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Linear control systems on the homogeneous spaces of the 2D Lie group

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Abstract

In this paper, we classify all the possible linear control systems on the homogeneous spaces of the 2D solvable Lie group and study their controllability and control sets.

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1. Introduction

Let S be a connected Lie group and $L \subset S$ a closed subgroup. A *linear control system (in short, LCS)* defined on the homogeneous space $L \setminus S$ is a family of ordinary differential equations

$$\dot{x}(t) = f_0(x(t)) + \sum_{j=1}^{m} u_j(t) f_j(x(t)), \quad u \in \mathcal{U} \qquad (\Sigma_{L\setminus S})$$

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parametrized by $u = (u_1, \dots, u_m)$ in $\mathcal{U}$, the set of piecewise constant functions, with $u(t) \in \Omega$ where $\Omega \subset \mathbb{R}^m$ is a compact and convex subset satisfying $0 \in \operatorname{int}\Omega$. Here, the vector fields $f_0, f_1, \dots, f_m$ satisfy

$$\pi_*\mathcal{X} = f_0 \circ \pi \quad \text{and} \quad \pi_* Y^j = f_j \circ \pi,$$

where $\pi : S \to L \setminus S$ is the canonical projection, $\mathcal{X}$ is a linear vector field and $Y^1, \dots, Y^m$ are left-invariant vector fields.

When the state space $L \setminus S$ is Euclidean, the system $\Sigma_{L\setminus S}$ is a well known LCS on an Euclidean space appearing in several physical applications (see for instance [11,13]). Linear control systems defined on more general Lie groups first appeared for matrices groups in [12], and subsequently for any Lie group in [6]. In the Lie group context, the dynamical behavior of LCSs were extensively studied by using inherent geometric richness present in a Lie group (see [1–6,8–10] and references therein). The extension of LCS to homogeneous spaces first appeared in [10]. Therein it is shown their importance by proving an equivalence theorem which, roughly speaking, states that any control-affine system on a connected manifold, whose associated vector fields are complete and generate a finite Lie algebra is diffeomorphic equivalent to a LCS on a homogeneous space.

The present paper is a starting point in the study of the controllability properties and the control sets for LCSs in the general context where the homogeneous spaces in question are not necessarily connected Lie groups. By considering the two-dimensional solvable Lie group S we are able to show that, up to diffeomorphisms, there are only four types of LCSs on the homogeneous spaces of S. A full analysis of the controllability property and of the control sets of a LCS on a homogeneous space of S is here provided. We obtain that one of the four cases is quite different from what happens at the group level, by showing that the dynamics in such systems is more complex than the one on Lie groups.

The paper is structured as follows: In Section 2 we present definitions and properties about control-affine systems and LCS of the two-dimensional solvable Lie group. Since our aim is to reach a broad public, we only define LCS in coordinates, as systems evolving on the semi-direct product of $\mathbb{R}$ by itself. In Section 3, we characterize every closed subgroup L of S and those invariant under the flow of a linear vector field. Such results allow us to show that there are only four types of LCSs on the homogeneous spaces of S depending on the dimension of the manifold $L \setminus S$. In the case where $L \setminus S$ is one-dimensional, we obtain a LCS on the real line $\mathbb{R}$ or, a homogeneous system on the one-dimensional torus $\mathbb{T}$. Now, if $L \setminus S$ is two-dimensional we obtain systems on vertical and horizontal cylinders. Finally, Section 4 contains an exhaustive analysis of the obtained systems. In each case, we analyze the controllability property or, when the previous does not hold, explicitly determine their control sets. One of these cases reveals something exciting. Already on low-dimensional groups, the properties of control sets on groups and homogeneous spaces are quite different. For instance, we may have LCSs on homogeneous spaces that admit a unique bounded control set with nonempty interior, and LCSs that admit two unbounded control sets with nonempty interior. In comparison, any LCS on S admit at most one unbounded control set with nonempty interior.

2. Preliminaries

2.1. Control-affine systems, controllability and control sets

Let M be a finite dimensional smooth manifold. A *control-affine system* in M is determined by the family of ODE's

$$\dot{x}(t) = f_0(x(t)) + \sum_{j=1}^{m} u_j(t) f_j(x(t)), \quad u \in \mathcal{U} \qquad (\Sigma_M)$$

here, $f_0, f_1, \ldots, f_m$ are smooth vector fields on M and $u = (u_1, \ldots, u_m)$ belongs to the set $\mathcal{U}$ of the piecewise constant functions satisfying $u(t) \in \Omega$ where $\Omega \subset \mathbb{R}^m$ is a compact, convex subset satisfying $0 \in \operatorname{int}\Omega$. For any $x \in M$ and $u \in \mathcal{U}$ the solution of Σ_M is the unique curve $t \mapsto \phi(t, x, u)$ on M satisfying $\phi(0, x, u) = x$. For $x \in M$ the *positive* and *negative orbits* of Σ_M at x are defined as

$$\mathcal{O}^+(x) = \{\phi(t, x, u), t \geq 0, u \in \mathcal{U}\} \quad \text{and} \quad \mathcal{O}^-(x) = \{\phi(-t, x, u), t \geq 0, u \in \mathcal{U}\},$$

respectively. We say that Σ_M satisfies the Lie algebra rank condition (LARC) if the Lie algebra $\mathcal{L}$ generated by the vector fields $f_0, f_1, \ldots, f_m$, satisfies $\mathcal{L}(x) = T_x M$ for all $x \in M$. The system Σ_M is said to *controllable* if $M = \mathcal{O}^+(x)$ for all $x \in M$.

A set $D \subset M$ is a *control set* of Σ_M if it is maximal, w.r.t. set inclusion, with the following properties:

1. For any $x \in D$, there is $u \in \mathcal{U}$ with $\phi(\mathbb{R}_+, x, u) \subset D$;
2. For any $x \in D$, it holds that $D \subset \overline{\mathcal{O}^+(x)}$.

By [7, Proposition 3.2.4], any subset of M, with nonempty interior, which is maximal concerning property 2. above is a control set.

Let N be another smooth manifold and

$$\dot{y}(t) = g_0(x(t)) + \sum_{j=1}^{m} u_j(t) g_j(y(t)), \quad u \in \mathcal{U} \qquad (\Sigma_N)$$

a control-affine system on N. If $\psi : M \to N$ is a smooth map, we say that Σ_M and Σ_N are ψ-conjugated if their respective vector fields are ψ-conjugated, that is,

$$\psi_* f_j = g_j \circ \psi, \quad j = 0, 1, \ldots, m.$$

If such ψ exists, we say that Σ_M and Σ_N are conjugated. If ψ is a diffeomorphism, Σ_M and Σ_N are called *equivalent*. It is straightforward to see that controllability, topological properties of control sets and topological properties of positive/negative orbits are preserved by equivalent systems.

2.2. LCSs on the 2D solvable Lie group

Let us consider S to be the 2D solvable Lie group seeing as the semi-direct product $S = \mathbb{R} \times_\rho \mathbb{R}$, where $\rho_x = \mathrm{e}^x$. Under this interpretation, the product in S is given by

$$(x_1, y_1) * (x_2, y_2) = (x_1 + x_2, y_1 + \mathrm{e}^{x_1} y_2).$$

In this setup, the Lie algebra of S is given by the semi-direct product $\mathfrak{s} = \mathbb{R} \times_\theta \mathbb{R}$, where $\theta = \mathrm{id}_{\mathbb{R}}$. In particular, the bracket in $\mathfrak{s}$ reads as

$$[(\alpha_1, \beta_1), (\alpha_2, \beta_2)] = (0, \alpha_1\beta_2 - \alpha_2\beta_1).$$

The exponential map is explicitly given by,

$$\exp(a, b) = \begin{cases} (0, b), & \text{if } a = 0 \\ \left(a, \frac{1}{a}(\mathrm{e}^a - 1)b\right) & \text{if } a \neq 0 \end{cases},$$

and the group of automorphisms of $\mathfrak{s}$ and S are in bijection with $\mathbb{R} \times \mathbb{R}^*$ and given, respectively, by

$$P(\alpha, \beta) = (\alpha, a\alpha + b\beta) \qquad \text{and} \qquad \psi(x, y) = (x, (\mathrm{e}^x - 1)a + by),$$

where $(a, b) \in \mathbb{R} \times \mathbb{R}^*$.

A left-invariant vector field and a linear vector field on S, respectively, reads

$$Y(x, y) = (\alpha, \mathrm{e}^x\beta) \quad \text{and} \quad \mathcal{X}(x, y) = (0, by + (\mathrm{e}^x - 1)a),$$

where $(\alpha, \beta), (a, b) \in \mathbb{R}^2$. As we know, the flow of a linear vector field is a one-parameter group of automorphisms (see [6, Theorem 1]) which in this case is given by

$$\varphi_t(x, y) = \begin{cases} (x, y + t(\mathrm{e}^x - 1)a) & \text{if } b = 0, \\ \left(x, \mathrm{e}^{tb}y + \frac{1}{b}(\mathrm{e}^{tb} - 1)(\mathrm{e}^x - 1)a\right) & \text{if } b \neq 0. \end{cases}$$

It turns out that a *linear control system (LCS)* on S is defined by the family of ODE's as follows

$$\begin{cases} \dot{x} = u\alpha \\ \dot{y} = by + (\mathrm{e}^x - 1)a + u\mathrm{e}^x\beta \end{cases}, \quad \text{where} \quad u \in \Omega, \tag{Σ_S}$$

with $\Omega = [u_*, u^*]$ and $u_* < 0 < u^*$. As shown in [3, Section 2.2] the LCS Σ_S satisfies the LARC if and only if $\alpha(a\alpha + b\beta) \neq 0$.

For any $\psi \in \mathrm{Aut}(S)$, we have that $\psi_*(\mathcal{X} \circ \psi^{-1})$ is a linear vector field if $\mathcal{X}$ is linear and $\psi_*(Y \circ \psi^{-1})$ is left-invariant if Y is left invariant. In particular, automorphisms of S give rise to equivalent LCSs for any given initial LCS. Such facts will be used ahead in order to simplify the study of our LCSs.

2.1 Remark. Let us remark that in [3] the analysis on the two-dimensional solvable Lie group was made by considering $\mathbb{R}^+ \times \mathbb{R}$ as subjacent manifold and in the present paper we use $\mathbb{R} \times \mathbb{R}$. Such choices are made only in order to simplify the calculations. The relationship between both choices is made by considering the isomorphism

$$(x, y) \in \mathbb{R} \times \mathbb{R} \mapsto (e^x, y) \in \mathbb{R}^+ \times \mathbb{R}.$$

3. Linear systems on homogeneous spaces

We shall now define and characterize all possible LCS on the homogeneous spaces of S.

Let $L \subset S$ be a closed subgroup and $\pi : S \to L \setminus S$ be the canonical projection. It is worth noting that a LCS on the homogeneous space $L \setminus S$ is a control-affine system on $L \setminus S$ which is π-conjugated to a LCS on S. That is, a family of ODE's

$$\dot{P} = f_0(P) + u f_1(P), \quad \text{with} \quad u \in \Omega, P \in L \setminus S \tag{$\Sigma_{L\setminus S}$}$$

where f_0, f_1 are vector fields on $L \setminus S$ satisfying

$$\pi_* \mathcal{X} = f_0 \quad \text{and} \quad \pi_* Y = f_1,$$

with $\mathcal{X}$ linear and Y left-invariant. By [10, Proposition 4], the vector field $\pi_*\mathcal{X}$ is well defined on $L \setminus S$ if and only if L is invariant by the flow of $\mathcal{X}$, that is,

$$\varphi_t(L) = L, \quad \forall t \in \mathbb{R}, \tag{1}$$

where $\{\varphi_t\}_{t\in\mathbb{R}}$ is the flow of $\mathcal{X}$. Therefore, in order to characterize all the possible LCSs on the homogeneous spaces of S we need to find the possible φ_t-invariant closed subgroups L of S, which we will do in the next sections.

3.1. Subgroups invariant by linear vector fields

The natural question that arises, is whether it is possible to build a homogeneous space $L \setminus S$ of S. The first known condition is that L needs to be topologically closed in S. In this section, we classify all possible closed subgroups of S. We exclude the trivial cases $L = \{(0,0)\}$ and $L = S$, which leaves us with the analysis of the nontrivial zero-dimensional and one-dimensional subgroups of S. We start classifying the one-dimensional Lie subalgebras of $\mathfrak{s}$, up to isomorphisms. The principal meaning of the subsequent propositions is to reduce the difficulty of building homogeneous spaces of dimension one.

3.1 Proposition. *Up to isomorphisms, the only one-dimensional subalgebras of $\mathfrak{s}$ are*

$$\mathbb{R} \times \{0\} \quad \textit{and} \quad \{0\} \times \mathbb{R}.$$

Proof. Let us consider a one-dimensional subalgebra $\mathfrak{l} \subset \mathfrak{s}$ and a nonzero vector $(\alpha_0, \beta_0) \in \mathfrak{l}$ with $\alpha_0 \neq 0$. The automorphism of $\mathfrak{s}$ given by

$$P(\alpha, \beta) = \left(\alpha, -\frac{\beta_0}{\alpha_0}\alpha + \beta\right),$$

is such that $P(\alpha_0, \beta_0) = (\alpha_0, 0)$ implying, by linearity, that $P(\mathfrak{l}) = \mathbb{R} \times \{0\}$. Therefore, if $\mathfrak{l} \neq \{0\} \times \mathbb{R}$, it is isomorphic to $\mathbb{R} \times \{0\}$ concluding the proof. □

Using the previous statement we are able to classify all the possible zero and one dimensional subgroups of S.

3.2 Proposition. *Up to isomorphisms, a subgroup L of S satisfies:*

1. $\dim L = 0$ *and* $L = \{0\} \times \mathbb{Z}$ *or* $L = \mathbb{Z}c \times \{0\}$ *for some* $c \in \mathbb{R}$*;*
2. $\dim L = 1$ *and* $L = \mathbb{R} \times \{0\}$ *or* $L = \mathbb{Z}c \times \mathbb{R}$ *for some* $c \in \mathbb{R}$*.*

Proof. 1. Let us denote by $\pi : S \to (\{0\} \times \mathbb{R}) \setminus S$ the canonical projection. Needless to say, that $\{0\} \times \mathbb{R}$ is a normal subgroup and π is a homomorphism, implying that $\pi(L)$ is a zero-dimensional subgroup of $\{0\} \times \mathbb{R}$. Now, the fact that $(\{0\} \times \mathbb{R}) \setminus S$ is isomorphic to $\mathbb{R}$ gives us that $\pi(L) = \mathbb{Z}c'$ for some $c' \in \mathbb{R}$. Therefore, we get

$$L \subset (\{0\} \times \mathbb{R}) * (\mathbb{Z}c \times \{0\}) = \mathbb{Z}c' \times \mathbb{R}.$$

If $c = 0$ then L is a discrete subgroup of $\{0\} \times \mathbb{R}$ and hence $L = \{0\} \times \mathbb{Z}c'$ for some $c' \in \mathbb{R}$. In this case, $c' = 0$ and $L = \{(0, 0)\}$ or $c' \neq 0$ and the automorphism of S,

$$f(x, y) = (x, y(c')^{-1}), \quad \text{satisfies} \quad f(\{0\} \times \mathbb{Z}c') = \{0\} \times \mathbb{Z}.$$

On the other hand, if $c \neq 0$ and there exists $(nc, y) \in L$ we get

$$L \ni (-nc, 0) * (nc, y) = (0, \mathrm{e}^{-nc} y) \to (0, 0).$$

However, L is discrete implying that $y = 0$ and hence $L = \mathbb{Z}c \times \{0\}$.

2. Let us denote by L_0 the connected component of L. Since $\dim L_0 = 1$, we have that L_0 is a 1-parameter subgroup of G, that is,

$$L_0 = \{\exp s(\alpha, \beta),\ s \in \mathbb{R}\},$$

where $(\alpha, \beta) \in \mathfrak{l}$ is a nonzero vector. Moreover, by Proposition 3.1, we know that up to isomorphism, $\mathfrak{l} = \{0\} \times \mathbb{R}$ or $\mathfrak{l} = \mathbb{R} \times \{0\}$ implying that

$$L_0 = \mathbb{R} \times \{0\} \quad \text{or} \quad L_0 = \{0\} \times \mathbb{R}.$$

Again, the projection $\pi : S \to S/(\{0\} \times \mathbb{R})$ takes L onto the subgroup $\pi(L)$ with $\dim \pi(L) \in \{0, 1\}$. If $L_0 = \{0\} \times \mathbb{R}$. Thus, necessarily $\pi(L)$ is a discrete subgroup of $S/(\{0\} \times \mathbb{R})$ and hence $\pi(L) = \mathbb{Z}\bar{c}$ for some $\bar{c} \in \mathbb{R}$. Consequently, $L \subset \mathbb{Z}\bar{c} \times \mathbb{R}$. Moreover, the set $L \cap \mathbb{Z}\bar{c} \times \{0\}$ is a subgroup of $\mathbb{Z}\bar{c} \times \{0\}$ and hence, has the form $\mathbb{Z}p\bar{c} \times \{0\}$ for some $p \in \mathbb{Z}$. On the other hand, the following fact

$$L \ni (n\bar{c}, y) = (n\bar{c}, 0) * \underbrace{(0, \mathrm{e}^{-n\bar{c}} y)}_{\in L_0 \subset L} \implies (n\bar{c}, 0) \in L \cap \mathbb{Z}\bar{c} \times \{0\} = \mathbb{Z}p\bar{c} \times \{0\},$$

shows that $L = \mathbb{Z}c \times \mathbb{R}$ for $c = p\bar{c}$.

Assume now $L_0 = \mathbb{R} \times \{0\}$ and consider $(x, y) \in L$. Then,

$$(0, y) = (x, y) * (-x, 0) \in L.$$

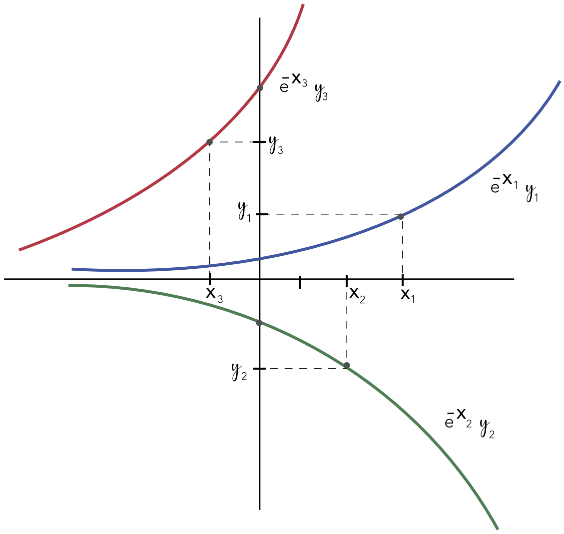

Fig. 1. The right cosets of $L_0 = \mathbb{R} \times \{0\}$.

On the other hand, a simple calculation shows that the right coset of L_0 by $(0, y)$ is given by (see Fig. 1)

$$L_0(0, y) = \{(s, \mathrm{e}^s y), \;\; s \in \mathbb{R}\}.$$

Since L is a subgroup, it holds that

$$[L_0(0, y)]^2 = \left\{(s, \mathrm{e}^s y) * (t, \mathrm{e}^t y), \quad s, t \in \mathbb{R}\right\} = \left\{(s + t, \mathrm{e}^s y + \mathrm{e}^{t+s} y), \quad s, t \in \mathbb{R}\right\} \subset L.$$

Furthermore, the map

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \quad (s, t) \in \mathbb{R}^2 \mapsto (s + t, \mathrm{e}^s y + \mathrm{e}^{t+s} y) \in \mathbb{R}^2,$$

is a differentiable map satisfying

$$Jg(s, t) = \begin{bmatrix} 1 & 1 \\ \mathrm{e}^s y + \mathrm{e}^{s+t} y & \mathrm{e}^{s+t} y \end{bmatrix} \quad \Longrightarrow \quad \det Jg(x, y) = -\mathrm{e}^s y.$$

The only point remaining concerned with the topological property of the map g. In fact, if $y \neq 0$, g is a local diffeomorphism, implying that $\mathrm{Im}(g) = [L_0(0, y)]^2$ has nonempty interior in $\mathbb{R}^2$, which is absurd since $\dim L = 1$. Therefore, $y = 0$ and hence $L = \mathbb{R} \times \{0\}$ concluding the proof. □

Let us now consider $L \subset S$ to be a subgroup and $\{\varphi_t\}_{t \in \mathbb{R}}$ a one-parameter group of automorphisms. Next, we obtain the conditions we need for L to be invariant by φ_t. That is, under which conditions equation (1) is satisfied.

By Proposition 3.2, up to automorphisms, we can assume that L is one of the following subgroups:

$$\{0\}\times\mathbb{Z},\ \ \mathbb{Z}c\times\{0\},\ \ \mathbb{R}\times\{0\}\ \text{ or }\ \mathbb{Z}c\times\mathbb{R}.$$

Assume first that $\varphi_t(x, y) = (x, y + t(e^x - 1)a)$. In this case, it is straightforward to see that $\{0\}\times\mathbb{Z}$ and $\mathbb{Z}c\times\mathbb{R}$ are φ_t-invariant. On the other hand, if $x \neq 0$, then

$$\varphi_t(x,0)\in\mathbb{R}\times\{0\},\ \forall t\in\mathbb{R}\iff t(e^x-1)a=0,\ \forall t\in\mathbb{R}\iff a=0.$$

Consequently, $\mathbb{Z}c\times\{0\}$ and $\mathbb{R}\times\{0\}$ are φ_t-invariant if and only if $a = 0$ if and only if $\varphi_t = \mathrm{id}_{\mathbb{R}^2}$ for all $t\in\mathbb{R}$.

If $\varphi_t(x, y) = \left(x, e^{bt}y + \frac{1}{b}(e^{tb} - 1)(e^x - 1)a\right)$ with $b \neq 0$, we have directly that $\mathbb{Z}c\times\mathbb{R}$ is φ_t-invariant and, as previously, if $x \neq 0$

$$\varphi_t(x,0)\in\mathbb{R}\times\{0\},\ \forall t\in\mathbb{R}\iff \frac{1}{b}(e^{tb}-1)(e^x-1)a,\ \forall t\in\mathbb{R}\iff a=0.$$

Now,

$$\varphi_t(0,n)\in\{0\}\times\mathbb{Z}\ \forall t\in\mathbb{R}\iff e^{bt}n\in\mathbb{Z},\ \forall t\in\mathbb{R}\iff b=0,$$

showing that $\mathbb{Z}\times\{0\}$ is not φ_t-invariant if $b \neq 0$. We have proved the following:

3.3 Proposition. *Let $\mathcal{X}$ be a linear vector field on S with associated flow $\{\varphi_t\}_{t\in\mathbb{R}}$. It holds:*

1. $\{0\}\times\mathbb{Z}$ *is φ_t-invariant if and only if $\mathcal{X} = (0, (e^x - 1)a)$;*
2. $\mathbb{Z}c\times\{0\}$ *or $\mathbb{R}\times\{0\}$ is φ_t-invariant if and only if $\mathcal{X} = (0, by)$;*
3. $\mathbb{Z}c\times\mathbb{R}$ *is always φ_t-invariant.*

3.2. *The possible LCSs on $L\setminus S$*

Let us consider

$$\dot{P} = f_0(P) + uf_1(P), \quad \text{where}\quad u\in\Omega,\ P\in L\setminus S \qquad (\Sigma_{L\setminus S})$$

a LCS on the homogeneous space $L\setminus S$ where $\mathcal{X}$ and Y, respectively, are the linear and left-invariant vector fields that are π-conjugated with f_0 and f_1, respectively. Let $\psi\in\mathrm{Aut}(S)$ such that $\widehat{L} = \psi(L)$ is one of the subgroups in Proposition 3.2 and consider $\widehat{\mathcal{X}}$, $\hat{Y}$ such that

$$\psi_*\widehat{\mathcal{X}} = \mathcal{X}\circ\psi \quad\text{and}\quad \psi_*\widehat{Y} = Y\circ\psi.$$

Since L is invariant by the flow of $\mathcal{X}$, it follows that $\widehat{L}$ is invariant by the flow of $\widehat{\mathcal{X}}$ and hence we have well defined vector fields $\widehat{f_0}$, $\widehat{f_1}$ on $\widehat{L}\setminus S$ determined by the relation

$$\widehat{f_0}\circ\widehat{\pi} = \widehat{\pi}_*\widehat{\mathcal{X}} \quad\text{and}\quad \widehat{f_1}\circ\widehat{\pi} = \widehat{\pi}_*\widehat{Y},$$

where $\widehat{\pi} : S \to \widehat{L}\setminus S$ is the canonical projection. By a straightforward calculation, we get that the map

$$\widehat{\psi} : L \setminus S \to \widehat{L} \setminus S, \quad \text{defined by the relation} \quad \widehat{\psi} \circ \pi = \widehat{\pi} \circ \psi,$$

is a diffeomorphism that satisfies

$$\widehat{\psi}_* f_0 = \widehat{f_0} \circ \widehat{\psi} \quad \text{and} \quad \Psi_* f_1 = \widehat{f_1} \circ \widehat{\psi},$$

showing that $\Sigma_{L\setminus S}$ is equivalent to the LCS

$$\dot{Q} = \widehat{f_0}(Q) + u\widehat{f_1}(Q), \quad \text{where} \quad u \in \Omega, Q \in \widehat{L} \setminus S. \tag{$\Sigma_{\widehat{L}\setminus S}$}$$

Therefore, we can assume w.l.o.g. that the subgroup L is one of the subgroups obtained in Proposition 3.2, and consequently, we just need to analyze the following cases.

3.2.1. $L = \{0\} \times \mathbb{Z}$

A simple calculation shows that for any $(x_1, y_1), (x_2, y_2) \in S$ we obtain

$$(x_1, y_1) * (x_2, y_2)^{-1} \in \{0\} \times \mathbb{Z} \iff \begin{cases} x_1 = x_2 \\ y_2 - y_1 \in \mathbb{Z} \end{cases} \iff (x_1, [y_1]) = (x_2, [y_2]),$$

where $[y] = y + \mathbb{Z}$. Therefore, $(\{0\} \times \mathbb{Z}) \setminus S$ is the horizontal cylinder on the x-axis $\mathcal{C}_H := \mathbb{R} \times \mathbb{R}/\mathbb{Z}$. The canonical projection reads as

$$\pi : S \to \mathcal{C}_H, \quad (x, y) \mapsto (x, [y]).$$

Let $(x_0, [y_0]) \in \mathcal{C}_H$ and $\varepsilon \in (0, 1/2)$ and define $U_\varepsilon := \mathbb{R} \times (y_0 - \varepsilon, y_0 + \varepsilon)$. If $(x_1, y_1), (x_2, y_2) \in U_\varepsilon$ we have that

$$\pi(x_1, y_1) = \pi(x_2, y_2) \iff x_1 = x_2 \quad \text{and} \quad y_1 - y_2 \in \mathbb{Z}.$$

However,

$$y_1, y_2 \in (y_0 - \varepsilon, y_0 + \varepsilon) \implies y_1 - y_2 \in (-1, 1) \implies y_1 = y_2.$$

Thus, $\pi_{|U_\varepsilon}$ is a diffeomorphism. Its inverse is the map

$$g : \pi(U_\varepsilon) \to U_\varepsilon, \quad g(x, [y]) = (x, y),$$

and, for any $(x, [y]) \in U_\varepsilon$ and $(\alpha, \beta) \in T_{(x,[y])}\mathcal{C}_H \simeq \mathbb{R}^2$ we have that

$$(dg)_{(x,[y])}(\alpha, \beta) = \frac{d}{ds}_{|s=0} g(x + s\alpha, [y + s\beta]) = \frac{d}{ds}_{|s=0} (x + s\alpha, y + s\beta) = (\alpha, \beta),$$

implying that $(d\pi)_{(x_0,y_0)} = \mathrm{id}_{\mathbb{R}^2}$ for all $(x_0, y_0) \in S$.

Now, if $\{0\} \times \mathbb{Z}$ is invariant by the flow of $\mathcal{X}$ we have by Proposition 3.3 that $\mathcal{X}(x, y) = (0, (\mathrm{e}^x - 1)a)$ and consequently,

$$f_0(\pi(x, y)) = (d\pi)_{(x,y)}\mathcal{X}(x, y) = (0, (\mathrm{e}^x - 1)a) \implies f_0(z, w) = (0, (\mathrm{e}^z - 1)a), \; (z, w) \in \mathcal{C}_H.$$

On the other hand, if $Y = (\alpha, \beta) \in \mathfrak{g}$ we get

$$f_1(\pi(x, y)) = (d\pi)_{(x,y)} Y(x, y) = (\alpha, \mathrm{e}^x \beta) \implies f_1(z, w) = (\alpha, \mathrm{e}^z \beta),\ (z, w) \in \mathcal{C}_H.$$

As a consequence, the linear system on $\mathcal{C}_H$ has the form

$$\begin{cases} \dot{z} = u\alpha \\ \dot{w} = (\mathrm{e}^z - 1)a + u\mathrm{e}^z\beta \end{cases}, \quad u \in \Omega, (z, w) \in \mathcal{C}_H \qquad (\Sigma_{\mathcal{C}_H})$$

Moreover, the Lie algebra generated by f_0 and f_1 is given by

$$\mathcal{L}_{\mathcal{C}_H}(z, w) = \operatorname{span}\{(\alpha, \mathrm{e}^z\beta), (0, a\alpha\mathrm{e}^z)\},$$

implying that $\Sigma_{\mathcal{C}_H}$ satisfies the LARC if and only if $a\alpha \neq 0$.

3.2.2. $L = \mathbb{Z}c \times \{0\}$

For any $(x_1, y_1), (x_2, y_2) \in S$ we have that

$$(x_1, y_1) * (x_2, y_2)^{-1} \in \mathbb{Z}c \times \{0\} \iff \begin{cases} x_1 - x_2 \in \mathbb{Z}c \\ y_1 - \mathrm{e}^{x_1 - x_2} y_2 = 0 \end{cases}$$
$$\iff ([x_1], \mathrm{e}^{-x_1} y_1) = ([x_2], \mathrm{e}^{-x_2} y_2),$$

where $[y] = y + \mathbb{Z}c$. Therefore, $(\mathbb{Z}c \times \{0\}) \setminus S$ is the vertical cylinder on the y-axis $\mathcal{C}_V := \mathbb{R}/\mathbb{Z}c \times \mathbb{R}$. The canonical projection is given by

$$\pi : S \to \mathcal{C}_V, \quad (x, y) \mapsto ([x], \mathrm{e}^{-x} y).$$

As previously, for any $([x_0], y_0) \in \mathcal{C}_V$ and $\varepsilon \in (0, c/2)$ the canonical projection restricted to the open set $W_\varepsilon = (x_0 - \varepsilon, x_0 + \varepsilon) \times \mathbb{R}$ is a diffeomorphism with inverse given by

$$g : \pi(W_\varepsilon) \to W_\varepsilon, \quad g([x], y) = (x, \mathrm{e}^x y),$$

and, for any $([x], y) \in \pi(W_\varepsilon)$ and $(\alpha, \beta) \in T_{(x,[y])}\mathcal{C}_V \simeq \mathbb{R}^2$, it holds that

$$\begin{aligned}(dg)_{([x],y)}(\alpha, \beta) &= \frac{d}{ds}_{|s=0} g([x + s\alpha], y + s\beta) \\ &= \frac{d}{ds}_{|s=0} (x + s\alpha, \mathrm{e}^{x+s\alpha}(y + s\beta)) = (\alpha, \mathrm{e}^x(\alpha y + \beta)).\end{aligned}$$

It turns out that

$$(d\pi)_{(x_0,y_0)}(\alpha, \beta) = (\alpha, -\alpha y_0 + \mathrm{e}^{-x_0}\beta), \quad \forall (x_0, y_0) \in S, (\alpha, \beta) \in T_{(x_0,y_0)} S \simeq \mathbb{R}^2.$$

By Proposition 3.3, the invariance of $\mathbb{Z}c \times \{0\}$ by the flow of $\mathcal{X}$ gives rise to $\mathcal{X}(x, y) = (0, by)$. Hence,

$$f_0(\pi(x,y)) = (d\pi)_{(x,y)}\mathcal{X}(x,y) = (d\pi)_{(x,y)}(0,by) = (0, b\mathrm{e}^{-x}y)$$
$$\implies f_0(z,w) = (0,bw),\ (z,w) \in \mathcal{C}_V.$$

In the sequel, the description is the same as before. If $Y = (\alpha, \beta) \in \mathfrak{g}$ we obtain

$$f_1(\pi(x,y)) = (d\pi)_{(x,y)}Y(x,y) = (d\pi)_{(x,y)}(\alpha, \mathrm{e}^x\beta) = (\alpha, -\alpha y + \beta)$$
$$\implies f_1(z,w) = (\alpha, -\alpha w + \beta),\ (z,w) \in \mathcal{C}_V.$$

As a consequence, the linear system on $\mathcal{C}_V$ is given by

$$\begin{cases} \dot{z} = u\alpha \\ \dot{w} = bw + u(-\alpha w + \beta) \end{cases}, \quad u \in \Omega, (z,w) \in \mathcal{C}_V \tag{$\Sigma_{\mathcal{C}_V}$}$$

The Lie algebra generated by f_0 and f_1 is in this case given by

$$\mathcal{L}_{\mathcal{C}_V}(z,w) = \operatorname{span}\{(\alpha, -\alpha w + \beta), (0, b\beta)\},$$

which says that $\Sigma_{\mathcal{C}_V}$ satisfies the LARC if and only if $b\alpha\beta \neq 0$.

3.2.3. $L = \mathbb{R} \times \{0\}$

Using the calculations for the previous case, we have that

$$(x_1,y_1) * (x_2,y_2)^{-1} \in \mathbb{R} \times \{0\} \iff \begin{cases} x_1 - x_2 \in \mathbb{R} \\ y_1 - \mathrm{e}^{x_1 - x_2}y_2 = 0 \end{cases} \iff \mathrm{e}^{-x_1}y_1 = \mathrm{e}^{-x_2}y_2.$$

Therefore, $(\mathbb{R} \times \{0\}) \setminus S$ coincides with the real line $\mathbb{R}$ and, the canonical projection and its differential are given by

$$\pi : S \to \mathbb{R}, \quad (x,y) \mapsto \mathrm{e}^{-x}y \quad \text{and} \quad (d\pi)_{(x,y)}(\alpha,\beta) = -\alpha y + \beta\mathrm{e}^x.$$

By using Proposition 3.3, the invariance of $\mathbb{R} \times \{0\}$ by the flow of $\mathcal{X}$ implies that $\mathcal{X}(x,y) = (0, by)$ and hence

$$(d\pi)_{(x,y)}\mathcal{X}(x,y) = (d\pi_2)_{(x,y)}(0,by) = b\mathrm{e}^{-x}y = b\pi(x,y) \implies f_0(z) = bz,\ z \in \mathbb{R}.$$

Analogously,

$$(d\pi)_{(x,y)}Y(x,y) = (d\pi)_{(x,y)}(\alpha, \mathrm{e}^x\beta) = \mathrm{e}^{-x}\mathrm{e}^x\beta = \beta \implies f_1(z) = \beta,\ z \in \mathbb{R},$$

where $Y = (\alpha,\beta) \in \mathfrak{g}$. The linear system on $\mathbb{R}$ is given by

$$\dot{z} = bz + u\beta, \quad u \in \Omega, z \in \mathbb{R}, \tag{$\Sigma_{\mathbb{R}}$}$$

and the LARC is equivalent to $\beta \neq 0$.

3.2.4. $L = \mathbb{Z}c \times \mathbb{R}$

The set $\mathbb{Z}c \times \mathbb{R}$ is a normal subgroup of S, implying that the homogeneous space $(\mathbb{Z}c \times \mathbb{R}) \setminus S$ is a Lie group. A simple calculation shows that

$$(x_1, y_1)(x_2, y_2)^{-1} \in \mathbb{Z}c \times \mathbb{R} \iff x_1 - x_2 \in \mathbb{Z}c,$$

thus, $(\mathbb{Z}c \times \mathbb{R}) \setminus S$ coincides with $\mathbb{R}$ if $x_0 = 0$ and with the torus $\mathbb{T}_c = \mathbb{R}/\mathbb{Z}c$ if $x_0 \neq 0$. The canonical projection and its differential are given, respectively, by

$$\pi(x, y) = [x] \quad \text{and} \quad (d\pi)_{(x,y)}(\alpha, \beta) = \alpha.$$

Consequently,

$$f_0(z) = 0 \quad \text{and} \quad f_1(z) = \alpha,$$

and hence,

$$\dot{z} = u\alpha, \quad u \in \Omega, \ z \in \mathbb{T}_c \qquad (\Sigma_{\mathbb{T}_c})$$

where for simplicity we define $\mathbb{T}_0 = \mathbb{R}$. For the system $\Sigma_{\mathbb{T}_c}$ the LARC is equivalent to $\alpha \neq 0$.

4. Controllability and control sets

In this section we completely classify controllability, and the control sets of LCSs on the homogeneous spaces of S satisfying the LARC.

By the previous sections, any LCS on a homogeneous space $L \setminus S$ of S is equivalent to one of the control-affine systems

$$\Sigma_{\mathcal{C}_H}, \ \Sigma_{\mathcal{C}_V}, \ \Sigma_{\mathbb{R}} \ \text{or} \ \Sigma_{\mathbb{T}_c},$$

obtained in the previous section. In what follows, we will analyze the four systems above in the following subsections.

By far, the most important result is given by Theorem 4.5. There, we show the face of two possible control sets with nonempty interior of a LCS on a homogeneous space.

4.1. The case $\Sigma_{\mathbb{T}_c}$

In this subsection, we characterize the controllability property of the corresponding control-affine system as follows.

4.1 Theorem. *The system* $\Sigma_{\mathbb{T}_c}$ *is controllable if and only if it satisfies the LARC.*

Proof. In this case, the solutions of $\Sigma_{\mathbb{T}_c}$ are given as concatenations of the curves

$$\phi(t, z_0, u) = [z_0 + u\alpha t].$$

Since by the LARC we have that $\alpha \neq 0$, it holds that:

1. If $c \neq 0$ we have that $\phi((0,+\infty), z_0, u) = \mathbb{T}_c$ and consequently the system is controllable.
2. If $c = 0$, let us consider $z_1, z_2 \in \mathbb{R}$ and assume w.l.o.g. that $z_1 < z_2$. In this case, any

$$u_1, u_2 \in \Omega; \quad \text{with} \quad u_1\alpha > 0 \quad \text{and} \quad u_2\alpha < 0 \quad \Longrightarrow \quad t_1 = \frac{z_2 - z_1}{u_1\alpha} > 0 \text{ and } t_2 = \frac{z_1 - z_2}{u_2\alpha} > 0.$$

Moreover,

$$\phi(t_1, z_1, u_1) = z_1 + u_1\alpha t_1 = z_2, \quad \text{and} \quad \phi(t_2, z_2, u_2) = z_2 + u_2\alpha t_2 = z_1,$$

which shows the controllability of $\Sigma_{\mathbb{T}_0}$ and concludes the proof. □

4.2. *The case* $\Sigma_{\mathbb{R}}$

Since $\Sigma_{\mathbb{R}}$ coincides with a linear system on $\mathbb{R}$, the general theory of LCSs on Euclidean spaces already gives us the controllability properties of $\Sigma_{\mathbb{R}}$ (see, for instance, [7, Section 3]). For completeness purposes, we present the proof also for this case.

4.2 Theorem. *If the LCS $\Sigma_{\mathbb{R}}$ satisfies the LARC it holds that:*

1. *$b = 0$ and $\Sigma_{\mathbb{R}}$ is controllable or*
2. *$b \neq 0$ and $\Sigma_{\mathbb{R}}$ admits exactly one control set given by*

$$-\frac{\alpha}{b}\Omega, \;\; \textit{if } b < 0 \qquad\qquad \textit{or} \qquad\qquad -\frac{\alpha}{b}\operatorname{int}(\Omega), \;\; \textit{if } b > 0.$$

Proof. Here we just prove the case where $b \neq 0$ since $b = 0$ was already proved in Theorem 4.1. Moreover, let us consider only the case $b < 0$ since the positive case is analogous.

Under the previous assumptions, the solutions of $\Sigma_{\mathbb{R}}$ are builded by concatenations of the curves

$$\phi(t, z_0, u) = \mathrm{e}^{tb}\left(z_0 + \frac{\alpha}{b}u\right) - \frac{\alpha}{b}u, \;\; u \in \Omega.$$

Assume $\alpha > 0$, since the case $\alpha < 0$ is analogous. In this case,

$$-\frac{\alpha}{b}\Omega = \left[-\frac{\alpha}{b}u_*, -\frac{\alpha}{b}u^*\right],$$

and, for any $z_0 \in -\frac{\alpha}{b}\Omega$ we have that

$$\begin{aligned}\phi(t, z_0, u) + \frac{\alpha}{b}u_* &= \mathrm{e}^{tb}\left(z_0 + \frac{\alpha}{b}u\right) - \frac{\alpha}{b}u + \frac{\alpha}{b}u_* \geq \mathrm{e}^{tb}\left(-\frac{\alpha}{b}u_* + \frac{\alpha}{b}u\right) - \frac{\alpha}{b}u + \frac{\alpha}{b}u_* \\ &= \underbrace{(\mathrm{e}^{tb} - 1)}_{<0}\underbrace{\left(\frac{\alpha}{b}u - \frac{\alpha}{b}u_*\right)}_{\leq 0} \geq 0\end{aligned}$$

and

$$\phi(t,z_0,u)+\frac{\alpha}{b}u^*=\mathrm{e}^{tb}\left(z_0+\frac{\alpha}{b}u\right)-\frac{\alpha}{b}u+\frac{\alpha}{b}u^*\leq \mathrm{e}^{tb}\left(-\frac{\alpha}{b}u^*+\frac{\alpha}{b}u\right)-\frac{\alpha}{b}u+\frac{\alpha}{b}u^*$$
$$=\underbrace{(\mathrm{e}^{tb}-1)}_{<0}\underbrace{\left(\frac{\alpha}{b}u-\frac{\alpha}{b}u^*\right)}_{\geq 0}\leq 0$$

showing that

$$\forall t\geq 0, u\in\Omega,\ \ \phi\left(t,-\frac{\alpha}{b}\Omega,u\right)\subset -\frac{\alpha}{b}\Omega \quad \text{which implies} \quad \mathcal{O}^+(z_0)\subset -\frac{\alpha}{b}\Omega, \quad \forall z_0\in -\frac{\alpha}{b}\Omega.$$

Let us consider $z_0, z_1 \in -\frac{\alpha}{b}\operatorname{int}(\Omega)$ and assume w.l.o.g. that $z_0 < z_1$. Since

$$\phi(t,z_0,u^*)=\mathrm{e}^{bt}\left(z_0+\frac{\alpha}{b}u^*\right)-\frac{\alpha}{b}u^*\to -\frac{\alpha}{b}u^* \ \text{ as } \ t\to+\infty,$$

there exists $t_0 > 0$ such that $\phi(t_0, z_0, u^*) = z_1$. Analogously,

$$\phi(t,z_1,u_*)=\mathrm{e}^{bt}\left(z_1+\frac{\alpha}{b}u_*\right)-\frac{\alpha}{b}u_*\to -\frac{\alpha}{b}u_* \ \text{ as } \ t\to+\infty,$$

assuring the existence of $t_1 > 0$ such that $\phi(t_1, z_1, u_*) = z_0$. Consequently,

$$\forall z_0\in -\frac{\alpha}{b}\operatorname{int}\Omega,\ \ \mathcal{O}^+(z_0)=-\frac{\alpha}{b}\operatorname{int}\Omega \ \text{ and by continuity } \ \mathcal{O}^+(z_0)=-\frac{\alpha}{b}\Omega,\ \ \forall z_0\in -\frac{\alpha}{b}\Omega,$$

showing that $-\frac{\alpha}{b}\Omega$ is a control set of $\Sigma_{\mathbb{R}}$. Let us show that $\Sigma_{\mathbb{R}}$ does not admit control sets in $\left(-\infty, -\frac{\alpha}{b}u_*\right)$. For any $z_0 \in \mathbb{R} \setminus -\frac{\alpha}{b}\Omega$, we have that

$$\phi(t,z_0,u)-z_0=\mathrm{e}^{tb}\left(z_0+\frac{\alpha}{b}u\right)-z_0=(\mathrm{e}^{tb}-1)\left(z_0-\frac{\alpha}{b}u\right),$$

which implies,

$$\phi(t,z_0,u)>z_0, \quad \text{if } z_0<-\frac{\alpha}{b}u_* \quad \text{and} \quad \phi(t,z_0,u)<z_0, \quad \text{if } z_0>-\frac{\alpha}{b}u_*,$$

and hence,

$$\mathcal{O}^+(z_0)\setminus\{z_0\}\subset(z_0+\infty) \ \text{ if } z_0<-\frac{\alpha}{b}u_* \quad \text{and} \quad \mathcal{O}^+(z_0)\setminus\{z_0\}\subset(-\infty,z_0) \ \text{ if } z_0>-\frac{\alpha}{b}u_*.$$

Consequently, if $z_0, z_1 \leq -\frac{\alpha}{b}u_*$ with $z_0 < z_1$, there are no trajectories of $\Sigma_{\mathbb{R}}$ starting at z_1 and approaching arbitrarily the point z_0, that is, any control set of $\Sigma_{\mathbb{R}}$ contained in $\left(-\infty, -\frac{\alpha}{b}u_*\right)$ cannot have two distinct points, or it would fail condition 2. in the definition of control sets. On the other hand, if $\{z_0\}$ is a control set of $\Sigma_{\mathbb{R}}$ contained in $\left(-\infty, -\frac{\alpha}{b}u_*\right)$, by the first condition in the definition of control sets, there exists $u \in \Omega$ such that

$$\forall t\in\mathbb{R},\ \ \phi(t,z_0,u)=z_0 \iff (\mathrm{e}^{tb}-1)\left(z_0+\frac{\alpha}{b}u\right)=0,$$

which is certainly not possible. Therefore, $\Sigma_{\mathbb{R}}$ does not admit control sets in $\left(-\infty, -\frac{\alpha}{b}u_*\right)$. Analogously $\Sigma_{\mathbb{R}}$ does not admit control sets in $\left(-\frac{\alpha}{b}u^*, +\infty\right)$ assuring the uniqueness of $-\frac{\alpha}{b}\Omega$ and concluding the proof. □

4.3. The case $\Sigma_{\mathcal{C}_H}$

This case allows characterizing the controllability property of the affine system through the LARC as the next result shows.

4.3 Theorem. *The system $\Sigma_{\mathcal{C}_H}$ is controllable if and only if it satisfies the LARC.*

Proof. Since $\Sigma_{\mathcal{C}_H}$ satisfies the LARC, we have that $a\alpha \neq 0$. Let us first consider $P_0 = (z_0, [w_0])$, $P_1 = (z_1, [w_1]) \in \mathcal{C}_H$ with $z_1 \neq 0$ and construct a trajectory of $\Sigma_{\mathcal{C}_H}$ joining P_0 and P_1 in positive time as follows:

1. If $z_1 = z_0 \neq 0$ consider $u = 0$. The solution starting at P_0 associated with the control $u = 0$ is given by

$$\phi(t, P_0, 0) = (z_0, [t(\mathrm{e}^{z_0} - 1)a]).$$

   Since $az_0 \neq 0$ we have that $\{[t(\mathrm{e}^{z_0} - 1)a],\ t \in [0, +\infty)\} = \mathbb{R}/\mathbb{Z}$ and hence, there exists $t_1 \geq 0$ such that $\phi(t_1, P_0, 0) = P_1$;
2. If $z_1 \neq z_0$ we can assume w.l.o.g. that $z_0 < z_1$. Since the first coordinate of the solution of $\Sigma_{\mathcal{C}_H}$ reads as

$$\phi_1(t, P_0, u) = z_0 + u\alpha t,$$

   if we consider $u_0 \in \Omega$ such that $u_0\alpha > 0$, there exists $t_0 > 0$ such that $\phi_1(t_0, P_0, u_0) = z_1$. Therefore, we go from P_0 to a point P_1' where the first coordinate of P_1' is equal to the first component of P_1. Since such a component is $z_1 \neq 0$, by concatenating with a curve constructed as in the first step we obtain a solution of $\Sigma_{\mathcal{C}_H}$ joining P_0 and P_1 as stated.

Arguing similarly, we can prove that the previous analysis is also true for negative time, that is, any two points $P_0, P_1 \in \mathcal{C}_H$ with $z_1 \neq 0$ can be joined by a solution of $\Sigma_{\mathcal{C}_H}$ in negative time. Therefore, if $z_1 = 0$, we can consider a point P_1' with nonzero first coordinate and connected P_0 to P_1' in positive time and P_1 to P_1' in negative time. Reversing the time in the second curve allows us to obtain a solution of $\Sigma_{\mathcal{C}_H}$ connecting P_0 to P_1 as desired. □

4.4. The case $\Sigma_{\mathcal{C}_V}$

A simple calculation shows that the solutions $\phi = (\phi_1, \phi_2)$ of $\Sigma_{\mathcal{C}_V}$ for constant controls $u \in \Omega$ are given by

$$\phi_1(t, [z_0], b\alpha^{-1}) = [z_0 + bt] \quad \text{and} \quad \phi_2(t, w_0, b\alpha^{-1}) = w_0 + b\beta\alpha^{-1}t,$$

if $b - u\alpha = 0$ and

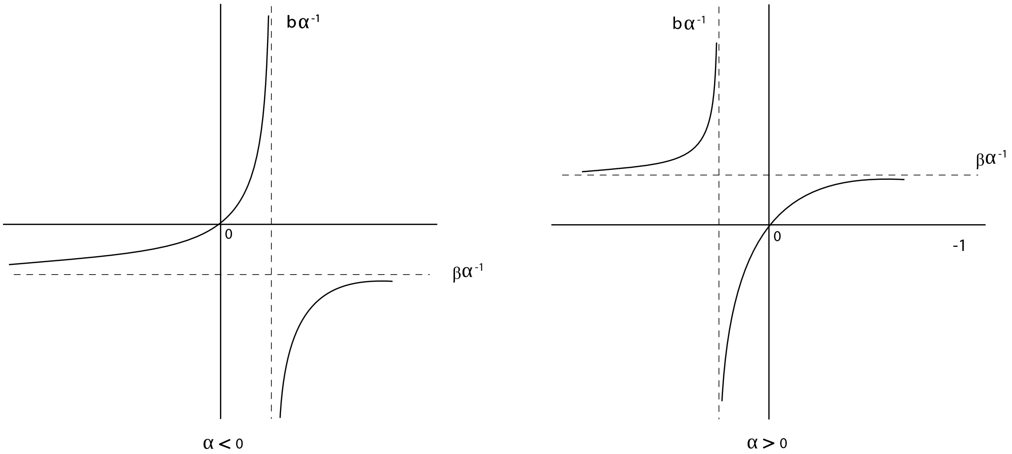

Fig. 2. Graphics of the function m.

$$\phi_1(t,[z_0],u)=[z_0+u\alpha t] \quad \text{and} \quad \phi_2(t,w_0,u)=\mathrm{e}^{t(b-u\alpha)}\left(w_0-m(u)\right)+m(u),$$

if $b-u\alpha\neq 0$, where

$$m:\mathbb{R}\setminus\{b\alpha^{-1}\}\to\mathbb{R} \quad u\mapsto \frac{u\beta}{u\alpha-b}.$$

The sign of b and β does not change the results to come, so we will assume w.l.o.g. that $b<0$ and $\beta>0$. Define the sets

$$B^+:=\{u\in\Omega;\ u\alpha-b>0\} \quad \text{and} \quad B^-:=\{u\in\Omega;\ u\alpha-b<0\},$$

and notice that $\Omega\setminus\{b\alpha^{-1}\}=B^+\cup B^-$. Since we are assuming $b<0$ we have that $0\in B^+$ and so $B^+\neq\emptyset$.

Next, we state the main properties of the function m defined previously (see Fig. 2):

1. $m'(u)=\frac{-b\beta}{(u\alpha-b)^2}>0$;
2. $\lim_{u\to\pm\infty}m(u)=\beta\alpha^{-1}$;
3. $\lim_{u\to(b\alpha^{-1})^-}m(u)=+\infty$ and $\lim_{u\to(b\alpha^{-1})^+}m(u)=-\infty$;
4. $m(v_1)>\beta\alpha^{-1}>m(v_2)$ for any $v_1\in(-\infty,b\alpha^{-1})$ and $v_2\in(b\alpha^{-1},+\infty)$;
5. m changes sign on B^+ and has constant sign in B^-.

We shall now describe a central lemma in the proof of our main result for the control sets of $\Sigma_{\mathcal{C}_V}$.

4.4 Lemma. *For all $t\geq 0$ and $u\in\Omega$, it holds that*

$$\phi_2(\pm t,m(B^\pm),u)\subset m(B^\pm).$$

Proof. Since both cases are analogous, we only show the relation for B^+.

Assume first that $b \notin \alpha\Omega$. In this case $B^+ = \Omega$, $B^- = \emptyset$ and

$$m(B^+) = m(\Omega) = [m(u_*), m(u^*)] \implies \forall u \in \Omega, \; m(u_*) \leq m(u) \leq m(u^*). \tag{2}$$

If $v, u \in \Omega = B^+$ we have that

$$\phi_2(t, m(v), u) - m(u^*) = \mathrm{e}^{t(b-u\alpha)}(m(v) - m(u)) + m(u) - m(u^*)$$
$$\overset{(2)}{\leq} \mathrm{e}^{t(b-u\alpha)}(m(u^*) - m(u)) + m(u) - m(u^*) = \underbrace{(\mathrm{e}^{t(b-u\alpha)} - 1)}_{\leq 0,\, u \in B^+} \underbrace{(m(u^*) - m(u))}_{\geq 0} \leq 0,$$

implying that $\phi_2(t, m(v), u) \leq m(u^*)$. Analogously,

$$\phi_2(t, m(v), u) - m(u_*) = \mathrm{e}^{t(b-u\alpha)}(-m(v) - m(u)) + m(u) - m(u_*)$$
$$\overset{(2)}{\geq} \mathrm{e}^{t(b-u\alpha)}(m(u_*) - m(u)) + m(u) - m(u_*) = \underbrace{(\mathrm{e}^{t(b-u\alpha)} - 1)}_{\leq 0,\, u \in B^+} \underbrace{(m(u_*) - m(u))}_{\leq 0} \geq 0,$$

implying that $\phi_2(t, m(v), u) \geq m(u_*)$ and hence

$$m(u_*) \leq \phi_2(t, m(v), u) \leq m(u^*) \implies \forall t \geq 0, u \in \Omega, \; \phi_2(t, m(B^+), u) \subset m(B^+).$$

Now, if $b \in \alpha\Omega$, it turns out that $u_* \in B^+$ or $u^* \in B^+$. We consider the case $u_* \in B^+$, since the case $u^* \in B^+$ is analogous. Using the fact that $u \mapsto b - u\alpha$ is a line, and u_* and 0 belongs to B^+ by assumption, we get that $b\alpha^{-1} > 0$. Therefore,

$$u \to (b\alpha^{-1})^{\mp} \implies m(u) \to \pm\infty$$

implying that $m(B^+) = [m(u_*), +\infty)$. Also, since m is strictly increasing we have that

$$\forall v \in B^+, \; m(u_*) \leq m(v) \quad \text{and} \quad \forall u \in B^- \quad m(u_*) > \beta\alpha^{-1} > m(u). \tag{3}$$

Consider now $v \in B^+$, $u \in \Omega$ and $t \geq 0$, and let us analyze the following possibilities:

- $u \in B^+$: Analogously as in the case $B^+ = \Omega$, we have that

$$\phi_2(t, m(v), u) - m(u_*) = \mathrm{e}^{t(b-u\alpha)}(m(v) - m(u)) + m(u) - m(u_*)$$
$$\overset{(3)}{\geq} \mathrm{e}^{t(b-u\alpha)}(m(u_*) - m(u)) + m(u) - m(u_*) = \underbrace{(\mathrm{e}^{t(b-u\alpha)} - 1)}_{\leq 0, u \in B^+} \underbrace{(m(u_*) - m(u))}_{\leq 0} \geq 0,$$

implying that $\phi_2(t, m(v), u) \geq m(u_*)$ or, equivalently, $\phi_2(t, m(v), u) \in m(B^+)$.

- $u = b\alpha^{-1}$: In this case,

$$\phi(t, m(v), b\alpha^{-1}) = m(v) + \underbrace{b\alpha^{-1}\beta}_{>0} t > m(v) \geq m(u_*) \implies \varphi_2(t, m(v), b\alpha^{-1}) \in m(B^+).$$

- $u \in B^-$: In this case, $b - u\alpha > 0$ and we have that

$$\phi_2(t, m(v), u) - m(u_*) = \mathrm{e}^{t(b-u\alpha)}(m(v) - m(u)) + m(u) - m(u_*)$$

$$\overset{(3)}{\geq} \mathrm{e}^{t(b-u\alpha)}(m(u_*) - m(u)) + m(u) - m(u_*) = \underbrace{(\mathrm{e}^{t(b-u\alpha)} - 1)}_{\geq 0, u \in B^-} \underbrace{(m(u_*) - m(u))}_{\geq 0, (3)} \geq 0,$$

which determines $\phi_2(t, m(v), u) \leq m(u_*)$, and again we obtain $\phi_2(t, m(v), u) \in m(B^+)$. Since $v \in B^+$, $u \in \Omega$ and $t \geq 0$ were arbitrary, we get that

$$\phi_2(t, m(B^+), u) \subset m(B^+), \quad \forall t \geq 0, u \in \Omega,$$

showing that $\phi_2(t, m(B^+), u) \subset m(B^+)$ for all $t \geq 0$, concluding the proof. $\square$

We can now prove the main result of this section.

4.5 Theorem. *If $\Sigma_{\mathcal{C}_V}$ satisfies the LARC and $b < 0$, the sets*

$$D^+ = \mathbb{R}/\mathbb{Z}c \times m(B^+) \quad \textit{and} \quad D^- = \mathrm{int}\Big(\mathbb{R}/\mathbb{Z}c \times m(B^-)\Big),$$

are the only possible control sets of $\Sigma_{\mathcal{C}_V}$.

Proof. Let us show that D^+ is a control set.

By Lemma 4.4, for all $t \geq 0$ and $u \in \Omega$,

$$\phi(t, \mathbb{R}/\mathbb{Z}c \times m(B^+), u) = \phi_1(t, \mathbb{R}/\mathbb{Z}c, u) \times \phi_2(t, m(B^+), u) \subset \mathbb{R}/\mathbb{Z}c \times m(B^+) = D^+,$$

showing that D^+ is invariant in positive time. Let us show that controllability holds on $\mathrm{int}\, D^+$ that is, for any $P_1, P_2 \in \mathrm{int}\, D^+$ there exists a trajectory of the system $\Sigma_{\mathcal{C}_V}$ connecting P_1 to P_2 in positive time.

Since $P_1, P_2 \in \mathrm{int}\, D^+$ we can write

$$P_1 = ([z_1], m(u_1)) \quad \text{and} \quad P_2 = ([z_2], m(u_2)), \quad \text{with } u_1, u_2 \in B^+ \setminus \{u_*, u^*\}.$$

A trajectory connecting P_1 and P_2 can be constructed using the following procedures:

1. If $m(u_1) = m(u_2) \neq 0$ we have that $u_1 = u_2 \neq 0$. Therefore,

$$\phi_1(t, [z_1], u_1) = [z_1 + u\alpha t] \quad \text{and} \quad \phi_2(t, m(u_1), u_1) = m(u_1),$$

   and in this case $\phi_1((0, +\infty), [z_1], u_1) = \mathbb{R}/\mathbb{Z}c$, implying that

$$\phi(t_0, P_1, u_1) = P_2, \quad \text{for some} \quad t_0 > 0.$$

2. If $m(u_1) \neq m(u_2)$ with $m(u_2) \neq 0$, the fact that $u_2 \in B^+ \setminus \{u_*, u^*\}$ implies the existence of $u_1' \in B^+$ with $m(u_1') > m(u_2)$. Then,
$$\phi_2(t, m(u_1), u_1') \to m(u_1'), \;\; t \to +\infty,$$
and by continuity, there exists $t_1 > 0$ such that $\phi_2(t_1, m(u_1), u_1') = m(u_2)$. Consequently, we go from P_1 to $P_2' = ([z_2'], m(u_2))$. Since $m(u_2) \neq 0$ we can apply the first step to obtain a solution of $\Sigma_{\mathcal{C}_V}$ starting at P_2' and passing through P_2 and by concatenation, we obtain a solution starting at P_1 and passing through P_2;
3. If $m(u_1) = m(u_2) = 0$, for any $0 \neq u_0 \in B^+$ there is, by continuity, $t_0 > 0$ such that
$$P_2'' = \phi(-t_0, P_2, u_0) \in \operatorname{int} D^+ \quad \text{and} \; \varphi_2(t_0, 0, u_0) \neq 0.$$
In particular, $\phi(t_0, P_2'', u_0) = P_2$ and $P_2'' = ([z_2''], m(u_2''))$ with $m(u_2'') \neq 0$.
4. If $m(u_1) \neq m(u_2)$ with $m(u_2) = 0$, we can consider by step 3 a point $P_2'' = ([z_2''], m(u_2''))$ with $m(u_2'') \neq 0$ and $t_0 > 0$, $u_0 \in \operatorname{int} B^+$ with $\phi(t_0, P_2'', u_0) = P_2$. Now, using step 1 if $m(u_1) = m(u_2'')$ or step 2 if $m(u_1) \neq m(u_2'')$, we can construct a trajectory from P_1 to P_2''. As previously, by concatenation we obtain a solution starting at P_1 and passing through P_2 in positive time.

Since P_1, P_2 were arbitrary points we conclude that controllability holds in $\operatorname{int} D^+$. In particular, we get that
$$\operatorname{int} D^+ = \operatorname{int}\Big(\mathbb{R}/\mathbb{Z}c \times m(B^+)\Big) = \mathcal{O}^+(P), \quad \forall P \in \operatorname{int} D^+,$$
which implies by continuity that $D^+ = \overline{\mathcal{O}^+(P)}$ for all $P \in D^+$. Since maximality is trivial in this case, we conclude that D^+ is a control set.

Analogously, by considering negative time we can show that $D^- = \mathcal{O}^-(P)$ for all $P \in D^-$, implying that D^- is also a control set (see Fig. 3).

Let us now show that D^+ and D^- are the only possible control sets of $\Sigma_{\mathcal{C}_V}$ by analyzing the possible cases:

• $b \notin \alpha\Omega$: In this case we have that $B^+ = \Omega$ and $B^- = \emptyset$ implying that
$$D^+ = \mathbb{R}/\mathbb{Z}c \times [m(u_*), m(u^*)].$$
Consequently, for any $w < m(u_*)$, $u \in \Omega$ and $t > 0$ we have that
$$\phi_2(t, w, u) - w = \underbrace{(\mathrm{e}^{t(b-u\alpha)} - 1)}_{<0,\, u\in B^+} \underbrace{(w - m(u))}_{w<m(u_*)\leq m(u)} > 0,$$
implying that
$$\mathcal{O}^+([z], w) \setminus \{([z], w)\} \subset \mathbb{R}/\mathbb{Z}c \times (w, +\infty).$$

As a consequence, if $P_1 = ([z_1], w_1)$, $P_2 = ([z_2], w_2) \in \mathcal{C}_V$ satisfies $w_1 < w_2 < m(u_*)$, there are no trajectories of $\Sigma_{\mathcal{C}_V}$ starting at P_1 and coming arbitrarily close to P_2, implying that any possible control set intersecting $\mathbb{R}/\mathbb{Z}c \times (-\infty, m(u_*))$ has only one point. However, if $\{P\}$ is a control set, there exists $u \in \Omega$ such that

$$\phi(t, P, u) = P, \quad \forall t \in \mathbb{R},$$

which only happens for $u = 0$ and $P \in \mathbb{R}/\mathbb{Z}c \times \{0\} \subset D^+$. Therefore, $\Sigma_{\mathcal{C}_V}$ does not admit control sets in $\mathbb{R}/\mathbb{Z}c \times (-\infty, m(u_*))$.

Analogously we show that $\Sigma_{\mathcal{C}_V}$ does not admit control sets on $\mathbb{R}/\mathbb{Z}c \times (m(u^*), +\infty)$, implying that D^+ is the only control set of $\Sigma_{\mathcal{C}_V}$ in this case.

• $b \in \partial(\alpha\Omega)$: Let us assume that $u^* = b\alpha^{-1}$. In this case $B^+ = [u_*, u^*)$ and $B^- = \emptyset$, implying that

$$D^+ = \mathbb{R}/\mathbb{Z}c \times [m(u_*), +\infty).$$

Analogously as the previous case, any solutions starting at $P = ([z], w)$ with $w < m(u_*)$ cannot "go down" the cylinder if $u \in B^+$ and we only have to analyze the case $u^* = b\alpha^{-1}$. In this case,

$$\phi_2(t, w, b\alpha^{-1}) = w + \underbrace{b\alpha^{-1}\beta}_{>0} t > w,$$

which implies that

$$\mathcal{O}^+([z], w) \setminus \{([z], w)\} \subset \mathbb{R}/\mathbb{Z}c \times (w, +\infty).$$

Arguing as in the last case, we are able to conclude that D^+ is still the only control set of $\Sigma_{\mathcal{C}_V}$.

• $b \in \mathrm{int}(\alpha\Omega)$: Let us assume that $u_* \in B^+$ and $u^* \in B^-$. For any $m(u_*) > w > m(u^*)$ and $u \in \Omega$ we have as previously that

$$\phi_2(t, w, u) > w, \ \ \forall t > 0 \ \text{ if } \ u \in \overline{B^+}.$$

On the other hand, if $u \in B^-$

$$\phi_2(t, w, u) - w = \underbrace{(\mathrm{e}^{t(b-u\alpha)} - 1)}_{>0,\, u \in B^-} \underbrace{(w - m(u))}_{w > m(u^*) \geq m(u)} > 0,$$

showing also that

$$\phi_2(t, w, u) > w, \ \ \forall t > 0, u \in B^+.$$

As previously, we obtain that

$$\mathcal{O}^+([z], w) \setminus \{([z], w)\} \subset \mathbb{R}/\mathbb{Z}c \times (w, +\infty),$$

which implies that $\Sigma_{\mathcal{C}_V}$ admits no control sets in $\mathcal{C}_V \setminus (D^+ \cup D^-)$, concluding the proof. □

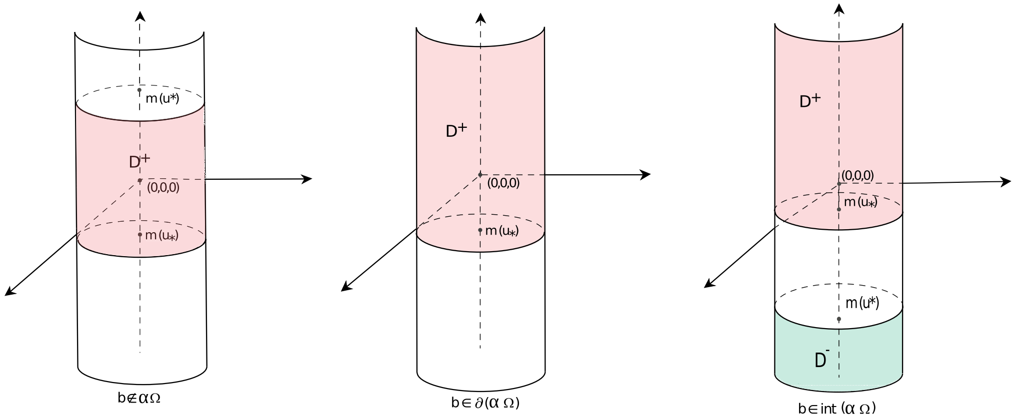

Fig. 3. The control sets of $\Sigma_{\mathcal{C}_V}$.

4.6 Remark. Let us mention an important consequence of Theorem 4.5. On low dimensional groups, the properties of control sets for LCS on Lie groups and homogeneous spaces are quite different. For instance, under the LARC, one can find LCSs on a homogeneous space of S that admits bounded control sets with nonempty interior and LCSs with two control sets with nonempty interior. In comparison, any LCS on S that satisfies the LARC admits exactly one unbounded control set (see [3, Theorem 3.6]).

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