Structural properties of the bounded control set of a linear system

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Abstract

The present paper shows that the bounded control set of a linear system on a connected Lie group G contains all the bounded orbits of the system. As a consequence, its closure is the continuous image of the cartesian product of the set of control functions by the central subgroup associated with the drift of the system.

Key words: Linear systems, control set, central subgroup

1 Introduction

Consider a vector field $\mathcal{X}$ on a connected differentiable manifold M of dimension n which model a dynamical system. A control system Σ_M on M is determined by a family of controlled differential equations

$$\dot{x}(t) = \mathcal{X}(x(t)) + \sum_{j=1}^{m} u_j(t) Y^j(x(t)), \qquad (\Sigma_M)$$

which allows to modify the behavior of $\mathcal{X}$ according to the control vectors $Y^1, Y^j, ..., Y^m$, and the set $\mathcal{U}$ of admissible control functions

$$\mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m \ : \ u \text{ is measurable with } u(t) \in \Omega \text{ a.e.}\},$$

where $\Omega \subset \mathbb{R}^m$ is a closed and convex set with $0 \in \operatorname{int} \Omega$.

The controllability notion of Σ_M is one of the most relevant properties of the system. In fact, it allows to connect any two point of M through a concatenation of integral curves of Σ_M, in positive time. For instance, a neccesary condition to solve any optimization problem between two states, like a time optimal, or minimum energy trajectory is the existence of a control $u \in \mathcal{U}$ such that the corresponding solution connect these two states. Despite the fact that are nice examples of controllable systems, this global property is hard to satisfy in general. A well known example is the class of linear system on Euclidean spaces $\Sigma_{\mathbb{R}^n}$, where $\mathcal{X} = A$ is a

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matrix of order n, and $Y^j = b^j \in \mathbb{R}^n$ are constant vector fields. In this context, the Kalman rank condition characterize controllability. However, to obtain that you need to consider $\Omega = \mathbb{R}^m$, which is far from real life. A more realistic approach consider the case when $\Omega \subset \mathbb{R}^m$ is compact, and the notion of control set for Σ_M, which is roughly speaking a maximal subset where controllability. Recently, several papers are focused in the study of the controllability and the existence, uniqueness and topological properties of the control sets of the class Σ_G of linear systems on a connected Lie group G. In this case, the drift $\mathcal{X}$ is a linear vector field in the sense that its flow $\{\varphi_t\}_{t\in\mathbb{R}}$ is a 1-parameter group of automorphisms of G, and the control vectors are elements of the Lie algebra. It turns out that many properties of the system depends strongly of the dynamical behavior of $\mathcal{X}$. In fact, associated with the flow of $\mathcal{X}$ there are connected subgroups, called unstable, central and stable subgroups, which have a nice a relationship with the set of the reachable points of Σ_G and hence with its controllability (see [1, 2, 4, 6, 8, 9]). In fact, such relation implies that the control set of the identity of Σ_G is in fact the only control set of Σ_G ([4, Theorem 3.11]) and it is bounded if and only if the central subgroup is compact ([1, Theorem 4.2]).

Our aim here is to study structural properties of the control set of the identity of Σ_G in the case it is compact. As our main result shows, in this case it holds that all bounded orbits of Σ_G are contained in the closure of the control set. This allows us to show that in fact its closure is the continuous image of the cartesian product of $\mathcal{U}$ by the central subgroup.

The article is organized as follows: Section 2 contains the preliminaries of the paper and some results concerning decompositions induced by automorphism at the algebra and group level. We also introduce here the concept of linear systems and state their relation with the dynamics of $\mathcal{X}$. In Section 3 we prove our main result. We prove here several properties concerning bounded orbits and the central subgroup of $\mathcal{X}$ which implies the main result. Moreover, such properties also allows us to create a continuous function from the cartesian of $\mathcal{U}$ by the central subgroup in the closure of the control set.

2 Preliminaries

This section is devoted to present the main background needed in order to establish the main theorem. We also prove some new results that will be useful ahead. Finally, we introduce the notion of a linear control system on Lie groups and the concept of control set.

2.1 Decompositions at the algebra level

Let $\psi \in \mathrm{Aut}(\mathfrak{g})$ be an automorphism of $\mathfrak{g}$ and $\alpha \in \mathbb{C}$ an eigenvalue of ψ. The real generalized eigenspaces of ψ are given by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : (\psi - \alpha I)^n X = 0 \text{ for some } n \geq 1\}, \quad \text{if} \quad \alpha \in \mathbb{R} \quad \text{and}$$

$$\mathfrak{g}_\alpha = \mathrm{span}\{\mathrm{Re}(v), \mathrm{Im}(v);\ v \in \bar{\mathfrak{g}}_\alpha\}, \quad \text{if} \quad \alpha \in \mathbb{C}$$

where $\bar{\mathfrak{g}} = \mathfrak{g} + i\mathfrak{g}$ is the complexification of $\mathfrak{g}$ and $\bar{\mathfrak{g}}_\alpha$ the generalized eigenspace of $\bar{\psi} = \psi + i\psi$, the extension of ψ to $\bar{\mathfrak{g}}$.

We define the *unstable, central* and *stable* ϕ-invariant subspaces of $\mathfrak{g}$, respectively, by

$$\mathfrak{g}^+ = \bigoplus_{\alpha:\, |\alpha|>1} \mathfrak{g}_\alpha, \qquad \mathfrak{g}^0 = \bigoplus_{\alpha:\, |\alpha|=1} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{g}^- = \bigoplus_{\alpha:\, |\alpha|<1} \mathfrak{g}_\alpha.$$

Following [3] the fact that $[\bar{\mathfrak{g}}_\alpha, \bar{\mathfrak{g}}_\beta] \subset \bar{\mathfrak{g}}_{\alpha\beta}$ when $\alpha\beta$ is an eigenvalue of ϕ and zero otherwise implies that $\mathfrak{g}^+, \mathfrak{g}^0, \mathfrak{g}^-$ are in fact ψ-invariant Lie subalgebras with $\mathfrak{g}^+$, $\mathfrak{g}^-$ nilpotent ones. Moreover, it turns out that $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$.

2.2 Decompositions at the group level

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and consider $\varphi \in \mathrm{Aut}(G)$. The *unstable, central* and *stable* ψ-invariant subgroups of G, denoted respectively by G^+, G^- and G^0, are the connected Lie subgroups of G with

Lie algebras induced by $\phi := (d\psi)_e$ given by $\mathfrak{g}^+$, $\mathfrak{g}^-$ and $\mathfrak{g}^0$, respectively. We denote also by $G^{+,0}$, and $G^{-,0}$ the connected Lie subgroup whose Lie algebras are given by $\mathfrak{g}^{+,0} := \mathfrak{g}^+ \oplus \mathfrak{g}^0$ and $\mathfrak{g}^{-,0} := \mathfrak{g}^- \oplus \mathfrak{g}^0$, respectively.

We say that G is *decomposable* by ψ if

$$G = G^{+,0}G^- = G^{-,0}G^+ = G^{+,-}G^0.$$

If $G^0 = \{e\}$ we say that ψ is *hyperbolic*. In particular, if ψ is hyperbolic, $G = G^{+,-} := G^+G^-$ is decomposable.

By [3, Proposition 3.4] the compactness of the central subgroup G^0 is a sufficient condition in order to G be decomposable. On the other hand, if ψ is hyperbolic then G is a nilpotent Lie group. If G is simply connected, the dynamics subgroups are closed in G, however that is not true in general (see Example 2.3). In particular, if $\{\varphi_t\}_{t\in\mathbb{R}} \subset \mathrm{Aut}(G)$ is a 1-parameter subgroup, the dynamical subgroups associated with each φ_t coincides. Moreover, in this case the subgroups are closed and have trivial intersection (see [6, Proposition 2.9]).

Next we analyze a map associated with hyperbolic automorphisms.

2.1 Proposition: *If $\psi \in \mathrm{Aut}(G)$ is hyperbolic, the map*

$$f_\psi : \quad g \in G \mapsto g\psi(g^{-1}) \in G$$

is an onto local diffeomorphism. Furthermore, f_ψ is injective if and only if $\mathrm{fix}(\psi) = \{e\}$.

Proof: The differentiability of f_ψ follows direct from its definition. Moreover, for any G it holds that

$$f_\psi(hg) = hg\psi((hg)^{-1}) = h\left(g\psi(g^{-1})\right)\psi(h^{-1}) = hf_\psi(g)\psi(h^{-1}) \implies f_\psi \circ L_h = L_h \circ R_{\psi(h^{-1})} \circ f_\psi, \qquad (1)$$

and since left and right translations are diffeomorphims we get in particular that f_φ has constant rank. On the other hand,

$$(df_\psi)_e X = \frac{d}{dt}_{|t=0} \mathrm{e}^{tX}\psi\left(\mathrm{e}^{-tX}\right) = \left((dL_{\mathrm{e}^{tX}})_{\psi(\mathrm{e}^{-tX})}\frac{d}{dt}\psi(\mathrm{e}^{-tX}) + (dR_{\psi(\mathrm{e}^{-tX})})_{\mathrm{e}^{tX}}\frac{d}{dt}\mathrm{e}^{tX}\right)_{|t=0} = X - (d\psi)_e X$$

and $(d\psi)_e$ does not have 1 as an eigenvalue, it holds that $(df_\psi)_e(\mathfrak{g}) = \mathfrak{g}$ and hence f_ψ is a local diffeomorphism.

Let us prove the surjectiveness of f_ψ by induction on the dimension of G.

If G is an abelian group then for any $x, y \in G$ we get from equation (1) that

$$f_\psi(hg) = f_\psi(h)f_\psi(g)$$

showing that f_ψ is an homomorphism. Being f_ψ a local diffeomorphism we must have that $f_\psi(G)$ is a subgroup with nonempty interior in G and hence $f_\psi(G) = G$. In particular, the result is true when $\dim G = 1$.

Assume now that the result is true for any Lie group G with $\dim G < n$ and consider G to be a nonabelian Lie group of dimension n and $\psi \in \mathrm{Aut}(G)$ to be a hyperbolic. Since on nilpotent Lie groups the closed normal subgroup $Z(G)_0$ is nontrivial, the Lie group $\hat{G} = G/Z(G)_0$ satisfies $\dim \hat{G} = \dim G - \dim Z(G) < n$. Let then $\hat{\psi} \in \mathrm{Aut}(\hat{G})$ such that

$$\hat{\psi} \circ \pi = \pi \circ \psi, \quad \text{where} \quad \pi : G \to \hat{G} \ \text{ is the canonical projection.}$$

A simple calculation show that the associated map

$$f_{\hat{\psi}} : \hat{g} \in \hat{G} \mapsto \hat{g}\hat{\psi}(\hat{g}^{-1}) \in \hat{G}$$

also verifies $f_{\hat{\psi}} \circ \pi = \pi \circ f_\psi$. Since $\pi(G^{+,-}_\psi) = \hat{G}^{+,-}_{\hat{\psi}}$ and $\pi(G^0_\psi) = \hat{G}^0_{\hat{\psi}}$ it holds that $\hat{\psi}$ is hyperbolic and by the induction hypothesis that

$$\pi(f_\psi(G)) = f_{\hat{\psi}}(\pi(G)) = f_{\hat{\psi}}(\hat{G}) = \hat{G} = \pi(G). \qquad (2)$$

On the other hand, since $\dim Z(G)_0 < n$ and $\psi|_{Z(G)_0} \in \mathrm{Aut}(Z(G)_0)$ is hyperbolic, by the abelian case we get

$$f_{\psi|_{Z(G)_0}} = f_\psi|_{Z(G)_0} \quad \text{is surjective.}$$

Therefore, for any $g\in G$ there are by (2) elements $x\in G$ and $h\in Z(G)_0$ such that $g=f_\psi(x)h$. Since $f_\psi|_{Z(G)_0}$ is surjective it holds that $h=f_\varphi(y)$ for some $y\in Z(G)_0$ and so

$$g=f_\psi(x)h=f_\psi(x)f_\psi(z)=f_\psi(xz),$$

where for the last equality we used (1) and the fact that $z\in Z(G)_0$. Therefore, f_ψ is surjective as stated.

For the last assertion, note that

$$f_\psi(g)=f_\psi(h)\iff g\psi(g^{-1})=h\psi(h^{-1})\iff \psi(g^{-1}h)=g^{-1}h\iff g^{-1}h\in \mathrm{fix}(\psi)$$

and consequently f_ψ is injective if and only if $\mathrm{fix}(\psi)=\{e\}$. □

Let us consider $\{\varphi_t\}_{t\in\mathbb{R}}$ be a 1-parameter flow of automorphisms. The next results shows that the dynamical subgroups of $\{\varphi_t\}_{t\in\mathbb{R}}$ have nice properties.

2.2 Proposition: *With the previous notation, the map*

$$f: X+Y\in\mathfrak{g}^+\times\mathfrak{g}^- \quad\mapsto \mathrm{e}^X\mathrm{e}^Y\in G^{+,-},$$

is a diffeomorphism. Moreover,

$$\forall t\in\mathbb{R},\quad f\circ(d\varphi_t)_e=\varphi_t\circ f. \tag{3}$$

Proof: By the very definition, f is differentiable. Morever, by [7, Proposition 3], the Lie subgroups G^+ and G^- associated with $\{\varphi_t\}_{t\in\mathbb{R}}$ are simply connected and hence the exponential map restricted to both $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are diffemorphisms. Let $X\in\mathfrak{g}^+$ and define the isomorphism

$$T_X^+:\mathfrak{g}^+\to\mathfrak{g}^+,\quad T_X^+:=(dL_{\mathrm{e}^{-X}})_{\mathrm{e}^X}\circ(d\exp|_{\mathfrak{g}^+})_X.$$

Let $Z\in\mathfrak{g}^+$ and define the curve

$$\gamma:\mathbb{R}\to U,\quad \gamma(s):=\exp|_{\mathfrak{g}^+}^{-1}\left(\mathrm{e}^X\mathrm{e}^{sT_X^+Z}\right),$$

is well define, differentiable with $\gamma(0)=X$ and

$$(d\exp|_{\mathfrak{g}^+})_X\gamma'(0)=\frac{d}{ds}_{|s=0}\mathrm{e}^{\gamma(s)}=\frac{d}{ds}_{|s=0}\mathrm{e}^X\mathrm{e}^{sT_XZ}=(dL_{\mathrm{e}^X})_eT_X^+Z.$$

In particular, $\gamma'(0)=Z$. Analogously, for $Y,W\in\mathfrak{g}^-$ we construct a curve $\beta:\mathbb{R}\to\mathfrak{g}^-$ satisfying

$$\mathrm{e}^{\beta(s)}=\mathrm{e}^{sT_Y^-W}\mathrm{e}^Y,\quad \beta(0)=Y\quad\text{ and }\quad \beta'(0)=W.$$

Therefore,

$$(df)_{X+Y}(Z+W)=\frac{d}{ds}_{|s=0}f(\gamma(s)+\beta(s))=\frac{d}{ds}_{|s=0}\mathrm{e}^{\gamma(s)}\mathrm{e}^{\beta(s)}$$
$$=\frac{d}{ds}_{|s=0}\mathrm{e}^X\mathrm{e}^{sT_X^+Z}\mathrm{e}^{sT_Y^-W}\mathrm{e}^Y=(d(L_{\mathrm{e}^X}\circ R_{\mathrm{e}^Y}))_e(T_X^+Z+T_Y^-W),$$

which shows that f is a local diffeomorphism.

Let $X,Z\in\mathfrak{g}^+$ and $Y,W\in\mathfrak{g}^+$. Then,

$$f(X+Y)=f(Z+W)\quad\iff\quad \mathrm{e}^X\mathrm{e}^Y=\mathrm{e}^Z\mathrm{e}^W\quad\iff\quad G^+\ni\mathrm{e}^{-Z}\mathrm{e}^X=\mathrm{e}^W\mathrm{e}^{-Y}\in G^-.$$

Since G^+ and G^- are associated with a 1-parameter group of automorphisms it holds that $G^+\cap G^-=\{e\}$ and hence $\mathrm{e}^X=\mathrm{e}^Z$ and $\mathrm{e}^Y=\mathrm{e}^W$ implying that $X=Z$ and $Y=W$ and showing the injectiveness of f.

For the last assertion, the fact that $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are $(d\varphi_t)$-invariant implies

$$f\left((d\varphi_t)_e(X+Y)\right)=f\left((d\varphi_t)_eX+(d\varphi_t)Y\right)=\mathrm{e}^{(d\varphi_t)_eX}\mathrm{e}^{(d\varphi_t)_eY}$$
$$=\varphi_t(\mathrm{e}^X)\varphi_t(\mathrm{e}^Y)=\varphi_t(\mathrm{e}^X\mathrm{e}^Y)=\varphi_t(f(X+Y)),$$

ending the proof. □

The next example shows that for discrete flows of automorphisms the previous result is in general not true.

2.3 Example: Let us consider $\mathbb{T}x^2 = \mathbb{R}^2/\mathbb{Z}^2$ the 2-dimensional torus. As a general result, the group of automorphisms $\mathrm{Aut}(\mathbb{T}^2)$ is given by

$$\mathrm{Aut}(\mathbb{T}^2) = \{A \in \mathrm{Gl}(\mathbb{R}^2);\ \ A\mathbb{Z}^2 = \mathbb{Z}^2\}.$$

In particular,

$$\psi = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \in \mathrm{Aut}(\mathbb{T}), \quad \text{with} \quad \mathfrak{g}^+ = \mathbb{R}(1, \sqrt{2}) \quad \text{and} \quad \mathfrak{g}^- = \mathbb{R}(1, -\sqrt{2}).$$

In particular, both G^+ and G^- are images of the well known irrational flows on $\mathbb{T}^2$, and are henceforth dense in $\mathbb{T}^2$. Since $G^{+,-} = \mathbb{T}^2$ is compact, the map f from the previous proposition cannot be a diffeomorphism.

The previous proposition implies the following result.

2.4 Lemma: *Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a 1-parameter group of automorphisms. For any compact subset $K \subset G$ and any $x \in G^{+,-}$ with $x \neq e$ there exists $t_0 \in \mathbb{R}$ such that $\varphi_t(x) \notin K$ for $t \geq t_0$.*

Proof: By Proposition 2.2 it is enough to show the equivalent assertion for $(d\varphi_t)_e$ restricted to $\mathfrak{g}^{+,-}$. Following [7], there exists $c, \mu > 0$ such that

$$|(d\varphi_t)_e v| \geq \mathrm{e}^{\mu t}|v|, \ \ t > 0, v \in \mathfrak{g}^+ \quad \text{and} \quad |(d\varphi_t)_e v| \geq \mathrm{e}^{\mu t}|v|, \ \ t < 0, v \in \mathfrak{g}^-,$$

which implies the assertion. □

2.3 Linear systems

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ identified with the right-invariant vector fields. A *linear system* on G is given a family of ordinary differential equations

$$\dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^{m} u_j(t) Y^j(g(t)), \tag{Σ_G}$$

where the *drift* $\mathcal{X}$ is a linear vector field, that is, its associated flow $\{\varphi_t\}_t \in \mathbb{R}$ is a 1-parameter group of automorphisms, $Y^1, \ldots, Y^m \in \mathfrak{g}$ and $u = (u_1, \ldots, u_m) \in \mathcal{U}$. The set $\mathcal{U}$ of admissible control functions is given by

$$\mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m \ : \ u \text{ is measurable with } u(t) \in \Omega \text{ a.e.}\},$$

where $\Omega \subset \mathbb{R}^m$ is a compact and convex set with $0 \in \mathrm{int}\,\Omega \neq$. Then $\mathcal{U}$, endowed with the weak*-topology of $L^\infty(\mathbb{R}, \mathbb{R}^m) = L^1(\mathbb{R}, \mathbb{R}^m)^*$, is a compact metrizable space and the shift flow

$$\theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \quad (t, u) \mapsto \theta_t u = u(\cdot + t),$$

is a continuous dynamical system, which is chain transitive (see [5, Section 4.2]). We will denote by $\mathcal{U}_{\mathrm{per}}$ the set of periodic point of θ in $\mathcal{U}$, that is, $u \in \mathcal{U}_{\mathrm{per}}$ if there exists $\tau > 0$ such that $\theta_\tau u = u$. By [5, Lemma 4.2.2] it holds that $\mathcal{U}_{\mathrm{per}}$ is dense in $\mathcal{U}$.

For any $g \in G$ and $u \in \mathcal{U}$, the solution $t \mapsto \phi(t, g, u)$ of Σ_G is complete and satisfies *cocycle property*

$$\phi(t + s, g, u) = \phi(t, \phi(s, g, u), \theta_s u)$$

for all $t, s \in \mathbb{R}$, $g \in G$, $u \in \mathcal{U}$. Together with the shift flow it constitutes a skew-product flow

$$\phi : \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \quad (t, u, x) \mapsto \phi_t(u, x) = (\theta_t u, \varphi(t, x, u)),$$

called the *control flow* of the system (see [5, Section 4.3]). We also write $\phi_{t,u} : G \to G$ for the map $g \mapsto \phi(t, g, u)$. In the particular case of linear systems, the intrinsic relations between the vector fields involved and the group structure implies the following relation

$$\phi_{\tau,u} \circ R_g = R_{\varphi_\tau(g)} \circ \phi_{\tau,u}, \quad \text{for any} \quad \tau \in \mathbb{R}, g \in G, \tag{4}$$

where $\{\varphi_t\}_{t\in\mathbb{R}}$ is the flow of $\mathcal{X}$. We define the *set of points reachable from $g \in G$ at time $\tau > 0$* and the *reachable set of g*, respectively, by

$$\mathcal{A}_\tau(g) := \{\phi_{\tau,u}(g), \quad u \in \mathcal{U}\} \quad \text{and} \quad \mathcal{A}(g) := \bigcup_{\tau>0} \mathcal{A}_\tau(g).$$

The set of *points controllable to g at time $\tau > 0$* and the *controllable set to g* are defined similarly. They are denoted by $\mathcal{A}^*_\tau(g)$ and $\mathcal{A}^*(g)$, respectively.

The next result relates the reachable and controllable set from the identity $e \in G$ with the dynamical subgroups associated with the flow of $\mathcal{X}$ (see [6, Lemma 3.1] and [2, Theorem 3.8]).

2.5 Theorem: *For a linear system Σ_G, it holds:*

1. *If $\varphi_t(g) \in \mathcal{A}(e)$ for all $t \in \mathbb{R}$ then $\mathcal{A}(e)g \subset \mathcal{A}(e)$;*
2. *If $\varphi_t(g) \in \mathcal{A}^*(e)$ for all $t \in \mathbb{R}$ then $\mathcal{A}^*(e)g \subset \mathcal{A}^*(e)$;*
3. *If $\mathcal{A}(e)$ is open, then $G^{+,0} \subset \mathcal{A}(e)$ and $G^{-,0} \subset \mathcal{A}^*(e)$.*

Next we define the concept of control set.

2.6 Definition: *A nonempty set $\mathcal{C} \subset G$ is said to be a control set of Σ_G if it is maximal (w.r.t. set inclusion) satisfying*

(i) For any $g \in \mathcal{C}$ there exists $u \in \mathcal{U}$ such that $\phi(\mathbb{R}^+, g, u) \subset \mathcal{C}$;

(ii) For any $g \in \mathcal{C}$ it holds that $\mathcal{C} \subset \mathrm{cl}(\mathcal{A}(g))$.

Since the identity $e \in G$ is a singularity of $\mathcal{X}$ and $0 \in \mathrm{int}\,\Omega$, there exists a control set $\mathcal{C}$ of Σ_G containing the identity. Moreover, if $e \in \mathrm{int}\,\mathcal{C}$ then $\mathcal{C}$ is the only control set of Σ_G and it satisfies $G^0 \subset \mathrm{int}\,\mathcal{C}$ ([4]). Furthermore, by the recent work [1] it holds that $\mathcal{C}$ is bounded if and only if G^0 is a compact subgroup.

3 The main result and its consequences

Our aim here is to study structural properties of $\mathcal{C}$ under the assumptions that it is bounded, or equivalently, that the central subgroup associated with the flow of $\mathcal{X}$ is compact.

Throughout the whole section we will then consider Σ_G be a linear system on such that the subgroup G^0 associated with the flow of the drift $\mathcal{X}$ of Σ_G is a compact subgroup. Moreover we will assume that $e \in \mathrm{int}\,\mathcal{C}$ and will consider the notation $Q := \mathrm{cl}(\mathcal{C})$.

3.1 The main result

We start by characterizing bounded orbits of linear system.

3.1 Proposition: *For any $u \in \mathcal{U}$ there exists at most one $x \in G^{+,-}$ such that*

$$\{\phi_{t,u}(x), \quad t \in \mathbb{R}\} \quad \textit{is bounded.}$$

Proof: In fact, if $x_1, x_2 \in G^{+,-}$ are such that $\{\phi_{t,u}(x_i), \quad t \in \mathbb{R}\}$ is bounded for $i = 1, 2$ then

$$\varrho\left(\phi_{t,u}(x_1), \phi_{t,u}(x_2)\right) = \varrho\left(\varphi_t(x_1), \varphi_t(x_2)\right) = \varrho(\varphi_t(x_2^{-1}x_1), e)$$

is bounded which by Lemma 2.4 implies necessarely $x_2^{-1}x_1 = e$ or $x_1 = x_2$. □

Next we show that any element of $\mathcal{U}_{\mathrm{per}}$ is associated with a bounded orbit.

3.2 Proposition: *With the previous assumption, for any $u \in \mathcal{U}_{\mathrm{per}}$ there exists a unique $x(u) \in G^{+,-}$ such*

$$\phi_{n\tau,u}(x(u)) \in x(u)G^0, \quad \text{for all} \quad n \in \mathbb{Z}.$$

In particular, the orbit $\{\phi_{t,u}(x(u)), \; t \in \mathbb{R}\}$ is bounded.

Proof: Assume first that $G^{+,-}$ is a subgroup of G. Let $u \in \mathcal{U}_{\mathrm{per}}$ be a τ-periodic funcion and decompose $\phi_{\tau,u}(e) = xy$ with $x \in G^{+,-}$ and $y \in G^0$. Since G^0 normalizes $G^{+,-}$ we have that $\psi := C_y \circ \varphi_\tau|_{G^{+,-}}$ is an automorphism of $G^{+,-}$. Moreover, the compactness of G^0 implies that G^+ and G^- are the unstable and stable subgroups of ψ and hence ψ is hyperbolic. Also, a simple calculation shows that

$$\psi^{n+1}(x) = (y\varphi_\tau(y)\cdots\varphi_{n\tau}(y))\,\varphi_{(n+1)\tau}(x)\,(y\varphi_\tau(y)\cdots\varphi_{n\tau}(y))^{-1}$$

implying that

$$x \in \mathrm{fix}(\psi) \quad \iff \quad (\varphi_{n\tau}(x))_{n\in\mathbb{N}} \quad \text{is bounded.}$$

However, as a consequence of Lemma 2.4, the only point $x \in G^{+,-}$ such that $(\varphi_{n\tau}(x))_{n\in\mathbb{N}}$ is bounded is the identity and hence $\mathrm{fix}(\psi) = \{e\}$.

By Proposition 2.1, the map

$$f_\psi : \quad g \in G^{+,-} \mapsto g\psi(g^{-1}) \in G^{+,-} \quad \text{is a diffeomorphism.}$$

Let then $x(u) \in G^{+,-}$ be the unique element satisfying $x = f_\psi(x(u))$. It holds that

$$x = f_\psi(x(u)) = x(u)\psi(x(u)^{-1}) = x(u)C_y(\varphi_\tau(x(u)^{-1})) = x(u)y\varphi_\tau(x(u)^{-1})y^{-1}$$

and hence

$$\phi_{\tau,u}(x(u)) = \phi_{\tau,u}(e)\varphi_\tau(x(u)) = xy\varphi_\tau(x(u)) = x(u)y \in x(u)G^0.$$

Also, if $\phi_{n\tau,u}(x(u)) \in x(u)G^0$ then

$$\phi_{(n+1)\tau,u}(x(u)) = \phi_{\tau,u}(\phi_{n\tau,u}(x(u)) \in \phi_{\tau,u}\left(x(u)G^0\right) = \phi_{\tau,u}(x(u)) \cdot \varphi_\tau(G^0) \in x(u)G^0,$$

where for the last equality we used equation (4). Let $n \in \mathbb{Z}^+$ and $g \in G^0$ such that $\phi_{n\tau,u}(x(u)) = x(u)g$. Since $\phi_{\tau,u}^{-1} = \phi_{-\tau,\theta_t u} = \phi_{-\tau,u}$ we have that

$$x(u) = \phi_{-n\tau,u}\left(\phi_{n\tau,u}(x(u))\right) = \phi_{-n\tau,u}(x(u)g) = \phi_{-n\tau,u}(x(u))\varphi_{-n\tau}(g)$$

$$\implies \phi_{-n\tau,u}(x(u)) = x(u)\varphi_{-n\tau}(g^{-1}) \in x(u)G^0$$

and hence

$$\phi_{n\tau,u}(x(u)) \in x(u)G^0, \quad \text{for all} \quad n \in \mathbb{Z}.$$

For the general case, let us consider $N^0 := N \cap G^0$. Following [1, Lemma 2.2] N^0 is a compact, connected normal subgroup of G. Moreover, the nilradical N of G satisfies $N = G^{+,-}N^0$. By considering the induced linear system $\Sigma_{\hat{G}}$ on the Lie group $\hat{G} = G/N^0$, the fact that $\hat{G}^{+,-} = \pi(G^{+,-}) = \pi(G^{+,-}N^0) = \pi(N)$ implies that $\hat{G}^{+,-}$ is a subgroup. By the previous case, for any $u \in \mathcal{U}_{\mathrm{per}}$ there exists a unique $\hat{x}(u) \in \hat{G}^{+,-}$ such that

$$\hat{\phi}_{n\tau,u}(\hat{x}(u)) \in \hat{x}(u)\hat{G}^0, \;\; n \in \mathbb{Z}.$$

Since G is decomposable, there exists a unique $x(u) \in G^{+,-}$ such that $\pi(x(u)) = \hat{x}(u)$. Moreover,

$$\pi\left(\phi_{n\tau,u}(x(u))\right) = \hat{\phi}_{n\tau,u}(\hat{x}(u)) \in \hat{x}(u)\hat{G}^0 = \pi(x(u)G^0)$$

$$\implies \phi_{n\tau,u}(x(u)) \in \pi^{-1}\left(\pi(x(u)G^0)\right) = x(u)G^0N^0 = x(u)G^0$$

and hence

$$\phi_{n\tau,u}(x(u)) \in x(u)G^0, \quad \text{for all} \quad n \in \mathbb{Z}.$$

For the boundedness of the whole orbit $\{\phi_{t,u}(x(u)),\ t \in \mathbb{R}\}$, let ud consider $t \in \mathbb{R}$ and write it as $t = n\tau + r$ with $n \in \mathbb{Z}$ and $|r| \in [0, \tau)$. Then,

$$\phi_{t,u}(x(u)) = \phi_{r,u}(\phi_{n\tau,u}(x(u)) \in \phi_{r,u}\left(x(u)G^0\right) = \phi_{r,u}(x(u)) \cdot G^0,$$

showing that

$$\{\phi_{t,u}(x(u)),\ \ t \in \mathbb{R}\} \subset \mathcal{A}_\tau(x(u))G^0 \cup \mathcal{A}^*_\tau(x(u))G^0,$$

and consequently that $\{\phi_{t,u}(x(u)),\ \ t \in \mathbb{R}\}$ is bounded. □

The next lemma shows that the orbits associated with $\mathcal{U}_{\text{per}}$ are contained in Q.

3.3 Lemma: *For any $u \in \mathcal{U}_{\text{per}}$ it holds that $\phi_{t,u}\left(x(u)G^0\right) \subset Q$, for any $t \in \mathbb{R}$.*

Proof: Let us first prove that $x(u)G^0 \subset Q$. By Proposition 3.2 we have that,

$$\phi_{n\tau,u}(x(u)) \in x(u)G^0, \quad \text{for all} \quad n \in \mathbb{Z}.$$

Decomposing $x(u) = gh$ with $g \in G^+$ and $h \in G^-$ gives us that

$$\varrho(\phi_{n\tau,u}(x(u)), \phi_{n\tau,u}(g)) = \varrho(\phi_{n\tau,u}(g)\varphi_{n\tau}(h), \phi_{n\tau,u}(g)) = \varrho(\varphi_{n\tau}(h), e) \to 0 \quad \text{as} \quad n \to +\infty.$$

In particular, $\phi_{n\tau,u}(g) \to x(u)g_1$ for some $g_1 \in G^0$. However, the fact that $G^+ \subset \mathcal{A}$ implies by invariance that $\phi_{n\tau,u}(g) \in \mathcal{A}$ and hence $x(u)g_1 \in \text{cl}(\mathcal{A})$. Let then $g_2 \in G^0$ arbitrary. By Theorem 2.5 it holds that

$$x(u)g_2 = x(u)g_1(g_1^{-1}g_2) \in \text{cl}(\mathcal{A})g_1^{-1}g_2 = \text{cl}(\mathcal{A}g_1^{-1}g_2) \subset \text{cl}(\mathcal{A}) \implies x(u)G^0 \subset \text{cl}(\mathcal{A}).$$

Consider now the decomposition $x(u) = h'g'$ with $h'^{-,0}G^+$. By the same arguments as previously, we have that $\phi_{-n\tau,u}(h') \to x(u)h_1$, for some $h_1 \in G^0$. Using that $G^{-,0} \subset \mathcal{A}^*$ gives us that $x(u)h_1 \in \text{cl}(\mathcal{A}^*)$ and again by Theorem 2.5 we get, for any $h_2 \in G^0$, that

$$x(u)h_2 = x(u)h_1(h_1^{-1}h_2) \in \text{cl}(\mathcal{A}^*)h_1^{-1}h_2 = \text{cl}(\mathcal{A}^*h_1^{-1}h_2) \subset \text{cl}(\mathcal{A}^*), \implies x(u)G^0 \subset \text{cl}(\mathcal{A}^*).$$

Therefore,

$$x(u)G^0 \subset \text{cl}(\mathcal{A}) \cap \text{cl}(\mathcal{A}^*) = Q.$$

Consider now $t > 0$. By invariance in positive time we already have that $\phi_{t,u}(x(u)G^0) \subset \text{cl}(\mathcal{A})$. On the other hand, for any $n \in \mathbb{Z}$ we have that

$$\phi_{t,u}(x(u)) = \phi_{t,u}\left(\phi_{-n\tau,u}\left(\phi_{n\tau,u}(x(u))\right)\right) = \phi_{t-n\tau,u}\left(\phi_{n\tau,u}(x(u))\right) \in \phi_{t-n\tau,u}\left(x(u)G^0\right),$$

where for the last equality we used the cocycle property and the fact that u is τ-periodic.
Since $x(u)G^0 \subset Q \subset \text{cl}(\mathcal{A}^*)$, if we take $n \in \mathbb{Z}$ such that $t - n\tau \leq 0$, the invariance in negative time of $\text{cl}(\mathcal{A}^*)$ implies that

$$\phi_{t,u}((x(u))) \in \phi_{t-n\tau,u}(\text{cl}(\mathcal{A}^*)) \subset \text{cl}(\mathcal{A}^*)$$

and hence, for any $g \in G^0$, we get that

$$\phi_{t,u}(x(u)g) = \phi_{t,u}(x(u))\varphi_t(g) \in \text{cl}(\mathcal{A}^*)\varphi_t(g) = \text{cl}(\mathcal{A}\varphi_t(g)) \subset \text{cl}(\mathcal{A}^*) \implies \phi_{t,u}(x(u)G^0) \subset \text{cl}(\mathcal{A}^*),$$

and consequently

$$\phi_{t,u}(x(u)G^0) \subset Q, \quad \text{for all} \quad t \geq 0.$$

By arguing analogously, we get that $\phi_{t,u}(x(u)G^0) \subset Q$ for all $t < 0$ and so

$$\phi_{t,u}(x(u)G^0) \subset Q, \quad \text{for all} \quad t \in \mathbb{R},$$

concluding the proof. □

The previous lemma allows us to construct a continuous functions from the set of control function $\mathcal{U}$ to $G^{+,-}$ as follows: Let $u \in \mathcal{U}$ arbitrary. Since $\mathcal{U}_{\text{per}}$ is dense in $\mathcal{U}$ there exists $u_k \in \mathcal{U}_{\text{per}}$ with $u_k \to u$. Denote by $x_k := x(u_k)$ the unique point in $G^{+,-}$ given by Proposition 3.2. By Lemma 3.3 the sequence $(x_k)_{k\in\mathbb{N}}$ is contained in Q and hence is bounded. Moreover, if $x \in G^{+,-}$ is such that $x_{k_n} \to x$, Lemma 3.3 also implies that

$$\phi_{t,u}(x) = \lim_{n\to+\infty} \phi_{t,u_{k_n}}(x_{k_n}) \in Q, \quad \text{for all} \quad t \in \mathbb{R}.$$

Consequently, $\{\phi_{t,u}(x), \quad t \in \mathbb{R}\} \subset Q$ is a bounded orbit and by Proposition 3.1 the point $x \in G^{+,-}$ is the unique point with such property. Therefore, the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded and has only one adherent point implying that $(x_k)_{k\in\mathbb{N}}$ converges to a point $x(u) \in G^{+,-}$.

We can now state and prove the main result of this paper.

3.4 Theorem: *All bounded orbits of Σ_G are contained in Q.*

Proof: Let $h \in G$ and decompose it as $h = xg$ with $x \in G^{+,-}$ and $g \in G^0$. If for some $u \in \mathcal{U}$ we have that $\{\phi_{t,u}(h),\ t \in \mathbb{R}\}$ is bounded, then

$$\phi_{t,u}(x) = \phi_{t,u}(hg^{-1}) = \phi_{t,u}(h)\varphi_t(g) \in \{\phi_{t,u}(h),\ \ t \in \mathbb{R}\}G^0.$$

Since $x(u) \in G^{+,-}$ has also bounded orbit, Proposition 3.1 implies that $x = x(u)$ which by Lemma 3.3 and the previous discussion implies that

$$\{\phi_{t,u}(h),\ t \in \mathbb{R}\} = \{\phi_{t,u}(x(u)g),\ t \in \mathbb{R}\} \subset Q,$$

as stated. □

3.2 Consequence of the main result

In this section we show that the lift of the bounded control set of a linear system is the continuous image of $\mathcal{U} \times G^0$.

Let us as previously consider $Q = \text{cl}(\mathcal{C})$. We define the *lift* of the control set Q as the set

$$L(Q) := \{(u, g) \in \mathcal{U} \times G;\ \ \phi(\mathbb{R}, g, u) \subset Q\}.$$

By the main result, the map $(u, g) \in \mathcal{U}\times \mapsto x(u)g$ is continuous and the orbit $\{\phi_{t,u}(x(u)g),\ \ t \in \mathbb{R}\}$ is contained in Q. Therefore,

$$H : \mathcal{U} \times G^0 \to L(\mathcal{C}), \qquad (u, g) \mapsto (u, x(u)g),$$

is a well defined continuos map. Moreover,

$$(u, x) \in L(Q) \quad \iff \quad \phi(\mathbb{R}, x, u) \subset Q \quad \iff \quad \{\phi_{t,u}(x),\ \ t \in \mathbb{R}\} \ \text{ is bounded} \quad \iff \quad (u, x) \in H(\mathcal{U} \times G^0).$$

Hence $H(\mathcal{U} \times G^0) = L(Q)$ and, since $\mathcal{U} \times G^0$ is compact and H is continuous, H is a closed map and hence a homeomorphism between $\mathcal{U} \times G^0$ and $L(Q)$. In particular, Q is the continuous image of $\mathcal{U} \times G^0$.

Let us assume that $G^{+,-}$ is a subgroup of G. Since G is decomposable it actually holds that $G^{+,-}$ is a normal subgroup of G and hence the canonical projection $\pi : G \to G/G^{+,-}$ induces a linear system on $G/G^{+,-}$. However, the fact that G is decomposable implies that $\pi|_{G^0}$ is a group isomorphism and hence we can assume w.l.o.g. that $G/G^{+,-} = G^0$. Let us denote by Σ_{G^0} the linear system induced by π on G^0 under such identifications and by Φ^0 its control flow. The next result shows that under the extra assumption that $G^{+,-}$ is a subgroup the dynamics of $\Phi_t|_{L(\mathcal{C})}$ is basically the same as the one from the Φ^0_t.

3.5 Theorem: *If $G^{+,-}$ is a subgroup, then H conjugates $\Phi_t|_{L(\mathcal{C})}$ and Φ^0_t.*

Proof: By the cocycle property,

$$\phi_{t+s,u}(x(u)) = \phi_{t,\theta_s u}\left(\phi_{s,u}(x(u))\right).$$

Hence,

$$\{\phi_{t,\theta_s u}\left(\phi_{s,u}(x(u))\right),\ \ t \in \mathbb{R}\} \quad \text{is a bounded orbit.}$$

In particular, we have that $\phi_{s,u}(x(u)) = x(\theta_s u)h$ for an unique $h \in G^0$. Furthermore, under the previous identification, it holds that

$$h = \pi(h) = \pi(h(h^{-1}x(\theta_s u)h)) = \pi(x(\theta_s)h) = \pi(\phi_{s,u}(x(u))) = \pi(\phi_{s,u}(e)) = \phi^0_{s,u}(e),$$

showing that

$$\forall s \in \mathbb{R}, \quad \phi_{s,u}(x(u)) = x(\theta_s u)\phi^0_{s,u}(e).$$

Therefore,

$$\begin{aligned} H\left(\Phi^0_s(u,g)\right) &= H\left(\theta_s u, \phi^0_{t,u}(g)\right) = \left(\theta_s u, x(\theta_s u)\phi^0_{t,u}(e)\varphi_s(g)\right) \\ &= (\theta_s u, \phi_{s,u}(x(u))\varphi_s(g)) = (\theta_s u, \phi_{s,u}(x(u)g)) = \Phi_s(H(u,g)), \end{aligned}$$

showing that H conjugates $\Phi_t|_{L(\mathcal{C})}$ and Φ^0_t as stated. □

3.6 Remark: In [10, Theorem 3.4] the author shows that $L(Q)$ is the graph of a continuous function from $\mathcal{U}$, under a hyperbolicity assumption. In the context of linear systems, such assumption is equivalent to $G^0 = \{e\}$. Therefore our previous result shows that when the system is not necessarily hyperbolic but has a well behaved central manifold (that is G^0 is compact) we still have a characterization of $L(Q)$ in terms of the central manifolds and the set $\mathcal{U}$.

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