Opial-type inequalities for interval-valued functions

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Abstract

This paper establishes Opial-type inequalities for generalized Hukuhara differentiable interval-valued functions. Numerical examples are presented to illustrate the applicability of theory developed herein.

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1. Introduction

The importance of the study of set-valued analysis from a theoretical point of view as well as from their applications is well known [5,6]. Many advances in set-valued analysis have been motivated by control theory and dynamical game theory [6]. Optimal control theory and mathematical programming were one of the motivating forces behind set-valued analysis since the sixties [6]. Interval Analysis is a particular case and it was introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. The first monograph dealing with interval analysis was given by Moore [19]. Moore is recognized to be the first to use intervals in computational mathematics, now called numerical analysis. He also extended and implemented the arithmetic of intervals to computers. One of his major achievements was to show that Taylor series methods for solving differential equations not only are more tractable, but also more accurate [20].

On the other hand, several integral inequalities involving integrable functions and their derivatives, such as Wirtinger’s inequality, Ostrowski’s inequality and Opial’s inequality, among others, have been well studied during the past century (see [3,18] and their references). All these works have provided fundamental tools to the development of many areas in mathematical analysis.

Recently, some differential-integral inequalities have been extended to the set-valued context. For example Anastassiou in [3], by using Hukuhara derivative (H-derivative, for short), extended an Ostrowski-type inequality to context of fuzzy-valued functions. Also, Chalco-Cano et al. in [10,9], using the concept of generalized Hukuhara derivative

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(\(gH\)-derivative, for short), established some Ostrowski-type inequalities for interval-valued functions and presented an application to numerical integration.

We remark that \(H\)-derivative is a very restrictive concept (see [7,11]). On the other hand, it is well-known that \(gH\)-derivative is a very general concept on derivative for interval-valued functions, see [7,11,28]. In this direction, motivated by [3,9,10] and by [7,11,17,28], we establish some Opial-type integral inequalities for \(gH\)-differentiable interval-valued functions which is the main objective of this article.

This paper is organized as follows: Section 2 contains a brief summary of the standard interval arithmetic and provides a short discussion about the phenomenon of interval dependency founded in this arithmetic. Moreover the algebraic meaning of the square of an interval is established which becomes a fundamental part for obtaining the Opial-type integral inequalities for interval-valued functions. Section 3 recalls the Pompeiu–Hausdorff metric on the interval space as well as the concepts of \(gH\)-differentiable interval-valued functions and switching points for the \(gH\)-differentiability. Also the concept of piecewise continuously \(gH\)-differentiable interval-valued function is introduced in Section 4. In the same section, a result involving concepts of the switching points and the piecewise continuously \(gH\)-differentiable interval-valued function is presented. This result plays a key role in the development of this study. Section 4 recalls the Opial’s inequality and some of its variations, which are extended to context of piecewise continuously \(gH\)-differentiable interval-valued functions in Section 5. Section 6 presents some numerical examples to illustrate the applicability of the inequalities developed herein. Finally, Section 7 contains the final considerations and the conclusion of this study.

2. Basic concepts

Let \(\mathbb{R}\) be the one-dimensional Euclidean space. Following [14], let \(\mathcal{K}_C\) denote the family of all bounded and closed intervals of \(\mathbb{R}\), that is,

\[
\mathcal{K}_C = \{ [a, b] \mid a, b \in \mathbb{R} \text{ and } a \leq b \}.
\]

The interval arithmetic best known and most often used is standard interval arithmetic (SIA), which was introduced by Moore–Warmus–Sunaga ([22,29,30]) as follows: let \(*\) be one of the operations \(\{+,-,\times,\div\}\), then given \(A, B \in \mathcal{K}_C\) we have

\[
A *_{SIA} B = \left[ \min_{a \in A, b \in B} a \ast b, \max_{a \in A, b \in B} a \ast b \right].
\]

From (1), given \(A = [a, \bar{a}]\) and \(B = [b, \bar{b}]\) in \(\mathcal{K}_C\) and \(\lambda \in \mathbb{R}\), we have

\[
A +_{SIA} B = A + B = [a + b, a + \bar{b}] ;
\]

\[
\lambda \cdot_{SIA} A = \lambda \cdot A = [\min \{ \lambda a, \lambda \bar{a} \}, \max \{ \lambda a, \lambda \bar{a} \}] ;
\]

\[
A -_{SIA} B = A + (-1)B = [a - \bar{b}, a - b] ;
\]

\[
A \times_{SIA} B = AB = [\min \{ab, a\bar{b}, \bar{a}b, \bar{a}\bar{b}\}, \max \{ab, a\bar{b}, \bar{a}b, \bar{a}\bar{b}\}] ;
\]

\[
\frac{A}{B} = \frac{A}{B} = \left[ \min \left\{ \frac{a}{b}, \frac{a}{\bar{b}}, \frac{\bar{a}}{b}, \frac{\bar{a}}{\bar{b}} \right\}, \max \left\{ \frac{a}{b}, \frac{a}{\bar{b}}, \frac{\bar{a}}{b}, \frac{\bar{a}}{\bar{b}} \right\} \right] \quad \text{if } 0 \notin B.
\]

In the interval analysis literature, it is well-known that SIA may overestimate the length of the resultant interval due to the phenomenon of interval dependence (see [16,19,21]). In other words, given \(A, B \in \mathcal{K}_C\), the SIA always operates these intervals as if they were independent elements, even when they represent the same range (uncertainty), that is, \(A = B\). For example, for \(A = [-1, 1]\) we have that \(A -_{SIA} A = [-1, 1] \neq [0, 0]\) and \(A \times_{SIA} A = [-1, 1] \neq [0, 1]\). Therefore, since interval arithmetic operations are fundamental in interval analysis, some concepts, such as differentiability for interval-valued functions, where it is crucial to consider an arithmetic operation of difference between intervals, they may present shortcomings if are formulated by means of SIA.

In order to avoid some shortcomings caused by the phenomenon of interval dependency some additional operations have been introduced. For example, one of the first attempts in obtaining a useful concept of differentiability of interval-valued functions avoiding the phenomenon of interval dependence is credited to Hukuhara [15] who defined the arithmetic operation of difference between intervals called the Hukuhara difference (H-difference, for short).
is, given two intervals $A$ and $B$, if there exists another interval $C$ such that $A = B + C$, then $C$ is called the $H$-difference of $A$ and $B$, and it is denoted by $A - H B$. Then differently from SIA, it follows that $A - H A = [0, 0]$ for any interval $A$. However, this definition is very restrictive since the $H$-difference of two intervals does not exist for arbitrary intervals. That is, $A - H B$ only exists if

$$\text{len}(A) \geq \text{len}(B),$$

where $\text{len}(K) = \bar{k} - \underline{k}$ is the length of the interval $K = [\underline{k}, \bar{k}]$.

Recently, Stefanini and Bede [28] introduced the concept of generalized Hukuhara difference of two sets $A, B \in K_C$ ($gH$-difference for short) as follows

$$A \ominus gH B = C \Leftrightarrow \begin{cases} \text{(a)} & A = B + C \\ \text{(b)} & B = A + (-1)C. \end{cases} \quad (7)$$

In case (a), the $gH$-difference is coincident with the $H$-difference. Thus, the $gH$-difference is a generalization of the $H$-difference. On the other hand, $gH$-difference exists for any two intervals $A = [a, \bar{a}], B = [\bar{b}, b] \in K_C$ and

$$A \ominus gH B = \left[ \min\{a - \bar{b}, \bar{a} - b\}, \max\{a - \bar{b}, \bar{a} - b\} \right]. \quad (8)$$

Using the $gH$-difference, Stefanini and Bede [28] introduced a differentiability concept for interval-valued functions, which is more suitable than the $H$-differentiability. This concept is exposed in the next section. For more details and properties of $gH$-difference see [27,28].

We now focus on the interval product $A \times_{SIA} A$. It is known that the operations given by SIA were introduced with the purpose of handling computational errors, which may be caused by the limited capacity of digital representation of the machines. Each arithmetic operation provided by SIA it is an interval extension of its respective version for real numbers. This means that, for any $A, B \in K_C$, it follows that (see [21])

$$a * b = [a, a] \ast_{SIA} [b, b]$$

for any $a \in A$ and $b \in B$, and $\ast \in \{+, -, \times, \div\}$. In particular, for any $A \in K_C$, it follows that $a^2 = [a, a] \times_{SIA} [a, a]$ for all $a \in A$. That is, the mapping $\times_{SIA} : K_C \times K_C \rightarrow K_C$ given by $\times_{SIA}(A, A) = A \times_{SIA} A$, it is an interval extension of the real-valued function $f(x) = x^2$, with $x \in \mathbb{R}$. Now, considering the mapping called the square of an interval, which is given by (see [21])

$$K_C \ni A \mapsto A^2 = [a, \bar{a}]^2 = \begin{cases} \left[ a^2, \bar{a}^2 \right], & \text{if } 0 \leq a, \\ \left[ \frac{a^2}{\bar{a}}, a^2 \right], & \text{if } \bar{a} < 0, \\ \left[ 0, \max\{a^2, \bar{a}^2\} \right], & \text{if } a < 0 < \bar{a} \end{cases}$$

for all $A = [a, \bar{a}] \in K_C$, this mapping coincides with the interval extension of $f(x) = x^2$, $x \in \mathbb{R}$, that is called the united extension (see [21]). Indeed,

$$A^2 = f(A) = \{a^2 : a \in A\} = \bigcup_{a \in A} \{f(a)\} \text{ for all } A \in K_C.$$ 

Consequently, the square of an interval defined by (9), seen as an interval extension of $f(x) = x^2$, where $x \in \mathbb{R}$, it is more suitable than the product $A \times_{SIA} A$. A simple example of this fact is obtained by considering the interval $A = [-1, 1]$. From (9), it follows that $A^2 = [0, 1]$ whereas from SIA, it follows that $A \times_{SIA} A = AA = [-1, 1]$ is an interval that contains negative numbers.

Next some properties about the square of an interval are presented.

**Lemma 2.1.** (see [21]) Let $A = [a, \bar{a}]$ be an element of $K_C$, then

(i) $A^2 = AA$ if $a \geq 0$ or $\bar{a} \leq 0$;
(ii) $A^2 \neq AA$ if $\frac{a}{\bar{a}} < 0 < \frac{\bar{a}}{a}$;
(iii) $A^2$ is nonnegative, i.e., $a \geq 0$;
(iv) $A^2 = [0, 0]$ if and only if $A = [0, 0]$.
Henceforth we make the following convention: the product of two intervals $A$ and $B$ is $A \times_{SIA} B = AB$ if $A \neq B$, and it is $A^2$ if $A = B$. Moreover, it is known that $(\mathcal{K}_C, H)$, where $H$ is the Pompeiu–Hausdorff metric defined by

$$H(A, B) = H([a, \overline{a}], [b, \overline{b}]) = \max\{|a-b|, |\overline{a} - \overline{b}|\},$$

for all $A, B \in \mathcal{K}_C$, is a complete metric space (see [4,5,14]). Although $(\mathcal{K}_C, +, \cdot)$ is not linear space, it is a quasilinear space (see [4,26]) where the Pompeiu–Hausdorff quasinorm $\| \cdot \|$ is defined by (see [4,26,27])

$$\|A\| = H(A, [0, 0]) = H([a, \overline{a}], [0, 0]) = \max\{|a|, |\overline{a}|\}$$

for all $A = [a, \overline{a}] \in \mathcal{K}_C$.

**Lemma 2.2.** (see [21,28]) Let $A, B$ be two intervals in $\mathcal{K}_C$. Then

(i) $H(A, B) = \|A \Theta g H B\|$;

(ii) $\|A^2\| = \max\{|a|^2, |\overline{a}|^2\} = \max\{|a|^2, |\overline{a}|^2\} = \|A\|^2$.

### 3. Calculus for interval-valued functions

This section recalls some concepts and results about differentiability of interval-valued functions, introduces the concept of piecewise continuously $gH$-differentiable interval-valued functions, and provides a result that involves such concept and the concept of switching points. This result plays a key role in the development of this presentation.

An application $F : T \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$ given by $F(x) = [f(x), g(x)]$ for all $x \in X$, where $f, g : T \rightarrow \mathbb{R}$ are real-valued functions, with $f(x) \leq g(x)$ for all $x \in T$, is called an *interval-valued function*. The functions $f$ and $g$ are called the lower and the upper (endpoint) functions of $F$, respectively.

**Definition 3.1.** (see, e.g., [5,6]) Let $F : T \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$ be an interval-valued function. $L \in \mathcal{K}_C$ is called a limit of $F$ at $x_0 \in T$ if for every $\varepsilon > 0$ there exists $\delta(x_0) = \delta > 0$ such that $H(F(x), L) < \varepsilon$ for all $x \in T$ with $0 < |x - x_0| < \delta$. This is denoted by $\lim_{x \rightarrow x_0} F(x) = L$.

**Theorem 3.1.** (see, e.g., [5,6]) Let $F : T \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$ be an interval-valued function such that $F(x) = [f(x), g(x)]$ for all $x \in T$. Then $L = [l_1, l_2] \in \mathcal{K}_C$ is a limit of $F$ at $x_0 \in T$ if and only if $l_i$ is the limit of $f_i$ at $x_0$, $i \in \{1, 2\}$. Moreover, if $L$ is the limit of $F$ at $x_0$, then

$$\lim_{x \rightarrow x_0} F(x) = \left[ \lim_{x \rightarrow x_0} f(x), \lim_{x \rightarrow x_0} g(x) \right].$$

**Definition 3.2.** (see, e.g., [5,6]) Let $F : T \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$ be an interval-valued function. $F$ is said to be continuous at $x_0 \in T$ if $\lim_{x \rightarrow x_0} F(x) = F(x_0)$.

**Theorem 3.2.** (see, e.g., [5,6]) Let $F : T \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$ be an interval-valued function such that $F(x) = [f(x), g(x)]$ for all $x \in T$. Then $F$ is continuous at $x_0 \in T$ if and only if $f$ and $g$ are continuous at $x_0$. Moreover, if $F$ is continuous at $x_0$, then $\lim_{x \rightarrow x_0} F(x) = [f(x_0), g(x_0)]$.

Henceforth $T$ denotes an open interval.

The $H$-derivative (differentiability in the sense of Hukuhara) for interval-valued functions was initially introduced in [15] and it is based on the $H$-difference of intervals.

**Definition 3.3.** ([15]) Let $F : T \rightarrow \mathcal{K}_C$ be an interval-valued function. We say that $F$ is $H$-differentiable at $x_0 \in T$ if there exists an element $F'_H(x_0) \in \mathcal{K}_C$ such that the limits

$$\lim_{h \rightarrow 0^+} \frac{F(x_0 + h) - H F(x_0)}{h} \text{ and } \lim_{h \rightarrow 0^+} \frac{F(x_0) - H F(x_0 - h)}{h}$$

exist and are equal to $F'_H(x_0)$. In this case $F'_H(x_0)$ is called the $H$-derivative of $F$ at $x_0$. 

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Theorem 3.3, if we consider the interval-valued function $F(x) = (1 - x^3)[-2, 1]$, since

$$F(0 + h) - H \ F(0) = (1 - h^3)[-2, 1] - H [-2, 1],$$

the $H$-difference $F(0 + h) - H \ F(0)$ does not exist as $h \to 0^+$. Therefore, the $H$-derivative of $F$ does not exist at $x = 0$. In general, if $F(x) = C \cdot h(x)$, where $C$ is an interval and $h : [a, b] \to \mathbb{R}^+$ is a function with $h'(x_0) < 0$, then $F$ is not $H$-differentiable at $x_0$ ([7,11]). To overcome this shortcoming, in [28] the authors introduced the following differentiability concept for interval-valued functions, which is more general than the $H$-differentiability. For more details see [11,12,28].

**Definition 3.4.** ([28]) The $gH$-derivative of an interval-valued function $F : T \to K_C$ at $x_0 \in T$ is defined as

$$F_{gH}'(x_0) = \lim_{h \to 0} \frac{F(x_0 + h) \ominus gH \ F(x_0)}{h}. \quad (10)$$

If $F_{gH}'(x_0) \in K_C$ satisfying (10) there exists, then we say that $F$ is generalized Hukuhara differentiable ($gH$-differentiable, for short) at $x_0$.

In connection with the endpoint functions of $F$ we have the following result.

**Theorem 3.3.** ([11]) Let $F : T \to K_C$ be an interval-valued function such that $F(x) = [f(x), g(x)]$ for all $x \in T$. Then $F$ is $gH$-differentiable at $x_0 \in T$ if and only if one of the following cases holds

(i) $f$ and $g$ are differentiable at $x_0$ and

$$F_{gH}'(x_0) = \left[ \min \{ f'(x_0), g'(x_0) \}, \ \max \{ f'(x_0), g'(x_0) \} \right];$$

(ii) $f_-'(x_0), \ f_+'(x_0), \ g_-'(x_0)$ and $g_+'(x_0)$ exist and satisfy $f_-'(x_0) = g_+'(x_0)$ and $f_+'(x_0) = g_-'(x_0)$. Moreover,

$$F_{gH}'(x_0) = \left[ \min \{ f_-'(x_0), g_-'(x_0) \}, \ \max \{ f_+'(x_0), g_+'(x_0) \} \right]$$

$$= \left[ \min \{ f_+'(x_0), g_+'(x_0) \}, \ \max \{ f_+'(x_0), g_+'(x_0) \} \right].$$

**Example 3.1.** ([11]) Let the interval-valued function $F : \mathbb{R} \to K_C$ be defined by $F(x) = [-|x|, |x|]$ for all $x \in \mathbb{R}$. Then $F$ is $gH$-differentiable in $\mathbb{R}$ but the endpoint functions $f$ and $g$ are not differentiable at 0. Moreover, from (i) of Theorem 3.3, it follows that $F_{gH}'(x) = [f'(x), f'(x)] = [-1, 1]$ for all $x \in (-\infty, 0)$ and $F_{gH}'(x) = [f'(x), g'(x)] = [-1, 1]$ for all $x \in (0, \infty)$. From (ii) of Theorem 3.3, it follows that $F_{gH}'(0) = [-1, 1]$.

**Remark 3.1.** ([9]) From Example 3.1, it follows that $\text{len}(F(x))$ is decreasing on the interval $(0, \infty)$ whereas $\text{len}(F(x))$ is increasing. Moreover, $x = 0$ it is a switching point for the monotonicity of $\text{len}(F(x))$. That is, $\text{len}(F(x))$ changes its monotonicity at $x = 0$. Thus, from Definition 3 in [13], it implies that

(I) $F$ is $gH$-differentiable at $x_0 \in T$ in the first form if $f$ and $g$ are differentiable at $x_0$ and

$$F_{gH}'(x_0) = \left[ f'(x_0), g'(x_0) \right];$$

(II) $F$ is $gH$-differentiable at $x_0 \in T$ in the second form if $f$ and $g$ are differentiable at $x_0$ and

$$F_{gH}'(x_0) = \left[ g'(x_0), f'(x_0) \right].$$

Furthermore, a point $t_0 \in T$ is said to be a switching point for the $gH$-differentiability of $F$, if in any neighborhood $V$ of $t_0$ there exist points $t_1 < t_0 < t_2$ such that (see [28]).
(type I) $F$ is $gH$-differentiable at $t_1$ in the first form while it is not $gH$-differentiable in the second form, and $F$ is $gH$-differentiable at $t_2$ in the second form while it is not $gH$-differentiable in the first form, or

(type II) $F$ is $gH$-differentiable at $t_1$ in the second form while it is not $gH$-differentiable in the first form, and $F$ is $gH$-differentiable at $t_2$ in the first form while it is not $gH$-differentiable in the second form.

**Remark 3.2.** Throughout this presentation, to say that an interval-valued function $F$ has a switching point at $t_0 \in T$ means that $t_0 \in T$ is a switching point for the $gH$-differentiability of $F$.

The following result provides a sufficient condition in order for the endpoint functions of an interval-valued function be absolutely continuous.

**Lemma 3.1.** Let $F : [a, b] \rightarrow K_C$ be a continuous interval-valued function with $F(x) = [f(x), g(x)]$ for all $x \in [a, b]$. If $F$ is continuously $gH$-differentiable on $(a, b)$, where it has (if there exists) a finite number of switching points, then $f$ and $g$ are absolutely continuous on $[a, b]$.

**Proof.** For simplicity we consider only one switching point $c \in (a, b)$ for the $gH$-differentiability of $F$, the case of a finite number of switching points follows similarly. Let $F$ be continuously $gH$-differentiable on $(a, c)$ in the first form (in the second form, respectively), and continuously $gH$-differentiable on $(c, b)$ in the second form (in the first form, respectively). Then

(i) $f$ and $g$ are continuous in $[a, b]$;

(ii) $f$ and $g$ are continuously differentiable on $(a, c)$ and $(c, b)$;

(iii) $f'_-(c), f'_+(c), g'_-(c)$ and $g'_+(c)$ exist and satisfy $f'_-(c) = g'_+(c)$ and $f'_+(c) = g'_-(c)$.

Consequently, there exist $L_1 > 0$ and $L_2 > 0$ such that $|f'(x)| \leq L_1$ and $|g'(x)| \leq L_2$ for all $x \in [a, b] \setminus \{c\}$. Then, for any $x, y \in [a, b]$ with $x < y$, from Fundamental Theorem of Calculus, it follows that

\[
|f(x) - f(y)| \leq |f(x) - f(a)| + |f(a) - f(c)| + |f(c) - f(y)| + |f(b) - f(y)| = |f(a) - f(x)| + |f(c) - f(a)| + |f(b) - f(c)| + |f(y) - f(b)|
\]

\[
= \int_a^x f'(x)dx + \int_x^c f'(x)dx + \int_c^b f'(x)dx + \int_b^y f'(x)dx
\]

\[
\leq \int_a^x |f'(x)|dx + \int_c^b |f'(x)|dx + \int_a^c |f'(x)|dx + \int_b^y |f'(x)|dx
\]

\[
\leq \int_a^x L_1 dx + \int_a^c L_1 dx + \int_c^b L_1 dx + \int_b^y L_1 dx = \int_a^x L_1 dx = L_1(y - x).
\]

Analogously, given $x, y$ in $[a, b]$ with $y < x$, it follows that $|f(x) - f(y)| \leq L_1(x - y)$. Thus, for any $x, y \in [a, b]$, it follows that $|f(x) - f(y)| \leq L_1|x - y|$. That is, $f$ is continuous Lipschitz on $[a, b]$. By using similar arguments to these, it is proved that $g$ is also continuous Lipschitz on $[a, b]$. Since all real continuous Lipschitz functions are absolutely continuous, then $f$ and $g$ are absolutely continuous on $[a, b]$. \qed

Next we introduce the concept of piecewise continuously $gH$-differentiable interval-valued functions and we show that this is also a sufficient condition for endpoint functions of an interval-valued function to be absolutely continuous. This is the main result of this section.

**Definition 3.5.** We say that an interval-valued function $F$ is piecewise continuously $gH$-differentiable on an interval $[a, b]$ if this interval can be partitioned into a finite number of points $a = x_0 < x_1 < \cdots < x_n = b$ such that
Proof. Let \( p_{0} \), \( p_{1} \), \( p_{2} \) be continuous results on \( [x_{i}, x_{i+1}] \) for all \( i \in \{0, 1, \cdots, n-1\} \), and \( \lim_{x \to x_{i}^{+}} F_{g}(x) \in K_{C} \) for all \( i \in \{0, 1, \cdots, n\} \).

Theorem 3.4. Let \( F : [a, b] \to K_{C} \) be a continuous interval-valued function with \( F(x) = [f(x), g(x)] \) for all \( x \in [a, b] \). If \( F \) is piecewise continuously \( gH \)-differentiable on \([a, b]\) and it has (if there exists) a finite number of switching points on \((a, b)\), then \( f \) and \( g \) are absolutely continuous on \([a, b]\).

Proof. Let \( F \) be piecewise continuously \( gH \)-differentiable on \([a, b]\). Thus, the interval \([a, b]\) can be partitioned by a finite number of points \( a = x_{0} < x_{1} < \cdots < x_{n} = b \) such that \( F \) is continuously \( gH \)-differentiable on each subinterval \((x_{i-1}, x_{i})\) for all \( i \in \{1, 2, \cdots, n\} \). From the hypotheses, for each \( i \in \{1, 2, \cdots, n\} \) \( F \) must have (if there exists) only a finite number of switching points \( c_{j} \in (x_{i-1}, x_{i}) \). Consequently, from Lemma 3.1, it follows that \( f \) and \( g \) are absolutely continuous on \([x_{i-1}, x_{i}]\) for all \( i \in \{1, 2, \cdots, n\} \). Therefore, \( f \) and \( g \) are absolutely continuous on \([a, b]\). \( \square \)

4. Opial-type inequalities

This section recalls Opial’s inequality and some of its variations, which are used in the next section. In 1960, Opial [25] proved the following integral inequality, which is known as Opial’s inequality.

Theorem 4.1. ([25]) Let it be \( b > 0 \). If \( f \) is a real and continuously differentiable function on \([0, b]\) with \( f(0) = f(b) = 0 \) and \( f(x) > 0 \) on \((0, b)\), then

\[
\int_{0}^{b} |f(x)f'(x)| \, dx \leq b \int_{0}^{b} |f'(x)|^2 \, dx ,
\]

where this constant \( \frac{b}{2} \) is the best possible.

Also in 1960, Olech [24] proved that (11) holds for any real function \( f \) which is absolutely continuous on \([0, b]\) and which satisfies \( f(0) = f(b) = 0 \). Moreover, it was shown that the equality holds if and only if \( f(x) = mx \) for \( 0 \leq x \leq \frac{b}{2} \) and \( f(x) = m(b - x) \) for \( \frac{b}{2} \leq x \leq b \), where \( m \) is a constant. In 1962, Beesack [8] proved the following results.

Theorem 4.2. ([8]) Let it be \( b > 0 \). If \( f \) is a real and absolutely continuous function on \([0, b]\) with \( f(0) = 0 \), then

\[
\int_{0}^{b} |f(x)f'(x)| \, dx \leq \frac{b}{2} \int_{0}^{b} |f'(x)|^2 \, dx ,
\]

where the equality holds if and only if \( f(x) = mx \), where \( m \) is a constant.

Theorem 4.3. ([8]) Let \( b > a > 0 \). If \( f \) is a real and absolutely continuous function on \([a, b]\) and \( f(b) = 0 \), then

\[
\int_{a}^{b} |f(x)f'(x)| \, dx \leq \frac{b-a}{2} \int_{a}^{b} |f'(x)|^2 \, dx ,
\]

where the equality holds if and only if \( f(x) = m(b - x) \) for \( x \in [a, b] \), where \( m \) is a constant.

Combining Theorem 4.2 with Theorem 4.3 Beesack [8] proved that:

Theorem 4.4. ([8]) Let it be \( b > 0 \). If \( f \) is a real function such that \( f \) is absolutely continuous on \([0, \frac{b}{2}]\) and on \([\frac{b}{2}, b]\), with possible discontinuity at \( \frac{b}{2} \) and \( f(0) = f(b) = 0 \), then
\[ \int_a^b |f(x)f'(x)| \, dx \leq \frac{b}{4} \int_a^b |f'(x)|^2 \, dx. \]

Although the Opial’s inequality and its variations presented above consider real functions whose domains are intervals \([a, b]\) such that \(b > a \geq 0\), there exist other variations which consider functions whose domains are intervals \([a, b]\) where \(b > a\) but \(a\) it is not necessarily a nonnegative number such as the following versions.

**Theorem 4.5.** ([23]) Let \(p\) be a positive and continuous real function on an interval \([a, c]\) with \(\int_a^c \frac{1}{p(x)} \, dx < \infty\), and let \(q\) be a bounded positive, continuous and nonincreasing real function on \([a, c]\). If \(f\) and \(g\) are absolutely continuous real functions on \([a, c]\) and \(f(a) = g(a) = 0\), then

\[ \int_a^c \left( |f(x)g'(x)| + |f'(x)g(x)| \right) \, dx \leq \frac{1}{2} \int_a^c \frac{1}{p(x)} \, dx \int_a^c p(x)q(x) \left( |f'(x)|^2 + |g'(x)|^2 \right) \, dx, \]

where the equality holds if and only if \(q\) is a constant function and \(f(x) = g(x) = M \int_a^x \frac{1}{p(t)} \, dt\) for \(x \in [a, c]\), where \(M\) is a constant.

**Corollary 4.1.** If \(f\) and \(g\) are absolutely continuous real functions on \([a, c]\) and \(f(a) = g(a) = 0\), then

\[ \int_a^c \left( |f(x)g'(x)| + |f'(x)g(x)| \right) \, dx \leq \frac{c-a}{2} \int_a^c \left( |f'(x)|^2 + |g'(x)|^2 \right) \, dx. \]

**Proof.** The inequality follows from Theorem 4.5 for the constant functions \(p(x) = q(x) = 1\). \(\square\)

**Corollary 4.2.** If \(f\) is an absolutely continuous real function on \([a, c]\) and \(f(a) = 0\), then, it follows that

\[ \int_a^c \left( |f(x)f'(x)| \right) \, dx \leq \frac{c-a}{2} \int_a^c \left( |f'(x)|^2 \right) \, dx. \]

**Proof.** This result follows directly from Corollary 4.1. \(\square\)

**Remark 4.1.** Corollary 4.2 is a generalization of Theorem 4.2.

**Theorem 4.6.** ([23]) Let \(p\) be a positive and continuous real function on an interval \([c, b]\) with \(\int_c^b \frac{1}{p(x)} \, dx < \infty\), and let \(q\) be a bounded positive, continuous and nonincreasing real function on \([c, b]\). If \(f\) and \(g\) are absolutely continuous real functions on \([c, b]\) and \(f(b) = g(b) = 0\), then

\[ \int_c^b \left( |f(x)g'(x)| + |f'(x)g(x)| \right) \, dx \leq \frac{1}{2} \int_c^b \frac{1}{p(x)} \, dx \int_c^b p(x)q(x) \left( |f'(x)|^2 + |g'(x)|^2 \right) \, dx, \]

where the equality holds if and only if \(q\) is a constant function and \(f(x) = g(x) = M \int_c^x \frac{1}{p(t)} \, dt\) for \(x \in [c, b]\), where \(M\) is a constant.

**Corollary 4.3.** If \(f\) and \(g\) are absolutely continuous real functions on \([c, b]\) and \(f(b) = g(b) = 0\), then

\[ \int_c^b \left( |f(x)g'(x)| + |f'(x)g(x)| \right) \, dx \leq \frac{b-c}{2} \int_c^b \left( |f'(x)|^2 + |g'(x)|^2 \right) \, dx. \]

**Proof.** The inequality follows from Theorem 4.6 for the constant functions \(p(x) = q(x) = 1\). \(\square\)
Corollary 4.4. If \( f \) is an absolutely continuous real function on \([c, b]\) and \( f(b) = 0\), then it follows that
\[
\int_{c}^{b} \left( |f(x)| + |f'(x)| \right) dx \leq \frac{b - c}{2} \int_{c}^{b} \left( |f'(x)|^2 + |g'(x)|^2 \right) dx.
\]

Proof. This result follows directly from Corollary 4.3. \(\square\)

Remark 4.2. Corollary 4.4 is a generalization of Theorem 4.3.

From Corollary 4.3 and Corollary 4.1, it follows that

Corollary 4.5. If \( f \) and \( g \) are absolutely continuous real functions on \([a, c]\) and on \([c, b]\), with \( f(a) = g(a) = 0 \) and \( f(b) = g(b) = 0 \), then
\[
\int_{a}^{b} \left( |f(x)| + |f'(x)| \right) dx \leq \frac{(b - a)}{2} \int_{a}^{b} \left( |f'(x)|^2 + |g'(x)|^2 \right) dx.
\]

Proof. Given \( c \in [a, b] \), it follows that
\[
\int_{a}^{b} \left( |f(x)| + |f'(x)| \right) dx = \int_{a}^{c} \left( |f(x)| + |f'(x)| \right) dx + \int_{c}^{b} \left( |f(x)| + |f'(x)| \right) dx + \int_{c}^{b} \left( |f(x)| + |f'(x)| \right) dx
\]
\[
\leq \frac{c - a}{2} \int_{a}^{c} \left( |f'(x)|^2 + |g'(x)|^2 \right) dx + \frac{b - c}{2} \int_{c}^{b} \left( |f'(x)|^2 + |g'(x)|^2 \right) dx
\]
\[
\leq \frac{b - a}{2} \int_{a}^{c} \left( |f'(x)|^2 + |g'(x)|^2 \right) dx + \frac{b - a}{2} \int_{c}^{b} \left( |f'(x)|^2 + |g'(x)|^2 \right) dx
\]
\[
= \frac{b - a}{2} \left( \int_{c}^{b} \left( |f'(x)|^2 + |g'(x)|^2 \right) dx + \int_{c}^{b} \left( |f'(x)|^2 + |g'(x)|^2 \right) dx \right)
\]
\[
= \frac{b - a}{2} \int_{a}^{b} \left( |f'(x)|^2 + |g'(x)|^2 \right) dx. \quad \square
\]

5. Opial-type inequalities for interval-valued functions

This section presents some Opial-type inequalities for \( gH \)-differentiable interval-valued functions. We again remark that the concept of \( gH \)-differentiability is more general concept of differentiability than other existing concepts of differentiability for interval-valued functions. For more details see [11,17,28].

Theorem 5.1 (Interval Opial’s inequality). Let \( b > 0 \), and let \( F : [0, b] \to K_C \) be a continuous interval-valued function, with \( F(x) = [f(x), g(x)] \) for all \( x \in [0, b] \), such that \( F(0) = F(b) = [0, 0] \). If \( F \) is piecewise continuously \( gH \)-differentiable on \([0, b]\) and it has (if there exists) a finite number of switching points on \((0, b)\), then
\[
\int_{0}^{b} \|F(x)F'_{H}(x)\|dx \leq \frac{3b}{2} \int_{0}^{b} \|F'_{gH}(x)\|^2dx.
\]
Proof. From the hypotheses we have that $F_{gH}'(x) = [f'(x), g'(x)]$ or $F_{gH}'(x) = [g'(x), f'(x)]$ for almost every $x \in [0, b]$. Then for almost every $x \in [0, b]$, from Lemma 2.2, it follows that
\[
\|F_{gH}'(x)\|^2 = \max\{|f'(x)|^2, |g'(x)|^2\} \quad \text{for almost every } x \in [0, b].
\] (12)

Moreover,
\[
F(x) \cdot F_{gH}'(x) = \left[ \min \left\{ f(x)f'(x), f(x)g'(x), g(x)f'(x), g(x)g'(x) \right\}, \max \left\{ f(x)f'(x), f(x)g'(x), g(x)f'(x), g(x)g'(x) \right\} \right],
\]
(13)

for almost every $x \in [0, b]$. Thus, from (13), it follows that
\[
\left\| F(x) \cdot F_{gH}'(x) \right\| = \max \left\{ \min \left\{ f(x)f'(x), f(x)g'(x), g(x)f'(x), g(x)g'(x) \right\} - 0, \right.
\]
\[
\left| \max \left\{ f(x)f'(x), f(x)g'(x), g(x)f'(x), g(x)g'(x) \right\} - 0 \right| \}
\]
\[
= \max \left\{ \min \left\{ f(x)f'(x), f(x)g'(x), g(x)f'(x), g(x)g'(x) \right\} |, \right.
\]
\[
\left| \max \left\{ f(x)f'(x), f(x)g'(x), g(x)f'(x), g(x)g'(x) \right\} \right| \}
\]
\[
\leq |f(x)f'(x)| + |f(x)g'(x)| + |g(x)f'(x)| + |g(x)g'(x)|,
\]
(14)

for almost every $x \in [0, b]$. Consequently,
\[
\int_0^b \left\| F(x) \cdot F_{gH}'(x) \right\| dx
\]
\[
\leq \int_0^b \left( |f(x)f'(x)| + |f(x)g'(x)| + |g(x)f'(x)| + |g(x)g'(x)| \right) dx
\]
\[
= \int_0^b (|f(x)g'(x)| + |g(x)f'(x)|) dx + \int_0^b |f(x)f'(x)| dx + \int_0^b |g(x)g'(x)| dx.
\]
(15)

Thus, from Theorem 3.4, Corollary 4.5, Theorem 4.4, and from (15), it follows
\[
\int_0^b \|F(x) \cdot F_{gH}'(x)\| dx \leq \frac{b}{2} \int_0^b (|f'(x)|^2 + |g'(x)|^2) dx + \frac{b}{4} \int_0^b |f'(x)|^2 dx + \frac{b}{4} \int_0^b |g'(x)|^2 dx
\]
\[
= \frac{b}{2} \int_0^b (|f'(x)|^2 + |g'(x)|^2) dx + \frac{b}{4} \int_0^b \left( |f'(x)|^2 + |g'(x)|^2 \right) dx
\]
\[
= \frac{3b}{4} \int_0^b (|f'(x)|^2 + |g'(x)|^2) dx = \frac{3b}{2} \int_0^b \frac{1}{2} \left( |f'(x)|^2 + |g'(x)|^2 \right) dx
\]
\[
\leq \frac{3b}{2} \int_0^b \max \left\{ |f'(x)|^2, |g'(x)|^2 \right\} dx.
\]
(16)
Then from (12) and (16), it follows that
\[ \int_0^b \| F(x) \cdot F'_g(x) \| dx \leq \frac{3b}{2} \int_0^b \| F'_g(x) \|^2 dx. \]

The next results require conditions that are weaker than those of Theorem 5.1. However, such results generate inequalities that are greater than those generated by Theorem 5.1.

**Theorem 5.2.** Let it be \( b > a \), and let \( F : [a, b] \to K_C \) be a continuous interval-valued function, with \( F(x) = [f(x), g(x)] \) for all \( x \in [a, b] \), such that \( F(a) = [0, 0] \). If \( F \) is piecewise continuously \( gH \)-differentiable on \([a, b]\) and it has (if there exists) a finite number of switching points on \((a, b)\), then
\[ \int_a^b \| F(x)F'_g(x) \| dx \leq 2(b - a) \int_a^b \| F'_g(x) \|^2 dx. \]

**Proof.** By using similar argumentations to those used in Theorem 5.1, we have that
\[ \| F'_g(x) \|^2 = \max \left\{ |f'(x)|^2, |g'(x)|^2 \right\} \text{ for almost every } x \in [a, b] \]
and
\[ \| F(x) \cdot F'_g(x) \| \leq |f(x)f'(x)| + |f(x)g'(x)| + |g(x)f'(x)| + |g(x)g'(x)| \text{ for almost every } x \in [a, b]. \]

Consequently,
\[
\int_a^b \| F(x) \cdot F'_g(x) \| dx \\
\leq \int_a^b \left( |f(x)f'(x)| + |f(x)g'(x)| + |g(x)f'(x)| + |g(x)g'(x)| \right) dx \\
= \int_a^b |f(x)g'(x)| dx + \int_a^b |g(x)f'(x)| dx + \int_a^b |f(x)g'(x)| dx + \int_a^b |g(x)g'(x)| dx.
\]

Thus, from Theorem 3.4, Corollary 4.1, Corollary 4.2, and from (19), it follows
\[
\int_a^b \| F(x) \cdot F'_g(x) \| dx \\
\leq \frac{b - a}{2} \int_a^b \left( |f'(x)|^2 + |g'(x)|^2 \right) dx + \frac{b - a}{2} \int_a^b |f'(x)|^2 dx + \frac{b - a}{2} \int_a^b |g'(x)|^2 dx \\
= \frac{b - a}{2} \int_a^b \left( |f'(x)|^2 + |g'(x)|^2 \right) dx + \frac{b - a}{2} \int_a^b \left( |f'(x)|^2 + |g'(x)|^2 \right) dx
\]
\[(b - a) \int_a^b \left( |f'(x)|^2 + |g'(x)|^2 \right) dx \leq 2(b - a) \int_a^b \max \left\{ |f'(x)|^2, |g'(x)|^2 \right\} dx. \tag{20} \]

Then from (17) and (20), it follows that
\[
\int_a^b \|F(x)F'_H(x)\| dx \leq 2(b - a) \int_a^b \|F'_H(x)\|^2 dx. \]

**Theorem 5.3.** Let it be \(b > a\), and let \(F : [a, b] \to K_C\) be a continuous interval-valued function, with \(F(x) = [f(x), g(x)]\) for all \(x \in [a, b]\), such that \(F(b) = [0, 0]\). If \(F\) is piecewise continuously \(gH\)-differentiable on \([a, b]\) and it has (if there exists) a finite number of switching points on \((a, b)\), then
\[
\int_a^b \|F(x)F'_H(x)\| dx \leq 2(b - a) \int_a^b \|F'_H(x)\|^2 dx. \]

**Proof.** The proof uses similar arguments to those used in the proof of Theorem 5.2 by replacing the use of Corollary 4.2 by the use of Corollary 4.4 in the argumentations. □

**Theorem 5.4.** Let it be \(b > a\), and let \(F : [a, b] \to K_C\) be a continuous interval-valued function, with \(F(x) = [f(x), g(x)]\) for all \(x \in [a, b]\), such that \(F(a) = F(b) = [0, 0]\). If \(F\) is piecewise continuously \(gH\)-differentiable on \([a, b]\), which has (if there exists) a finite number of switching points on \((a, b)\), then
\[
\int_a^b \|F(x)F'_H(x)\| dx \leq 2(b - a) \int_a^b \|F'_H(x)\|^2 dx. \]

**Proof.** By using similar arguments to those used in Theorem 5.1, we have that
\[
\|F'_H(x)\|^2 = \max \left\{ |f'(x)|^2, |g'(x)|^2 \right\} \text{ for almost every } x \in [a, b] \tag{21} \]
and
\[
\left\| F(x) \cdot F'_H(x) \right\| \leq |f(x)f'(x)| + |f(x)g'(x)| + |g(x)f'(x)| + |g(x)g'(x)| \text{ for almost every } x \in [a, b]. \tag{22} \]

Consequently,
\[
\int_a^b \left\| F(x) \cdot F'_H(x) \right\| dx \leq \int_a^b \left( |f(x)f'(x)| + |f(x)g'(x)| + |g(x)f'(x)| + |g(x)g'(x)| \right) dx
\]
\[
= \int_a^b (|f(x)g'(x)| + |g(x)f'(x)|) dx + \int_a^b |f(x)f'(x)| dx + \int_a^b |g(x)g'(x)| dx. \tag{23} \]

Thus, from Theorem 3.4 and Corollary 4.5, it follows that
Fig. 1. Upper (——) and lower (- - - -) functions of $F$ in Example 6.1.

\[
\int_{a}^{b} \|F(x) \cdot F'_{gH}(x)\| dx \\
\leq \frac{b-a}{2} \int_{a}^{b} \left( |f'(x)|^2 + |g'(x)|^2 \right) dx + \frac{b-a}{2} \int_{a}^{b} |f'(x)| dx + \frac{b-a}{2} \int_{a}^{b} |g'(x)| dx \\
= \frac{b-a}{2} \int_{a}^{b} \left( |f'(x)|^2 + |g'(x)|^2 \right) dx + \left( |f'(x)|^2 + |g'(x)|^2 \right) dx \\
= \int_{a}^{b} \left( |f'(x)|^2 + |g'(x)|^2 \right) dx \leq 2(b-a) \int_{a}^{b} \max \left\{ |f'(x)|^2, |g'(x)|^2 \right\} dx. \\
(24)
\]

Then from (21) and (24), it follows that $\int_{a}^{b} \|F(x) \cdot F'_{gH}(x)\| dx \leq 2(b-a) \int_{a}^{b} \|F'_{gH}(x)\|^2 dx$. 

6. Numerical examples

This Section presents some examples to illustrate our main results.

Example 6.1. Let $F : [0, 1] \rightarrow \mathbb{K}^n$ be the interval-valued function given by $F(x) = [-1, 1] \cdot (x - x^2)$ for all $x \in [0, 1]$ (see Fig. 1). Since $h(x) = x - x^2$ is continuously differentiable on $(0, 1)$, then $F$ is continuously $gH$-differentiable on $(0, 1)$. Moreover, $F$ has only one switching point at $x = 1/2$, $F(0) = F(1) = [0, 0]$, and

\[
F'_{gH}(x) = \begin{cases} 
[2x - 1, -2x + 1], & \text{if } x \in (0, \frac{1}{2}) \\
[-2x + 1, 2x - 1], & \text{if } x \in (\frac{1}{2}, 1) 
\end{cases}
\]

equivalently

$F'_{gH}(x) = [-1, 1] \cdot |1 - 2x|$. 

Thus,

\[
\|F'_{gH}(x)\|^2 = (2x - 1)^2 \quad \text{and} \quad \int_{0}^{1} \|F'_{gH}(x)\|^2 dx = 1/3.
\]

Then from Theorem 5.1, it follows that

\[
\int_{0}^{b} \|F(x) \cdot F'_{gH}(x)\| dx \leq \frac{3}{2} \cdot \frac{1}{3} = \frac{1}{2}.
\]
The interval-valued function considered in Example 6.1 is an element in the class of all interval-valued functions $F : [a, b] \to K_C$ given by $F(x) = A \cdot h(x)$, where $a < b$, $A \in K_C$, and $h : [a, b] \to \mathbb{R}$ is a real-valued function. In this sense, the next example generalizes Example 6.1.

**Example 6.2.** Let $F : [a, b] \to K_C$ be an interval valued function defined by $F(x) = A \cdot h(x) = [\underline{a}, \overline{a}] \cdot h(x)$, where $a < b$ and $h : [a, b] \to \mathbb{R}$ is a continuously differentiable real-valued function such that $|h|$ has a finite number of local minimum or maximum points. Then $F$ is an continuously $gH$-differentiable interval-valued function which has a finite number of switching points on $(a, b)$, and consequently, for almost every $x \in [a, b]$, we have that

$$\| F'_{gH}(x) \|^2 = \| [\underline{a}, \overline{a}] \cdot h'(x) \|^2$$

$$= \| \min \{ ah'(x), \overline{ah}'(x) \}, \max \{ ah'(x), \overline{ah}'(x) \} \|$$

$$= (\max \{ \min \{ ah'(x), \overline{ah}'(x) \}, |\max \{ ah'(x), \overline{ah}'(x) \} | \})^2$$

$$= (\max \{ |ah'(x)|, |\overline{ah}'(x)| \})^2$$

$$= (\max \{ |a|, |\overline{a}| \} \cdot |h'(x)|)^2$$

$$= \| A \|^2 \cdot |h'(x)|^2.$$

Thus, if $a = 0$ and $h(a) = h(b) = 0$, then from Theorem 5.1, it follows that

$$\int_a^b \| F(x) F'_{gH}(x) \| dx \leq \frac{3b}{2} \| A \|^2 \int_a^b |h'(x)|^2 dx.$$

Now, if $h(a) = 0$ (or $h(b) = 0$), then from Theorem 5.2 (or Theorem 5.3), we have that

$$\int_a^b \| F(x) F'_{gH}(x) \| dx \leq (b - a) \| A \|^2 \int_a^b |h'(x)|^2 dx.$$

Next we present an example in which the interval-valued function does not belong to the class of interval-valued functions considered in Example 6.2.

**Example 6.3.** Let $F : [0, 1] \to K_C$ be given by $F(x) = [-1, -1] \cdot x^2 + [x, 1]$ for all $x \in [0, 1]$. Thus, $F(x) = [x - x^2, 1 - x^2] = [f(x), g(x)]$ for all $x \in [0, 1]$ (see Fig. 2), and consequently, the endpoint functions of $F$ are continuously differentiable on $(0, 1)$. Therefore, $F$ is continuously $gH$-differentiable on $(0, 1)$. Moreover, $F'_{gH}(x) = [-2x, -2x + 1]$ for all $x \in (0, 1)$ and $F(1) = [0, 0]$. Now, from (9) and Lemma 2.1, we have that

$$\left( F_{gH}'(x) \right)^2 = \begin{cases} [0, \max \{ 4x^2, 4x^2 - 4x + 1 \}], & \text{if } x \in (0, \frac{1}{2}) \cap (0, 1) \\ [4x^2 - 4x + 1, 4x^2], & \text{if } x \in (\frac{1}{2}, 1) \cap (0, 1). \end{cases}$$

Thus,
Thus, \( x \) are differentiable \( g_H \). Consequently, taking into consideration Theorem 5.3, it follows that

\[
\int_0^1 \| F(x)F'_{gH}(x) \| dx \leq 2(1 - 0) \left( \int_0^{\pi/4} \| F'_{gH}(x) \|^2 dx + \int_{\pi/4}^1 \| F'_{gH}(x) \|^2 dx \right) = \frac{35}{12}.
\]

In Example 6.1, Example 6.2, and Example 6.3 the interval-valued functions \( F : [a, b] \rightarrow K_C \) considered are \( g_H \)-differentiable on \((a, b)\). In the next example we consider an interval-valued function which is not \( g_H \)-differentiable on \([a, b]\) but it is piecewise continuously \( g_H \)-differentiable on \([a, b]\).

**Example 6.4.** Let \( F : [-\pi, \pi] \rightarrow K_C \) be the interval-valued function given by \( F(x) = [f(x), g(x)] \), where \( f, g : [-\pi, \pi] \rightarrow \mathbb{R} \) are given by \( f(x) = (\sin(x))^2 - |\sin(x)| \) and \( g(x) = -(\cos(x))^2 + |\cos(x)| \) for all \( x \in [-\pi, \pi] \) (see Fig. 3). Then \( F \) is continuous on \([-\pi, \pi] \) and it is continuously \( g_H \)-differentiable on \((-\pi, \pi) \) except at the points \( x_1 = -\frac{\pi}{2}, x_2 = 0 \) and \( x_3 = \frac{\pi}{2} \). Moreover, \( F \) has seven switching points at \( -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4} \in (-\pi, \pi) \). Thus, from Definition 3.5, \( F \) is piecewise continuously \( g_H \)-differentiable on \([-\pi, \pi] \).

Since \( F(-\pi) = F(\pi) = [0, 0] \) and \( \int_{-\pi}^{\pi} \| F'_{gH}(x) \|^2 dx \approx 1.694037 < \frac{17}{10} \), then from Theorem 5.3, it follows that

\[
\int_{-\pi}^{\pi} \| F(x)F'_{gH}(x) \| dx \leq 4\pi \int_{-\pi}^{\pi} \| F'_{gH}(x) \|^2 dx < \frac{34\pi}{5}.
\]

Now, we also have that \( F(0) = F(\pi) = [0, 0] \). Consequently, from Theorem 5.1, it follows that

\[
\int_0^{\pi} \| F(x)F'_{gH}(x) \| dx \leq \frac{3\pi}{2} \int_0^{\pi} \| F'_{gH}(x) \|^2 dx.
\]

Since \( \int_0^{\pi} \| F'_{gH}(x) \|^2 dx \approx 0.847017 < \frac{17}{20} \), then

\[
\int_0^{\pi} \| F(x)F'_{gH}(x) \| dx < \frac{51\pi}{40}.
\]

7. Conclusion

In 1960 the Polish mathematician Zdzidlaw Opial (1930–1974) published an inequality (Opial’s inequality) involving integrals of a function and its derivative. This inequality and some generalizations, extensions and discretizations
has been exhaustively explored in the last years. Also, this inequality has important applications in Theory of Differential Equations, such as uniqueness of initial value problems, uniqueness of boundary value problems and upper bounds of solutions, among other topics (for a complete survey in this field see [1]).

On the other hand, Opial’s inequality have relevant applications in Theory of Integral Inequalities, for instance several new inequalities with weighted function of Hardy type with explicit constants have been recently proved using generalizations of Opial’s inequality (see [2]). In this article we prove some Opial-type inequalities for interval-valued functions and, in a forthcoming paper, we will explore some applications to Interval Differential Equations as well as some applications to Interval Integral Inequalities.

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