dt \mid f \in S(F) \right\}.
\]
If \( S(F) \neq \emptyset \), then the integral exists and \( F \) is said to be integrable (Aumann integrable).

Note that if \( F \) is measurable then it has a measurable selector (see [2,3]) which is integrable and \( S(F) \neq \emptyset \). More precisely.

Theorem 6 (Aubin and Cellina [2]). Let \( F : T \to \mathcal{K}_C \) be a measurable and integrably bounded interval-valued function. Then it is integrable and \( \int_{t_1}^{t_2} F(t) \, dt \in \mathcal{K}_C \).

Theorem 7 (Diamond and Kloeden [21]). Let \( F : T \to \mathcal{K}_C \) be a measurable and integrably bounded interval-valued function such that \( F(t) = [\underline{f}(t), \overline{f}(t)] \). Then \( f \) and \( \overline{f} \) are integrable functions and
\[
\int_{t_1}^{t_2} F(t) \, dt = \left[ \int_{t_1}^{t_2} f(t) \, dt, \int_{t_1}^{t_2} \overline{f}(t) \, dt \right].
\]

Corollary 1. A continuous interval-valued function \( F : T \to \mathcal{K}_C \) is integrable.

The Aumann integral satisfies the following properties.

Proposition 1 (Aubin and Cellina [2]). Let \( F, G : T \to \mathcal{K}_C \) be two measurable and integrable bounded interval-valued functions. Then
\begin{enumerate}[(i)]
\item \( \int_{t_1}^{t_2} (F(t) + G(t)) \, dt = \int_{t_1}^{t_2} F(t) \, dt + \int_{t_1}^{t_2} G(t) \, dt \),
\item \( \int_{t_1}^{t_2} F(t) \, dt = \int_{t_1}^{t} F(t) \, dt + \int_{t}^{t_2} F(t) \, dt, \) \( t_1 < \tau < t_2 \).
\end{enumerate}

For the special class \( (1) \), we have the following result.

Theorem 8. Let \( F : T \to \mathcal{K}_C \) be a measurable and integrable bounded interval-valued function such that \( F(t) = C \cdot g(t) \). Then \( g \) and \( |g| \) are integrable functions. Moreover,
\begin{enumerate}[(i)]
\item for \( a + b = 0 \),
\[
\int_{t_1}^{t_2} F(t) \, dt = C \cdot \int_{t_1}^{t_2} |g(t)| \, dt,
\]
\item otherwise,
\[
\int_{t_1}^{t_2} F(t) \, dt = \frac{1}{2} \left\{ C \cdot \int_{t_1}^{t_2} (g(t) + |g(t)|) \, dt + C \cdot \int_{t_1}^{t_2} (g(t) - |g(t)|) \, dt \right\}.
\]
\end{enumerate}

Proof. (i) If \( a + b = 0 \), then \( b = -a, b > 0 \) and \( C = [-b, b] \). Then \( F(t) = [-b, b] \cdot g(t) = [-b|g(t)|, b|g(t)|] \). Consequently, from Theorem 7, \( |g| \) is integrable and (i) holds.

(ii) Otherwise, when \( a + b \neq 0 \), the interval-valued function \( F \) can be rewritten as
\[ F(t) = [\min\{ag(t), bg(t)\}, \max\{ag(t), bg(t)\}] = [m_1(t), m_2(t)]. \]

From Theorem 7 \( m_1 \) and \( m_2 \) are integrable. Consequently \( (a + b)g(t) = m_1 + m_2 \) is integrable and so \( g \) and \( |g| \) are integrable. On the other hand, \( F \) can also be rewritten as
\[ F(t) = \frac{m_1 + m_2}{2} + \frac{m_2 - m_1}{2}[-1, 1] = \frac{a + b}{2}g(t) + \frac{b - a}{2} |g(t)| \cdot [-1, 1]. \]
From the result in part (i) and Proposition 1,
\[
\int_{t_1}^{t_2} F(t) \, dt = \frac{a + b}{2} \int_{t_1}^{t_2} g(t) \, dt + \frac{b - a}{2} \int_{t_1}^{t_2} |g(t)| \, dt \cdot [-1, 1]
\]
\[
= \frac{1}{2} \left\{ C \cdot \int_{t_1}^{t_2} (g(t) + |g(t)|) \, dt + C \cdot \int_{t_1}^{t_2} (g(t) - |g(t)|) \, dt \right\}.
\]
Therefore (ii) holds. □

**Corollary 2.** Let \( F : T \to \mathcal{K}_C \) be a measurable and integrable bounded interval-valued function such that \( F(t) = C \cdot g(t) \). If \( g(t) \geq 0, \) for all \( t \in T \) or \( g(t) \leq 0, \) for all \( t \in T, \) then
\[
\int_{t_1}^{t_2} F(t) \, dt = C \cdot \int_{t_1}^{t_2} g(t) \, dt.
\]
(13)

Note that the absolute value in Eq. (11) cannot be removed. Also, if \( g \) does not have the same sign \( \forall t \in T \) then (13) is not true. Next example illustrates these remarks.

**Example 3.** Consider the function \( g \) defined by \( g(t) = 1 - t \) and the interval \( C = [-1, 2] \). Then, from Theorem 8 part (ii), we have
\[
\int_{0}^{2} F(t) \, dt = \int_{0}^{2} [-1, 2] \cdot (1 - t) \, dt = [-3/2, 3/2],
\]
while
\[
[-1, 2] \cdot \int_{0}^{2} g(t) \, dt = [-1, 2] \cdot 0 = [0, 0].
\]

On the other hand, if \( C = [-2, 2] \) then by using Proposition 1 part (ii) we get
\[
\int_{0}^{2} [-2, 2] \cdot g(t) \, dt = \int_{0}^{1} [-2, 2] \cdot (1 - t) \, dt + \int_{1}^{2} [-2, 2] \cdot (-1 - t) \, dt = [-2, 2] 1/2 + [-2, 2] 1/2 = [-2, 2] = [-2, 2] \int_{0}^{2} |g(t)| \, dt.
\]

5. Differentiability

The relationships between the derivative of an interval-valued function \( F(t) = [f(t), \overline{f}(t)] \) and the derivatives of the associated real valued functions \( f(t) \) and \( \overline{f}(t) \) have been completely studied in Chalco-Cano et al. [13]. For the sake of completeness, here we particularize their results for the special case \( F(t) = C \cdot g(t) \). With this aim, we first write the definition of the \( gH \)-derivative given by Stefanini et al. [51].

**Definition 3.** The \( gH \)-derivative of an interval-valued function \( F : T \to \mathcal{K}_C \) at \( t_0 \in T \) is defined as
\[
F'(t_0) = \lim_{h \to 0} \frac{F(t_0 + h) \ominus_{gH} F(t_0)}{h}.
\]
(14)

If \( F'(t_0) \in \mathcal{K}_C \) satisfying (14) exists, we say that \( F \) is generalized Hukuhara differentiable (\( gH \)-differentiable) at \( t_0 \).

From the equivalence between \( gH \)-difference, \( \pi \)-difference and the \( M \)-difference it immediately follows the equivalence between \( gH \)-derivative, \( \pi \)-derivative and the derivative defined in Markov [41].
Theorem 9 (Chalco-Cano [13]). Let $F : T \to \mathbb{K}_C$ be an interval-valued function such that $F(t) = [\underline{f}(t), \overline{f}(t)]$. Then, $F$ is gH-differentiable at $t_0 \in T$ if and only if one of the following cases holds:

(a) $\underline{f}$ and $\overline{f}$ are differentiable at $t_0$ and

$$F'(t_0) = [\min\{(\underline{f})'(t_0), (\overline{f})'(t_0)\}, \max\{(\underline{f})'(t_0), (\overline{f})'(t_0)\}];$$

(b) $(\underline{f})'(t_0), (\overline{f})'(t_0)$ and $(\overline{f}')_1(t_0)$ exist and satisfy $(\underline{f})'(t_0) = (\overline{f})'(t_0)$ and $(\overline{f}')_1(t_0) = (\overline{f}')_1(t_0).$ Moreover

$$F'(t_0) = [\min\{(\underline{f})'(t_0), (\overline{f})'(t_0)\}, \max\{(\underline{f})'(t_0), (\overline{f})'(t_0)\}];$$

as a consequence of the above general result, we have the following.

Theorem 10. Let $F : T \to \mathbb{K}_C$ be a measurable and integrable bounded interval-valued function such that $F(t) = C \cdot g(t)$, with $C = [a, b]$, $a < b.$ Then $F$ is gH-differentiable at $t_0 \in T$ if and only if

(a) Case $a \neq -b$: $g$ is differentiable at $t_0$. In this case $F'(t_0) = [a, b]g'(t_0)$.

(b) Case $a = -b$: then the following limits exist

$$v'_-(t_0) = \lim_{h \to 0^-} \frac{v(t_0 + h) - v(t_0)}{h}, \quad v'_+(t_0) = \lim_{h \to 0^+} \frac{v(t_0 + h) - v(t_0)}{h},$$

where $v = |g|$, and one of the following cases holds:

(b.1) $v'_-(t_0) = v'_+(t_0)$ and thus $v$ is differentiable at $t_0$, with $v'(t_0) = v'_-(t_0) = v'_+(t_0).$ In this case $F'(t_0) = [a, b]v'(t_0)$.

(b.2) $v'_-(t_0) = -v'_+(t_0).$ In this case $F'(t_0) = [a, b]v'_-(t_0) = [a, b]v'_+(t_0)$.

From Theorem 10 it follows that the differentiability of $g$ is a sufficient, but not a necessary condition for the differentiability of $F$ (see [13, Example 5]).

Let $\text{len}(F)(t) = \text{len}(F(t)) = \overline{f}(t) - \underline{f}(t)$ be the length of the interval $F(t)$. Condition (a) of Theorem 9 is linked with the monotonicity of $\text{len}(F)$ while condition (b) of Theorem 9 is linked to switching points of the length of $F(t)$, that are points at which the monotonicity of $\text{len}(F)$ changes. For example, if we consider $F(t) = [−|t|, |t|]$, for all $t \in \mathbb{R}$, then $F$ is gH-differentiable in $\mathbb{R}$, $\text{len}(F)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, thus $t = 0$ is a switching point for the differentiability of $F$. Formally,

Definition 4 (Chalco-Cano and Román-Flores [15], Stefanini and Bede [51]). Let $F : T \to \mathbb{K}_C$ be an interval-valued function such that $F(t) = [\underline{f}(t), \overline{f}(t)]$.

(I) We say that $F$ is gH-differentiable at $t_0 \in T$ in the first form if $\underline{f}$ and $\overline{f}$ are differentiable at $t_0$ and

$$F'(t_0) = [(\underline{f})'(t_0), (\overline{f})'(t_0)].$$

(II) We say that $F$ is gH-differentiable at $t_0 \in T$ in the second form if $\underline{f}$ and $\overline{f}$ are differentiable at $t_0$ and

$$F'(t_0) = [(\overline{f})'(t_0), (\underline{f})'(t_0)].$$

Definition 5 (Stefanini and Bede [51]). Let $F : T \to \mathbb{K}_C$ be an interval-valued function such that $F(t) = [\underline{f}(t), \overline{f}(t)]$.

A point $t_0 \in T$ is said to be a switching point for the differentiability of $F$, if in any neighborhood $V$ of $t_0$ there exist points $t_1 < t_0 < t_2$ such that

(type I) $F$ is differentiable at $t_1$ in the first form while it is not differentiable in the second form, and $F$ is differentiable at $t_2$ in the second form while it is not differentiable in the first form, or

(type II) $F$ is differentiable at $t_1$ in the second form while it is not differentiable in the first form, and $F$ is differentiable at $t_2$ in the first form while it is not differentiable in the second form.
The following result is an immediate consequence of those in [51]

**Proposition 2.** Let $F : T \rightarrow \mathcal{K}_C$ be an interval-valued function.

(a) If $F$ is gH-differentiable in the first form then $\text{len}(F)$ is increasing.
(b) If $F$ is gH-differentiable in the second form then $\text{len}(F)$ is decreasing.

Next result follows from Theorem 9 and Proposition 2.

**Proposition 3.** Let $F : T \rightarrow \mathcal{K}_C$ be an interval-valued function such that $F(t) = [\underline{f}(t), \overline{f}(t)]$. If $F$ is gH-differentiable at $t_0 \in T$ then one of the following cases hold:

(a) $\text{len}(F)$ is differentiable at $t_0$.
(b) $\text{len}(F)'_-(t_0)$ and $\text{len}(F)'_+(t_0)$ exist and satisfy $\text{len}(F)'_-(t_0) = -\text{len}(F)'_+(t_0)$.

**Remark 1.** Note that if $F : T \rightarrow \mathcal{K}_C$ is gH-differentiable at $t_0$ and $\text{len}(F)'_-(t_0) = 0 = -\text{len}(F)'_+(t_0)$ then $\text{len}(F)$ is differentiable at $t_0$. Routine calculations show that in this case, $\underline{f}$ and $\overline{f}$ are both differentiable at $t_0$ and $(\overline{f})'(t_0) = (\underline{f})'(t_0)$.

The study of the points at which the monotonicity of $\text{len}(F)$ changes is a quite interesting problem, which is closely related to the critical points of $\text{len}(F)$.

**Definition 6 (Stefanini and Bede [51]).** Let $F : T \rightarrow \mathcal{K}_C$ be an interval-valued function. We say that a point $t_0 \in T$ is an $l$-critical point of $F$ if it is a critical point for the length function $\text{len}(F)$.

All switching points for the differentiability of $F$ are $l$-critical points of $F$. But an $l$-critical point is not necessarily a switching point (see some examples in [51]). Next result states the relationships between them.

**Theorem 11.** Let $F : T \rightarrow \mathcal{K}_C$ be a gH-differentiable interval-valued function.

(a) If part (a) of Proposition 3 is verified and $t_0$ is a strict local maximum point of $\text{len}(F)$ then $t_0$ is a switching point of type I for the differentiability of $F$.
(b) If part (a) of Proposition 3 is verified and $t_0$ is a strict local minimum point of $\text{len}(F)$ then $t_0$ is a switching point of type II for the differentiability of $F$.
(c) If part (b) of Proposition 3 is verified and $\text{len}(F)'_+(t_0) < 0$ then $t_0$ is a switching point of type I for the differentiability of $F$.
(d) If part (b) of Proposition 3 is verified and $\text{len}(F)'_+(t_0) > 0$ then $t_0$ is a switching point of type II for the differentiability of $F$.

**Proof.** The proof of parts (a) and (b) is straightforward, so we omit it.

(c) Now let us assume that part (b) of Proposition 3 is verified. Then, there exists $\delta > 0$ sufficiently small such that $t_0$ is the only point in $(t_0 - \delta, t_0 + \delta)$ that verifies part (b). In fact, if any $t \in (t_0 - \delta, t_0 + \delta)$ verifies part (b) then $F(t) = C$ for some $C \in \mathcal{K}_C$, i.e., $F$ is constant in $(t_0 - \delta, t_0 + \delta)$ and $\text{len}(F)$ is differentiable at $t_0$. Now, since $\text{len}(F)'_+(t_0) < 0$ and $\text{len}(F)'_-(t_0) > 0$, we have that $t_0$ is a switching point of type I for the differentiability of $F$.

(d) The proof is similar to that of part (c). □

Note that in the proof of Theorem 11 we have also shown that if $F$ is a gH-differentiable interval-valued function, and $T_0 \subset T$ is the set of points such that Theorem 9 part (b) is verified but part (a) is not, then the points in $T_0$ are isolated. It is necessary to assume that part (a) in Theorem 9 is not verified in order to exclude the case $F(t) = C$ for some $C \in \mathcal{K}_C$.

Note also that if the hypothesis in part (c) of Theorem 11 is verified then $t_0$ is a strict local maximum point of $\text{len}(F)$. In the same form, if the hypothesis in part (d) of Theorem 11 is verified then $t_0$ is a strict local minimum point of $\text{len}(F)$. Thus, we have the following immediate consequence.
Corollary 3. Let $F : T \to \mathcal{K}_C$ be a $gH$-differentiable interval-valued function.

(a) If $t_0$ is a strict local maximum point of $\text{len}(F)$ then $t_0$ is a switching point of type I for the differentiability of $F$.
(b) If $t_0$ is a strict local minimum point of $\text{len}(F)$ then $t_0$ is a switching point of type II for the differentiability of $F$.

Next we give an example illustrating the above results.

Example 4. Let $F : \mathbb{R}^+ \to \mathcal{K}_C$ be an interval-valued function defined by (Fig. 1)

$$F(t) = \begin{cases} [1 - t, 2t + 2] & \text{if } 0 < t \leq 1, \\ \left[3t - \frac{t^2}{2} - \frac{5}{2}, 4 \exp(1 - t) + 3t - 3\right] & \text{if } 1 \leq t < \infty. \end{cases}$$

Then $F$ is $gH$-differentiable. At $t = 1$ part (b) of Theorem 9 is verified and on $\mathbb{R}^+ \setminus \{1\}$ part (a) of Theorem 9 is verified. On the other hand, the length function $\text{len}(F)$ is defined by

$$\text{len}(F)(t) = \text{len}(F(t)) = \begin{cases} 3t + 1 & \text{if } 0 < t \leq 1, \\ 4 \exp(1 - t) + \frac{t^2}{2} - \frac{1}{2} & \text{if } 1 \leq t < \infty. \end{cases}$$

Since $(\text{len}(F))'_+(1) = -3 < 0$ then $t = 1$ is a strict local maximum of $\text{len}(F)$. So, $t = 1$ is a switching point of type I for the differentiability of $F$. Now, $t \geq 1.99$ is a strict local minimum point of $\text{len}(F)$ then $t \geq 1.99$ is a switching point of type II for the differentiability of $F$.

Remark 2. Note that in Example 4, $t = 3$ is a strict local maximum of the extreme function $f$; however, $t = 3$ is not an $l$-critical point of $F$. Therefore, if $t_0$ is an extreme point of any endpoint function of $F$, $\overline{f}$ or $\underline{f}$, this does not imply that $t_0$ is an $l$-critical point of $F$. The converse is also not valid.

For the class (1) we have the following result.

Theorem 12. Let $F : T \to \mathcal{K}_C$ be a $gH$-differentiable interval-valued function such that $F(t) = C \cdot g(t)$.

(a) If $t_0$ is a strict local maximum point of $|g|$, then $t_0$ is a switching point of type I for the differentiability of $F$.
(b) If $t_0$ is a strict local minimum point of $|g|$, then $t_0$ is a switching point of type II for the differentiability of $F$.

Proof. Let $C = [a, b]$. Then $\text{len}(F)(t) = |ag(t) - bg(t)| = (b - a)|g(t)|$. So, if $t_0$ is a strict local maximum of $|g|$ then $t_0$ is a strict local maximum $\text{len}(F)$. From Corollary 3, $t_0$ is a switching point of type I for the differentiability of $F$. This proves part (a). Part (b) follows similarly. □
Example 5. Let us consider the interval-valued function $F : (-10, 10) \to K_C$ defined by (Fig. 2)

$$F(t) = [2, 4] \left( \cos(t) - \frac{t^2}{32} \right).$$

$F$ is gH-differentiable because $g(t) = \cos t - t^2/32$ is a differentiable function. $g$ has got two roots, $t_1 \cong -1.5004$ and $t_2 \cong 1.5004$, which are strict local minimum points of $|g|$. Therefore $t_1$ and $t_2$ are switching points of type II for the differentiability of $F$.

On the other hand, $t_3 = 0$, $t_4 \cong -3.3527$ and $t_5 \cong 3.3527$ are strict local maximum points of $|g|$ while $t_6 \cong -5.9052$ and $t_7 \cong 5.9052$ are strict local minimum points of $|g|$. Therefore, $t_3$, $t_4$ and $t_5$ are switching points of type I for the differentiability of $F$ and $t_6$ and $t_7$ are switching points of type II for the differentiability of $F$ (Fig. 3).

6. The fundamental theorem of calculus

In classical (real) analysis, the fundamental theorem of calculus specifies the relationship between the two central operations of calculus: differentiation and integration. The first part of the theorem, sometimes called the first fundamental theorem of calculus, shows that an indefinite integration can be reversed by a differentiation. The second part, sometimes called the second fundamental theorem of calculus, allows one to compute the definite integral of a function by using any one of its infinitely many antiderivatives. Markov [41] gave an interval valued version of these theorems. We first particularize the so-called first fundamental theorem of calculus in [41] for the class of interval functions (1).
Theorem 13. If the interval function \( F(t) = C \cdot g(t) \) is continuous in \( T \), then the interval function \( G \) defined by \( G(t) = \int_{t_1}^t F(u) \, du \) is \( g_H \)-differentiable at \( t \) and \( G'(t) = F(t) \), \( t \in T \).

Before stating an interval version of the second fundamental theorem of calculus, we first give the following definition.

Definition 7. Let \( F : T \to K_C \) be an interval-valued function. \( F \) is said to be \( \mu \)-increasing (\( \mu \)-decreasing) if the function \( \text{len}(F) \) is increasing (decreasing).

The second fundamental theorem of calculus in [41] assumes that the interval function \( F \) is \( \mu \)-increasing, but this condition can be weakened. To state a more general result, we first give a preliminary useful lemma whose proof is omitted since it is rather simple.

Lemma 1. Let \( F : T \to K_C \) be an interval-valued function. Let \( [x, \beta] \subseteq T \) and let \( \tilde{F} : [x, \beta] \to K_C \) be defined as \( \tilde{F}(t) = F(t) \bigcirc_{g_H} F(x) \). If \( F \) is \( \mu \)-increasing or \( \mu \)-decreasing in \( [x, \beta] \), then \( \tilde{F} \) is \( \mu \)-increasing.

Next we state an interval version of the second fundamental theorem of calculus for general interval functions.

Theorem 14. Let \( F : T \to K_C \) be an interval-valued function. Assume that \( F \) is \( g_H \)-differentiable in an open interval containing \( [x, \beta] \subseteq T \) and that it is \( \mu \)-increasing or \( \mu \)-decreasing in \( [x, \beta] \). Then

\[
\int_x^\beta F'(t) \, dt = F(\beta) \bigcirc_{g_H} F(x).
\]

Proof. The proof follows the same steps as the proof of Theorem 16 in [41] by noting that, from Theorem 5 in the cited paper, \( \tilde{F} = F(t) \bigcirc_{g_H} F(x) \) is \( g_H \)-differentiable in \( I \) and \( \tilde{F}'(t) = F'(t), \forall t \in I \).

Next theorem gives a more general result.

Theorem 15. Let \( F : T \to K_C \) be an interval-valued function. Assume that \( F \) is \( g_H \)-differentiable in an open interval containing \( [x, \beta] \subseteq T \). If \( F \) has a finite number of switching points on \( [x, \beta] \), say \( c_1, \ldots, c_n \), and exactly at these points, then

\[
\int_x^\beta F'(x) \, dx = \sum_{i=1}^{n+1} F(c_i) \bigcirc_{g_H} F(c_{i-1}),
\]

where \( x = c_0 < c_1 < \cdots < c_n < c_{n+1} = \beta \).

Proof. For simplicity we consider only one switching point, the case of an arbitrary finite number of switching points follows similarly. Let us suppose that \( F \) is \( g_H \)-differentiable on \( [x, \beta] \) and on \( [c, \beta] \). Then from Proposition 1 and Theorem 14 we have

\[
\int_x^\beta F'(x) \, dx = \int_x^c F'(x) \, dx + \int_c^\beta F'(x) \, dx = F(c) \bigcirc_{g_H} F(x) + F(\beta) \bigcirc_{g_H} F(c).
\]

Thus the proof is completed.

Remark 3. Theorem 30 in [51] states a result which is quite close to that in Theorem 15, but rather different arguments are used in the proofs. Moreover if \( c \in [x, \beta] \) is the only switching point for the differentiability of \( F \) and \( F(c) \) is a singleton, then it does not imply that \( \int_x^\beta F'(x) \, dx = F(\beta) \bigcirc_{g_H} F(x) \). For instance, if \( F(t) = [-|t|, |t|] \), we have that \( F \) is \( g_H \)-differentiable and \( F(0) = \{0\} \) but \( \int_{-1}^1 F'(x) \, dx \neq F(1) \bigcirc_{g_H} F(-1) \). This observation corrects Theorem 30 in [51].

While this paper was being reviewed, an anonymous referee kindly alerted us that the authors of [51] had made some corrections to their paper (see [52]), among which is the one mentioned in this remark.
Next result characterizes the $\mu$-monotonicity for the class of interval functions (1).

**Proposition 4.** Let $F : T \to K_C$ be an interval-valued function such that $F(t) = C \cdot g(t)$. Then, $F$ is $\mu$-increasing ($\mu$-decreasing) iff $|g|$ is increasing (decreasing).

**Proof.** Then the result follows immediately from the fact $\mu(F(t)) = \mu(C)|g(t)|$. □

As a consequence of Lemma 1, Theorem 15 and Proposition 4, we next state the following version of the second fundamental theorem of calculus for the class of interval functions (1).

**Corollary 4.** Let $F : T \to K_C$ be an interval-valued function such that $F(t) = C \cdot g(t)$. Assume that $F$ is $gH$-differentiable in an open interval containing $[\alpha, \beta] \subset T$ and that $|g|$ has a finite number of extreme points on $[\alpha, \beta]$, say $c_1, \ldots, c_n$, and exactly at these points, then

$$\int_\alpha^\beta F'(t) \, dt = \sum_{i=1}^{n+1} F(c_i) \bigcap_{gH} F(c_{i-1}),$$

where $\alpha = c_0 < c_1 < \cdots < c_n < c_{n+1} = \beta$.

**7. Applications**

As an example of the wide range of possible practical applications of the obtained results, in this section we report two applications to two rather different fields: statistics, where a Delta method for interval valued random elements is derived, and interval differential equations, where a new method for solving them is proposed. This new method overcomes some drawbacks of existing procedures.

**7.1. A Delta method for interval valued random elements**

Central limit theorems (CLTs) for random compact sets have been derived in the literature. For random compact sets in $\mathbb{R}^d$, the CLT was first established in full generality by Weil [56]. Later on, the result was extended to compact sets in more abstract spaces (see for example [47] for a CLT for random compact sets in Banach spaces). All these CLTs share the feature that the limit distribution is derived for the Hausdorff distance between the mean of the random compact sets and its expectation, not for the difference, as it is the case of the CLT for random vectors. This is due to the fact that, as discussed in Section 2, a suitable difference was not defined by them.

When the random compact sets take values in $K_C$, the CLT in [56] can be equivalently expressed as follows: let $Y_1, Y_2, \ldots, Y_n$ be independent, identically distributed (iid) random compact sets taking values in $K_C$ with common expectation $E(Y_1) = [a, b]$. Let $(1/n)(Y_1 + Y_2 + \cdots + Y_n) = [A_n, B_n]$. Then

$$\sqrt{n} \begin{pmatrix} A_n - a \\ B_n - b \end{pmatrix} \xrightarrow{L} Z_2,$$

as $n \to \infty$, where $\xrightarrow{L}$ denotes convergence in distribution and $Z_2$ is a bivariate normal variable with zero mean, that is to say, the CLT for random elements taking values in $K_C$ is a CLT for the endpoint functions, i.e., for a sequence of bivariate random variables.

In classical Statistics (here by classical we refer to statistics for random elements taking values in an Euclidean space, that is to say, to random vectors) the Delta method (see for example Theorem 11.2.14 in Lehmann and Romano [40]) gives the asymptotic distribution of differentiable functions of a vector of means, where the transformed vectors also take values in an Euclidean space. The definition of the difference between elements in $K_C$ in (4) and (5) let us define the $gH$-differentiability of a function taking values in $K_C$. Taking advantage of these definitions we will derive an interval-valued version of the classical Delta method.
Let $X_1, X_2, \ldots, X_n$ be iid random variables taking values in $T$ with expectation $\mu = E(X_1)$ and variance $\text{var}(X_1) = \sigma^2 < \infty$. Then, the CLT states that
\begin{equation}
\sqrt{n}(\bar{X} - \mu) \xrightarrow{L} Z_1, \tag{15}
\end{equation}
as $n \to \infty$, where $\bar{X} = (1/n)\{X_1 + X_2 + \cdots + X_n\}$ is the sample mean and $Z_1 \sim N(0, \sigma^2)$. Let $F : T \to K_C$ be an interval-valued function. Next result gives the convergence in law of the sequence of random sets $\sqrt{n}\{F(\bar{X}) \ominus_{gH} F(\mu)\}$.

Recall that the convergence in law for random elements taking values in $K_C$ is tantamount to the convergence in law of the bivariate random vector whose components are the endpoint functions.

**Theorem 16.** Let $X_1, X_2, \ldots, X_n$ be iid random variables taking values in $T$ with expectation $\mu = E(X_1)$ and variance $\text{var}(X_1) = \sigma^2 < \infty$. Let $F : T \to K_C$ be an interval-valued function. Assume that $F$ is $gH$-differentiable at $\mu$ with $F'(\mu) \neq \{0\}$. Then,
\begin{equation}
\sqrt{n}\{F(\bar{X}) \ominus_{gH} F(\mu)\} \xrightarrow{L} Z_1 F'(\mu), \tag{16}
\end{equation}
as $n \to \infty$, where $Z_1 \sim N(0, \sigma^2)$.

**Proof.** Differentiability of $F$ at $\mu$ implies
\[ F(x) \ominus_{gH} F(\mu) = (x - \mu)F'(\mu) + R(x - \mu), \]
where $R(y) \in K_C$ satisfies
\begin{equation}
\lim_{y \to 0} \frac{R(y)}{y} = \{0\}. \tag{16}
\end{equation}

Now,
\[ \sqrt{n}\{F(\bar{X}) \ominus_{gH} F(\mu)\} = \sqrt{n}(\bar{X} - \mu)F'(\mu) + \sqrt{n}R(\bar{X} - \mu). \]

Note that
\begin{equation}
\sqrt{n}R(\bar{X} - \mu) = \sqrt{n}(\bar{X} - \mu) \frac{R(\bar{X} - \mu)}{\bar{X} - \mu}. \tag{17}
\end{equation}

Since $\bar{X} - \mu = o_p(1)$, from (16) it follows
\begin{equation}
\frac{R(\bar{X} - \mu)}{\bar{X} - \mu} = o_p(1). \tag{18}
\end{equation}

From (15),
\[ \sqrt{n}(\bar{X} - \mu) = O_p(1). \tag{19} \]

Now, from (17)–(19), we get $\sqrt{n}R(\bar{X} - \mu) = o_p(1)$. Finally the result is obtained by Slutsky’s Theorem.

**Remark 4.** In the light of the equivalence of the convergence in law of random elements taking values in $K_C$ and the convergence in law of the bivariate random vector whose components are the endpoint functions, it might seem that the result in Theorem 16 is an immediate consequence of the classical Delta method. But this is not true, since as shown in Example 1, the $gH$-differentiability of $F$ does not imply the differentiability of the endpoint functions. In this sense, our result is more general than that obtained by applying the classical Delta method to the endpoint functions, which require that they are differentiable.

Next we particularize the result in Theorem 16 for the class of interval-valued functions (1).
Corollary 5. Let \( X_1, X_2, \ldots, X_n \) be iid random variables taking values in \( T \) with expectation \( \mu = E(X_1) \) and variance \( \text{var}(X_1) = \sigma^2 < \infty \). Let \( F : T \to \mathcal{K}_C \) be an interval-valued function such that \( F(t) = C \cdot g(t) \), with \( C = [a, b] \). If \( g \) is differentiable at \( \mu \) or \( g_\mu^- = -g_\mu^+ \) and \( \alpha = -b \), then

\[
\sqrt{n} (F(X) \ominus gH F(\mu)) \xrightarrow{L^2} Z_1 F'(\mu),
\]

as \( n \to \infty \), where \( Z_1 \sim N(0, \sigma^2) \).

7.2. Interval differential equations

We consider the following linear interval differential equation:

\[
\begin{aligned}
X'(t) &= a(t)X(t), \\
X(0) &= X_0,
\end{aligned}
\]

where \( X_0 \in \mathcal{K}_C \) and \( a : [0, T] \to \mathbb{R} \) is a continuous function. We are interested in finding a solution of (20), i.e., in finding a \( gH \)-differentiable interval-valued function \( X : [0, T] \to \mathcal{K}_C \) satisfying (20).

It is well-known, see for instance [23,36,51], that if \( X' \) is the \( gH \)-derivative in the first form (second form) of \( X \) on \([0, T]\), then problem (20) can be rewritten as a system of differential equations in two variables. In fact, if we denote \( X(t) = [\underline{x}(t), \overline{x}(t)] \) then we have two cases:

(I) If \( X' \) is the \( gH \)-derivative in the first form then \( X'(t) = [\underline{x}'(t), \overline{x}'(t)] \). Thus, solving (20) is equivalent to solving the system

\[
\begin{aligned}
\underline{x}'(t) &= \min \{a(t)\underline{x}(t), a(t)\overline{x}(t)\}, \\
\overline{x}'(t) &= \max \{a(t)\underline{x}(t), a(t)\overline{x}(t)\}, \\
\underline{x}(0) &= \underline{x}_0, \\
\overline{x}(0) &= \overline{x}_0,
\end{aligned}
\]

where \( \underline{x}(t), \overline{x}(t) \) are real functions. So, \( X_1(t) = [\underline{x}(t), \overline{x}(t)] \) is a solution of (20).

(II) If \( X' \) is the \( gH \)-derivative in the second form, then \( X'(t) = [\underline{x}'(t), \overline{x}'(t)] \). Thus, solving (20) is equivalent to solving the system

\[
\begin{aligned}
\overline{x}'(t) &= \min \{a(t)\underline{x}(t), a(t)\overline{x}(t)\}, \\
\underline{x}'(t) &= \max \{a(t)\underline{x}(t), a(t)\overline{x}(t)\}, \\
\underline{x}(0) &= \underline{x}_0, \\
\overline{x}(0) &= \overline{x}_0,
\end{aligned}
\]

where \( \underline{x}(t), \overline{x}(t) \) are real functions. So, \( X_2(t) = [\underline{x}(t), \overline{x}(t)] \) is a solution of (20).

In both cases the solutions \( X_1 \) and \( X_2 \) have monotone length functions (see Proposition 2). Thus, by using the previous procedure we will never have periodic solutions, that is, \( X_1 \) and \( X_2 \) will not have switching points. If we consider the possible existence of switching points for the differentiability of a solution \( X \) on \([0, T]\), then we may have more solutions by alternating the procedures (I) and (II) (for more details see [9,51]). A very interesting problem is how to choose these switching points [9,51]. Next we give a new procedure for obtaining a solution of (20) which includes switching points in a natural form.

The interval differential equation (20) is considered as an interval correspondent of the linear differential equation (see [8])

\[
\begin{aligned}
x'(t) &= a(t)x(t), \\
x(0) &= x_0.
\end{aligned}
\]

If we let

\[
g_1(t) = \exp \left( \int_0^t a(s) \, ds \right)
\]

then the solution \( x \) of the ordinary differential equation (21) is given by

\[
x(t, x_0) = x_0 g_1(t).
\]
Clearly, \( x(t, x_0) \) is a continuous function of the parameter \( x_0 \). So, by applying the united extension (interval extension principle [54,55]) to the function \( x \), for each fixed \( t \in [0, T] \), we obtain the interval-valued function \( X : [0, T] \rightarrow K_C \), defined by

\[
X(t) = x^U(t, X_0, B) = \bigcup_{x_0 \in X_0} x(t, x_0).
\]

Equivalently,

\[
X(t) = g_1(t) \cdot X_0.
\]

(22)

Since \( g_1 \) is a differentiable function then, from Theorem 10, \( X \) is \( gH \)-differentiable. Also, \( X \) verifies (20), in fact,

\[
X'(t) = g'_1(t) \cdot X_0 = (a(t)g_1(t)) \cdot X_0 = a(t)(g_1(t) \cdot X_0) = a(t) \cdot X(t),
\]

for all \( t \in [0, T] \). Therefore \( X \) is a solution of (20).

Note that the interval-valued function \( X \) in (22) is equivalent to the interval-valued function presented in Theorem 6 of [8]. In [8] it was shown that \( X \) is a solution of (20) if \( a(t) > 0 \) or if \( a(t) < 0 \).

**Example 6.** We consider the following problem which was presented in [51]:

\[
\begin{align*}
X'(t) &= (\sin t)X, \quad t \in [0, 6], \\
X(0) &= [1, 2].
\end{align*}
\]

(23)

If \( X' \) is the \( gH \)-derivative in the first form, via procedure (I), we obtain the solution \( X_1 \). If \( X' \) is the \( gH \)-derivative in the second form, via procedure (II), we obtain the solution \( X_{II} \). Now, from Example 42 in [51], if the switching points are chosen from the set \( \{ t = k\pi, \; k \in \mathbb{Z} \} \), then we have only one switching point in the interval \( [0, 6] \), \( t = \pi \). Let \( X_{III} \) be the solution obtained by using the procedure (I) in the interval \( [0, \pi] \) and the procedure (II) in the interval \( [\pi, 6] \). We also obtain the solution \( X_{IV} \) by using the procedure (II) in the interval \( [0, \pi] \) and the procedure (I) in the interval \( [\pi, 6] \). Therefore, we have four solutions of problem (23). For more details see [51].

In this case, the solution \( X \) of (23) given by (22) coincides with solution \( X_{III} \) obtained by combining procedures (I) and (II) (Fig. 4). Clearly, this solution is more adequate than the other solutions since this solution preserves some characteristics of the solution of the correspondent linear differential equation.

From the previous results we know that the behavior of an interval-valued function \( F \) is closely related to the behavior of its length function \( \text{len}(F) \). Next we show that this is also the case for the solution \( X \) of (23) given by (22). Specifically we show that the length function of \( X \) satisfies a differential equation which is closely related to the original problem (20).
Proposition 5. Let $X$ be the interval-valued function given by (22), then $\text{len}(X)$ is a solution of the linear differential equation
\[ x' = a(t)x, \quad x(0) = \text{len}(X_0). \]  \hfill (24)

**Proof.** Note that $\text{len}(X)(t) = g_1(t)\text{len}(X_0)$. Then $\text{len}(X)'(t) = a(t)g_1(t)\text{len}(X_0) = a(t)\text{len}(X)(t)$. Thus $\text{len}(X)$ is a solution of the linear differential equation (24). \hfill \square

Proposition 6. Let $X$ be the interval-valued function given by (22). Then the switching points of $X$ are all points $t \in [0, T]$ such that $a(t) = 0$.

**Proof.** The length function $\text{len}(X)$ is differentiable. Then, from Theorem 12, all critical points of $\text{len}(X)$ are the switching points of $X$. In this case, all critical points of $\text{len}(X)$ are all points $t \in [0, T]$ such that $a(t) = 0$. \hfill \square

From Propositions 5 and 6 it can be concluded that the solution $X$ of (23) given by (22) is more adequate than the other solutions since its length function $\text{len}(X)$ has the same characteristics as the solution of the correspondent linear differential equation. We next show that an additional property of this solution is that it is equivalent to the solution obtained via differential inclusions.

Proposition 7. Let $X$ be the interval-valued function given by (22), then $X$ is a solution of the differential inclusion
\[ x' = a(t)x, \quad x(0) \in X_0. \]

**Proof.** It is an immediate consequence of Theorem 2 in [16]. \hfill \square

On the other hand, additionally to the interval differential equation (20) there exist other interval correspondents of the linear differential equation (21). For instance (see [8,24]), we can consider the interval differential equation $X' + (-a(t))X = 0$ or, by using the $g\mathcal{H}$-difference, we can also consider $X'\ominus_{g\mathcal{H}} a(t)X = 0$. In the first case, we do not have any nontrivial interval solution, i.e., $X(t)$ is a real-valued function, see [8]. In the second case, the interval-valued function $X$ given by (22) is a solution of the interval differential equation $X'\ominus_{g\mathcal{H}} a(t)X = 0$, $X(0) = X_0$. Consequently, by using the $g\mathcal{H}$-difference we have another interval correspondent of the linear differential equation (21), which is equivalent to (20).

Now we consider the following linear differential equation:
\[ x'(t) = a(t)x(t) + b, \quad x(0) = x_0. \]  \hfill (25)

What would be the problem correspondent of Eq. (25) in the interval context? The following three problems were considered in [8]:
\[ X'(t) = a(t)X(t) + B, \quad X'(t) + (-1)a(t)X(t) = B, \quad X'(t) + (-1)B = a(t)X(t), \]
with $X(0) = X_0, X_0, B \in \mathcal{K}_C$ and $X'$ is the $G$-derivative of the interval-valued function $X$. Another interval differential equation correspondent of the linear differential equation (25) can be obtained by using $g\mathcal{H}$-difference:
\[ X'(t) = a(t)X(t)\ominus_{g\mathcal{H}} (-1)B, \quad X'(t)\ominus_{g\mathcal{H}} a(t)X(t) = B, \quad X'(t)\ominus_{g\mathcal{H}} B = a(t)X(t). \]

Which of the six equations is appropriate? Which of the six equations give a solution that has some characteristics of the deterministic solution (solution of (25))? We think that an appropriate solution is an interval-valued function $X$ such that its length function $\text{len}(X)$ is a solution of (25), analogously to Proposition 5 for problem (20). The solution of these questions is out of the scope of this paper, it will be the topic of a future work.

8. Conclusions

In this paper we have presented several new results on calculus of interval-valued functions using generalized Hukuhara differentiability. Specifically, we have made a study on the switching points for differentiability of an
interval-valued function. Then, we have particularized our previous results for the important class of interval-valued functions $F(t) = C \cdot g(t)$, where $C$ is an interval and $g$ a real function.

Since fuzzy arithmetic, Zadeh’s extension principle and (mathematical) analysis are done on $\alpha$-cuts, which are intervals, the results obtained could be extended to the fuzzy context. This point will be the topic of a future article. We also want to point out that the applications obtained in Section 7 can be extend to the fuzzy context. For instance, in [16] it was proposed a procedure to obtain a solution $X_E$ of a fuzzy differential equation by applying Zadeh’s extension principle to a deterministic solution. Nevertheless, there are some criticisms on this approach since one cannot talk about the derivative of a fuzzy function. Fuzzy differential equations are directly interpreted with the help of Zadeh’s extension principle without having a derivative of fuzzy functions. However, in some cases, we can show that $X_E$ is $gH$-differentiable (for fuzzy functions see [7–9,13,18]) as it was shown in Section 7 for the case of interval differential equations. This point will be also the topic of a future article.

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