General Barnes–Godunova–Levin type inequalities for Sugeno integral

Hamzeh Agahi a,b, H. Román-Flores c,*, A. Flores-Franulíč c

a Department of Statistics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, 424, Hafez Ave., Tehran 15914, Iran
b Statistical Research and Training Center, Tehran, Iran
c Instituto de Alta Investigación, Universidad de Tarapacá, Casilla 7D, Arica, Chile

ABSTRACT

Integral inequalities play important roles in classical probability and measure theory. Non-additive measure is a generalization of additive probability measure. Sugeno’s integral is a useful tool in several theoretical and applied statistics which has been built on non-additive measure. For instance, in decision theory, the Sugeno integral is a median, which is indeed a qualitative counterpart to the averaging operation underlying expected utility. In this paper, Barnes–Godunova–Levin type inequalities for the Sugeno integral on abstract spaces are studied in a rather general form and, for this, we introduce some new technics for the treatment of concave functions in the Sugeno integration context. Also, several examples are given to illustrate the validity of this inequality. Moreover, a strengthened version of Barnes–Godunova–Levin type inequality for Sugeno integrals on a real interval based on a binary operation ⋆ is presented.

1. Introduction

In 1974, Sugeno [35] initiated research on non-additive measures and integrals. Non-additive measures and integrals can be used for modelling problems in nondeterministic environment. Sugeno’s integral is a useful tool in several theoretical and applied statistics. For instance, in decision theory, the Sugeno integral is a median, which is indeed a qualitative counterpart to the averaging operation underlying expected utility. The use of the Sugeno integral can be envisaged from two points of view: decision under uncertainty and multi-criteria decision-making [13]. Sugeno’s integral is analogous to Lebesgue integral which has been studied by many authors, including Pap [26], Railesuc and Adams [30] and, Wang and Klir [36], among others. The study of inequalities for Sugeno integral was initiated by Román-Flores et al. [14,31–34], and then followed by the authors [1,3–10,15,19–21,37].

In [33], Román-Flores, Flores-Franulíč and Chalco-Cano studied some properties of Sugeno integral for monotone functions, they also provided some Young type inequalities. Based on these results, Flores-Franulíč and Román-Flores [14] provided a Chebyshev type inequality for Sugeno integral of continuous and strictly monotone functions based on Lebesgue measure. Some other classical inequalities have also been generalized to Sugeno integral by them (see, for example [32,34]). Later on, Ouyang and Fang [20] generalized the main results of [33] to the case of monotone functions. Based on these results, Ouyang et al. further generalized the Sugeno Chebyshev type inequalities [14] to the case of arbitrary non-additive measure-based Sugeno integral [19,21–24]. Recently, Agahi and Yaghoobi [1] proved a Minkowski type inequality for monotone functions and arbitrary non-additive measure-based Sugeno integral on real line, and then Agahi et al. [3,4]...
further generalized it to comonotone functions and arbitrary non-additive measure-base Sugeno integral on an arbitrary measurable space. Recently, the study of integral inequalities has been extended to a more general class of pseudo-integral operators (see [2,11,27]).

Any integral inequality can be a very strong tool for applications. In particular, when we think of an integral operator as a predictive tool then an integral inequality can be very important in measuring and dimensioning such process. The classical Barnes–Godunova–Levin inequality [28,29] is one of the famous mathematical inequalities for concave functions. This inequality is an important tool for modern analysis. In general, due to the behavior of the Sugeno integral, the concept of concavity (or convexity) is not frequently used in the Sugeno integration context. Moreover, the most of the integral inequalities studied in the Sugeno integration context normally consider conditions such as monotonicity or comonotonicity. The aim of this paper is to study some general Barnes–Godunova–Levin type inequalities for Sugeno integrals of concave functions. We think that our results will be useful for those areas in which the classical Barnes–Godunova–Levin inequality plays a role whenever the environment is non-deterministic.

The paper is organized as follows. Some necessary preliminaries are presented in Section 2. We address the essential problems in Section 3. In Section 4, we construct a strengthened version of Barnes–Godunova–Levin type inequality for Sugeno integrals on abstract spaces. Finally, a conclusion is given in Section 5.

2. Preliminaries

In this section, we are going to review some well known results from the theory of non-additive measures, Sugeno’s integral. For details, we refer to [30,35,36].

As usual we denote by $\mathbb{R}$ the set of real numbers. Let $X$ be a non-empty set, $\Sigma$ be a $\sigma$-algebra of subsets of $X$. Let $\mathbb{N}$ denote the set of all positive integers and $\mathbb{R}_+$ denote $[0, +\infty]$. Throughout this paper, we fix the measurable space $(X, \Sigma)$, and all considered subsets are supposed to belong to $\Sigma$.

**Definition 1** (Ralescu and Adams [30]). A set function $\mu : \Sigma \to \mathbb{R}_+$ is called a non-additive measure if the following properties are satisfied:

1. (FM1) $\mu(\emptyset) = 0$;
2. (FM2) $A \subset B$ implies $\mu(A) \leq \mu(B)$;
3. (FM3) $A_1 \subset A_2 \subset \cdots$ implies $\mu(\bigcup_{n=1}^\infty A_n) = \lim_{n \to \infty} \mu(A_n)$; and
4. (FM4) $A_1 \supset A_2 \supset \cdots$, and $\mu(A_1) < +\infty$ imply $\mu(\bigcap_{n=1}^\infty A_n) = \lim_{n \to \infty} \mu(A_n)$.

When $\mu$ is a non-additive measure, the triple $(X, \Sigma, \mu)$ then is called a non-additive measure space.

Let $(X, \Sigma, \mu)$ be a non-additive measure space, by $\mathcal{F}^\mu(X)$ we denote the set of all nonnegative measurable functions $f : X \to [0, \infty)$ with respect to $\mu$. In what follows, all considered functions belong to $\mathcal{F}^\mu(X)$. If $f \in \mathcal{F}^\mu(X)$, we will denote the set $\{x \in X | f(x) \geq \alpha\}$ by $F_x$ for $\alpha \geq 0$. Clearly, $F_x$ is nonincreasing with respect to $\alpha$, i.e., $\alpha \leq \beta$ implies $F_x \supseteq F_\beta$.

**Definition 2** (Pap [26], Sugeno [35], Wang and Klir [36]). Let $(X, \Sigma, \mu)$ be a non-additive measure space and $A \in \Sigma$. If $f \in \mathcal{F}^\mu(X)$ then the Sugeno integral of $f$ on $A$, with respect to the non-additive measure $\mu$, is defined as

$$
\int_A f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)).
$$

When $A = X$, then

$$
\int_X f d\mu = \int d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(F_\alpha)).
$$

It is well known that Sugeno integral is a type of nonlinear integral [18], i.e., for general case,

$$
\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu
$$

does not hold. Some basic properties of Sugeno integral are summarized in [26,36], we cite some of them in the next theorem.

**Theorem 1** (Pap [26], Wang and Klir [36]). Let $(X, \Sigma, \mu)$ be a non-additive measure space, then

1. $\int_A f d\mu \leq \mu(A)$;
2. $\int_A k d\mu = k \wedge \mu(A)$ for all nonnegative constant $k$ and $A \in \Sigma$;
3. $\mu(A \cap F_\alpha) \geq \alpha \Rightarrow \int_A f d\mu \geq \alpha$;
4. $\mu(A \cap F_\alpha) \leq \alpha \Rightarrow \int_A f d\mu \leq \alpha$.
(v) If \( \mu(A) < \infty \), then \( \mu(A \cap Fx) = x \iff \int_A f d\mu = x \);
(vi) If \( f \leq g \), then \( \int f d\mu \leq \int g d\mu \).

Before stating our main results we need a definition.

**Definition 3.** Functions \( f, g : X \to \mathbb{R} \) are said to be comonotone if for all \( x, y \in X \),
\[
(f(x) - f(y))(g(x) - g(y)) \geq 0
\]
and \( f \) and \( g \) are said to be countermonotone if for all \( x, y \in X \),
\[
(f(x) - f(y))(g(x) - g(y)) \leq 0.
\]
The comonotonicity of functions \( f \) and \( g \) is equivalent to the nonexistence of points \( x, y \in X \) such that \( f(x) < f(y) \) and \( g(x) > g(y) \). Similarly, if \( f \) and \( g \) are countermonotone then \( f(x) < f(y) \) and \( g(x) < g(y) \) cannot happen.

Also, it is clear that if \( f \) and \( g \) are comonotone then (see [19,25]) for any real numbers \( s, t \) either \( F_i \subseteq G_i \) or \( F_i \supseteq G_i \).

Now, our results can be stated as follows.

**3. Barnes–Godunova–Levin type inequalities for Sugeno integrals**

The following is the classical Barnes–Godunova–Levin inequality [28,29]:
\[
\left( \frac{\int_a^b f(x) dx}{b-a} \right)^\frac{1}{p} \left( \frac{\int_a^b g(x) dx}{b-a} \right)^\frac{1}{q} \leq B(p, q) \int_a^b f(x)g(x) dx,
\]
where \( p, q > 1, B(p, q) = \frac{6b-a}{(1+p)(1+q)^\frac{1}{p} + 1} \) and \( f, g \) are nonnegative concave functions on \( [a, b] \).

Unfortunately, as we will see in the following example, in general, the classical Barnes–Godunova–Levin inequality is not valid for Sugeno integral.

**Example 1.** Let \( f, g \) be two real valued functions defined as \( f(x) = g(x) = \sqrt{x} \) where \( x \in [0, 100] \). Let \( m \) be the Lebesgue measure and \( p = q = 4 \) in (1). A straightforward calculus shows that

(i) \( \int_0^{100} f^4(x) dm = \int_0^{100} g^4(x) dm = \int_{x=0}^{100} [x \wedge m([0, 100] \cap \{ x \geq x \})] = \int_{x=0}^{100} [x \wedge (100 - x)] = 50 \),

(ii) \( \int_0^{100} f(x)g(x) dm = \int_{x=0}^{100} [x \wedge m([0, 100] \cap \{ \sqrt{x} \geq x \})] = \int_{x=0}^{100} [x \wedge (100 - x^2)] = 9.5125 \),

(iii) \( B(4, 4) = \frac{3}{250} \sqrt{500} = 0.26833 \).

Therefore:
\[
\left( \frac{\int_0^{100} f^4(x) dm}{100} \right)^\frac{1}{4} \left( \frac{\int_0^{100} g^4(x) dm}{100} \right)^\frac{1}{4} = 7.0711 > B(4, 4) \left( \frac{\int_0^{100} f(x)g(x) dm}{100} \right) = 2.5525
\]
and, consequently, inequality (1) is not valid for Sugeno integral.

The aim of this section is to show a Barnes–Godunova–Levin type inequality derived from (1) for the Sugeno integral.

**Theorem 2.** If \( p, q \in (0, \infty) \) and \( f, g : [0, 1] \to [0, \infty) \) are two concave functions and \( m \) be the Lebesgue measure on \( \mathbb{R} \). Then

(a) If \( f(0) < f(1) \) and \( g(0) < g(1) \), then
\[
\left( \frac{\int_0^1 f^p dx}{1} \right)^\frac{1}{p} \left( \frac{\int_0^1 g^q dx}{1} \right)^\frac{1}{q} \wedge \left( 1 - \frac{\int_0^1 f^p dx}{1} \frac{1}{f(1) - f(0)} \right) \wedge \left( 1 - \frac{\int_0^1 g^q dx}{1} \frac{1}{g(1) - g(0)} \right) \leq \int_0^1 f(x)g(x) dx.
\]
(b) If \( f(0) = f(1) \) and \( g(0) = g(1) \), then
\[
\left( \frac{\int_0^1 f^p dx}{1} \right)^\frac{1}{p} \left( \frac{\int_0^1 g^q dx}{1} \right)^\frac{1}{q} \wedge f(0)g(0) \leq \int_0^1 f(x)g(x) dx.
\]
(c) If \( f(0) > f(1) \) and \( g(0) > g(1) \), then
\[
\left( \int_0^1 f^p dx \right)^\frac{1}{p} \left( \int_0^1 g^q dx \right)^\frac{1}{q} \left( \frac{\left( \int_0^1 f^p dx \right)^\frac{1}{p} - f(0)}{f(1) - f(0)} \right)^\frac{p}{p-1} \left( \frac{\left( \int_0^1 g^q dx \right)^\frac{1}{q} - g(0)}{g(1) - g(0)} \right)^\frac{q}{q-1} \leq \int_0^1 f(x)g(x)dx.
\]

Proof. Let \( p, q \in (0, \infty) \). \( \left( \int_0^1 f^p dx \right)^\frac{1}{p} = t_1 \) and \( \left( \int_0^1 g^q dx \right)^\frac{1}{q} = t_2 \). Since \( f, g : [0, 1] \to [0, \infty) \) are two concave functions, for \( x \in [0, 1] \) we have
\[
f(x) = f((1 - x) \cdot 0 + x \cdot 1) \geq (1 - x) \cdot f(0) + x \cdot f(1) = h_1(x),
g(x) = g((1 - x) \cdot 0 + x \cdot 1) \geq (1 - x) \cdot g(0) + x \cdot g(1) = h_2(x).
\]

(a) If \( f(0) < f(1) \) and \( g(0) < g(1) \), then by Theorem 1 (vi) and the comonotonicity of \( h_1 \) and \( h_2 \), we have
\[
\int_0^1 f(x)g(x)dx \geq \int_0^1 f_1(x)h_2(x)dx = \int_0^1 f(0)g(0)dx = f(0)g(0) \cdot 1.
\]
Since \( \left( \int_0^1 f^p dx \right)^\frac{1}{p} = t_1 \) and \( \left( \int_0^1 g^q dx \right)^\frac{1}{q} = t_2 \) where \( p, q \in (0, \infty) \), Theorem 1(i) implies that \( t_1 \leq 1 \) and \( t_2 \leq 1 \). Therefore
\[
\int_0^1 f(x)g(x)dx \geq f(0)g(0) \cdot 1 \geq f(0)g(0) \cdot t_1 t_2.
\]

(b) If \( f(0) = f(1) \) and \( g(0) = g(1) \), then
\[
f(x) = f((1 - x) \cdot 0 + x \cdot 1) \geq f(0) = h_1(x),
g(x) = g((1 - x) \cdot 0 + x \cdot 1) \geq g(0) = h_2(x).
\]
Thus, by Theorem 1 (vi) and (ii) we have
\[
\int_0^1 f(x)g(x)dx \geq \int_0^1 f_1(x)h_2(x)dx = \int_0^1 f(0)g(0)dx = f(0)g(0) \cdot 1.
\]

(c) If \( f(0) > f(1) \) and \( g(0) > g(1) \), then by Theorem 1 (vi) and the comonotonicity of \( h_1 \) and \( h_2 \), we have
\[
\int_0^1 f(x)g(x)dx \geq \int_0^1 f_1(x)h_2(x)dx = \int_0^1 f_1(x)h_2(x) dx = \left\{ x : x \geq x \right\} \cap \{ h_1(x)h_2(x) \geq x \},
\]
and the proof is completed.

Remark 1. The last step in demonstration of Theorem 2 (part a) is valid whenever \( [0, 1] \cap \{ x : x \geq \frac{t_1f(0)}{f(1) - f(0)} \} \neq \emptyset \) and \( [0, 1] \cap \{ x : x \geq \frac{t_2g(0)}{g(1) - g(0)} \} \neq \emptyset \). Analogously, the last step in demonstration of Theorem 2 (part c) is valid whenever \( [0, 1] \cap \{ x : x \geq \frac{t_1f(0)}{f(1) - f(0)} \} \neq \emptyset \) and \( [0, 1] \cap \{ x : x \geq \frac{t_2g(0)}{g(1) - g(0)} \} \neq \emptyset \). Anyway, in any other case the inequality is trivially verified.
Example 2. Let \( f, g \) be two real valued functions defined as \( f(x) = g(x) = \sqrt{x} \) where \( x \in [0, 1] \). Let \( m \) be the Lebesgue measure and \( p = q = 2 \). A straightforward calculus shows that

\[
\int_0^1 f^2(x) dm = \int_0^1 g^2(x) dm = \int_0^1 f^2(x) g(x) dm = \int_0^1 (\sqrt{x} \wedge m([0, 1] \cap \{x \geq \alpha\})) = \int_0^1 (\sqrt{1-x}) = 0.5,
\]

\[
\left(1 - \frac{\left(\int_0^1 f^2 dm\right)}{\int_1 g(0)}\right) = \left(1 - \frac{\left(\int_0^1 f^2 dm\right)}{f(1) - f(0)}\right) = 0.29289.
\]

Therefore:

\[
\left(\int_0^1 f^2 dm\right)^{\frac{1}{2}} \left(\int_0^1 g^2 dm\right)^{\frac{1}{2}} \wedge \left(1 - \frac{\left(\int_0^1 f^2 dm\right)^{\frac{1}{2}} - f(0)}{f(1) - f(0)}\right) \wedge \left(1 - \frac{\left(\int_0^1 g^2 dm\right)^{\frac{1}{2}} - g(0)}{g(1) - g(0)}\right) \right) = 0.29289 \leq \int_0^1 f(x) g(x) dm = 0.5.
\]

Now, we will prove the general case of Theorem 2.

Theorem 3. Let \( p, q \in (0, \infty) \) and \( f, g : [a, b] \rightarrow [0, \infty) \) be two concave functions and \( m \) be the Lebesgue measure on \( \mathbb{R} \). Then

(a) If \( f(a) < f(b) \) and \( g(a) < g(b) \), then

\[
\left(\int_a^b f^p dx\right)^{\frac{1}{p}} \left(\int_a^b g^q dx\right)^{\frac{1}{q}} \wedge \left(1 - \frac{\left(\int_a^b f^p dx\right)^{\frac{1}{p}} - f(a)}{f(b) - f(a)}\right) \wedge \left(1 - \frac{\left(\int_a^b g^q dx\right)^{\frac{1}{q}} - g(a)}{g(b) - g(a)}\right) \right) \leq \int_a^b f(x) g(x) dx.
\]

(b) If \( f(a) = f(b) \) and \( g(a) = g(b) \), then

\[
\left(\int_a^b f^p dx\right)^{\frac{1}{p}} \left(\int_a^b g^q dx\right)^{\frac{1}{q}} \wedge f(a) g(a) \leq \int_a^b f(x) g(x) dx.
\]

(c) If \( f(a) > f(b) \) and \( g(a) > g(b) \), then

\[
\left(\int_a^b f^p dx\right)^{\frac{1}{p}} \left(\int_a^b g^q dx\right)^{\frac{1}{q}} \wedge \left(\frac{\left(\int_a^b f^p dx\right)^{\frac{1}{p}} - f(a)}{f(b) - f(a)}\right) \wedge \left(\frac{\left(\int_a^b g^q dx\right)^{\frac{1}{q}} - g(a)}{g(b) - g(a)}\right) \right) \leq \int_a^b f(x) g(x) dx.
\]

Proof. Let \( p, q \in (0, \infty) \), \( \left(\int_a^b f^p dx\right)^{\frac{1}{p}} = t_1 \) and \( \left(\int_a^b g^q dx\right)^{\frac{1}{q}} = t_2 \). Since \( f, g : [a, b] \rightarrow [0, \infty) \) are two concave functions, for \( x \in [a, b] \) we have

\[
f(x) = f\left(1 - \frac{x - a}{b - a}.a + \left(\frac{x - a}{b - a}.b\right)\right) \geq \left(1 - \frac{x - a}{b - a}.f(a) + \left(\frac{x - a}{b - a}.f(b) = h_1(x), \right)\right)
\]

\[
g(x) = g\left(1 - \frac{x - a}{b - a}.a + \left(\frac{x - a}{b - a}.b\right)\right) \geq \left(1 - \frac{x - a}{b - a}.g(a) + \left(\frac{x - a}{b - a}.g(b) = h_2(x). \right)\right)
\]

We will prove (a) and (b), the other case is similar.

(a) If \( f(a) < f(b) \) and \( g(a) < g(b) \), then by Theorem 1(vi) and the comonotonicity of \( h_1 \) and \( h_2 \), we have

\[
\int_a^b f(x) g(x) dx \geq \int_a^b h_1(x) h_2(x) dx = \int_{x=0}^{\infty} \left\{ \alpha \wedge m([a, b] \cap \{h_1(x) h_2(x) \geq \alpha\}) \right\} \geq t_1 t_2 \wedge m([a, b] \cap \{h_1(x) h_2(x) \geq t_1 t_2\})
\]

\[
= t_1 t_2 \wedge m([a, b] \cap \{h_1(x) \geq t_1 \} \cap \{h_2(x) \geq t_2\}) = t_1 t_2 \wedge m([a, b] \cap \{h_1(x) \geq t_1 \}) \wedge m([a, b] \cap \{h_2(x) \geq t_2\}) \wedge m([a, b] \cap \{h_1(x) \geq t_1 \} \cap \{h_2(x) \geq t_2\})
\]

\[
= t_1 t_2 \wedge m([a, b] \cap \{x \geq \frac{t_1 (b - a) + af(b) - bf(a)}{f(b) - f(a)}\} \wedge m([a, b] \cap \{x \geq \frac{t_2 (b - a) + af(b) - bf(a)}{g(b) - g(a)}\})\)
\]

\[
= \left(\frac{t_1 (b - a) + af(b) - bf(a)}{f(b) - f(a)}\right) \wedge \left(\frac{t_2 (b - a) + af(b) - bf(a)}{g(b) - g(a)}\right).
\]
(b) If \( f(a) = f(b) \) and \( g(a) = g(b) \), then \( h_1(x) = f(a) \) and \( h_2(x) = g(a) \). Thus, by Theorem 1(vi) and (ii) we have

\[
\int_a^b f(x)g(x)dx \geq \int_a^b h_1(x)h_2(x)dx = \int_a^b f(a)g(a)dx = f(a)g(a) \land (b-a).
\]

If \((\int_a^b f^p\,dx)^{\frac{1}{p}} = t_1 \) and \((\int_a^b g^q\,dx)^{\frac{1}{q}} = t_2 \) where \( p, q \in (0, \infty) \), Theorem 1(i) implies that \( t_1 \leq (b-a)^{\frac{1}{p}} \) and \( t_2 \leq (b-a)^{\frac{1}{q}} \), which implies that \( b-a \geq (t_1t_2)^{\frac{1}{pq}} \) and, consequently

\[
\int_a^b f(x)g(x)dx \geq f(a)g(a) \land (b-a) \geq f(a)g(a) \land (t_1t_2)^{\frac{1}{pq}}
\]

and the proof is completed. \( \Box \)

**Example 3.** Let \( f, g \) be two real valued functions defined as \( f(x) = \ln(1 + x) \) and \( g(x) = \ln(1 + x)^2 \) where \( x \in [0, 4] \). Let \( m \) be the Lebesgue measure and \( p = q = 1 \). A straightforward calculus shows that

\[
\begin{align*}
(\int_0^a f(x)\,dx) \land (\int_0^a g(x)\,dx) & \leq \int_0^a f(x)g(x)\,dx, \\
(\int_0^a f(x)\,dx) \land (\int_0^a g(x)\,dx) & = \int_0^a f(x)g(x)\,dx.
\end{align*}
\]

Therefore:

\[
\left( \frac{\int_0^a f(x)\,dx}{\int_0^a g(x)\,dx} \right)^{\frac{1}{3}} \left( \frac{\int_0^a g(x)\,dx}{\int_0^a f(x)\,dx} \right)^{\frac{1}{3}} \leq \left( \frac{\int_0^a f(x)\,dx}{\int_0^a g(x)\,dx} \right). 
\]

**4. A -based B–G–L-inequality**

In this section, we provide a strengthened version of Barnes–Godunova–Levin type inequality for Sugeno integrals on a real interval based on a binary operation \( \star \) (see [16,17]), where \( \star : [0, \infty)^2 \to [0, \infty) \) is continuous and nondecreasing in both arguments.

**Theorem 4.** Let \( p, q \in (0, \infty) \) and \( f, g : [a, b] \to [0, \infty) \) be two concave functions and \( m \) be the Lebesgue measure on \( \mathbb{R} \). If the binary operation \( \star : [0, \infty)^2 \to [0, \infty) \) is continuous and nondecreasing in both arguments, then

(a) If \( f(a) < f(b) \) and \( g(a) < g(b) \), then

\[
\left( \left( \int_a^b f^p\,dx \right)^{\frac{1}{p}} \star \left( \int_a^b g^q\,dx \right)^{\frac{1}{q}} \right)^{\frac{1}{pq}} \leq \int_a^b f(x)\,dx \star g(x)\,dx.
\]

(b) If \( f(a) = f(b) \) and \( g(a) = g(b) \), then

\[
\left( \left( \int_a^b f^p\,dx \right)^{\frac{1}{p}} \star \left( \int_a^b g^q\,dx \right)^{\frac{1}{q}} \right)^{\frac{1}{pq}} \leq \int_a^b f(x)\,dx \star g(x)\,dx.
\]
(c) If \( f(a) > f(b) \) and \( g(a) > g(b) \), then

\[
\left( \left[ \int_a^b f^p dx \right]^{\frac{1}{p}} \left[ \int_a^b g^q dx \right]^{\frac{1}{q}} \right)^{\frac{1}{\min(p,q)}} \leq \left( \frac{\int_a^b f^p dx}{f(b) - f(a)} \right)^{\frac{1}{p}} \left( \frac{\int_a^b g^q dx}{g(b) - g(a)} \right)^{\frac{1}{q}}.
\]

**Proof.** The proof is similar to that of Theorem 3. \( \square \)

**Example 4.** Consider the binary operation \( \star = + \). Then \( \star \) is continuous and nondecreasing in both arguments and, taking \( f \) and \( g \) as in Example 2, we have

(i) \( \int_0^1 f(x) + g(x) \, dm = \sqrt[4]{\frac{1}{2} \left( 1 - \frac{1}{4} x^2 \right)} = 0.82843 \).

(ii) \( \int_0^1 g^2(x) \, dm = \int_0^1 f^2(x) \, dm = 0.5 \).

(iii) \( \left( b - \left( \int_a^b f^p dx \right)^{\frac{1}{p}} \left( \int_a^b g^q dx \right)^{\frac{1}{q}} \right) = \int_0^1 \frac{\int_0^1 f^p dm}{f(1) - f(0)} - f(0) = 0.29289 \).

Therefore:

\[
\left( \left[ \int_a^b f^2 dx \right]^{\frac{1}{2}} \left[ \int_a^b g^2 dx \right]^{\frac{1}{2}} \right) \leq \left( 1 - \frac{\int_0^1 f^2 dm}{f(1) - f(0)} \right) \left( 1 - \frac{\int_0^1 g^2 dm}{g(1) - g(0)} \right) = 1.4142 \wedge 0.29289 \leq \int_0^1 f(x) + g(x) \, dm = 0.82843.
\]

5. Conclusion

In this paper, we have investigated some Barnes–Godunova–Levin type inequalities for Sugeno integral of concave functions defined on a real interval \( [a, b] \) (Section 3). Also, a strengthened version of Barnes–Godunova–Levin type inequality for Sugeno integrals on a real interval based on a binary operation \( \star \) is presented (Section 4). In the future research, we will continue exploring other integral inequalities for nonadditive measures and integrals.

**Acknowledgements**

The authors are grateful to the referees and Area Editor of the paper for their critical reading of the manuscript and many valuable recommendations for improvements.

**References**


