Time optimal control for Linear Systems on Lie Groups

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April 10, 2020

Abstract

This paper is devoted to the study of time optimal control of linear systems on Lie groups. General necessary conditions of existence of time minimizers are stated when the controls are unbounded. The results are applied to systems on various Lie groups.

Keywords: Lie groups; Linear systems; Time optimal control; Pontryagin Maximum Principle.

1 Introduction

In this paper we are interested in time optimal problems for linear systems on Lie groups.

By linear system is meant a controlled system

$$(\Sigma) \qquad \dot{g} = \mathcal{X}_g + \sum_{j=1}^{m} u_j Y_g^j$$

on a connected Lie group G where $\mathcal{X}$ is a linear vector field, that is a vector field whose flow is a one-parameter group of automorphisms, and the Y^j's are left-invariant.

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The class $\mathcal{U}$ of admissible controls is the set of measurable and locally bounded functions defined on intervals of $\mathbb{R}$ with values in a subset U of $\mathbb{R}^m$ that can be $U = [-B, B]^m$ (bounded case) or $U = \mathbb{R}^m$ (unbounded case). As usual the system is assumed to satisfy the Lie algebra rank condition.

Thanks to the Filippov Theorem, (see [1]), we know that time minimizers always exist in the bounded case. It is no longer true in the unbounded one.

Our goal is to study the existence of time minimizers when the controls are unbounded, and to provide some of their characteristics in all cases. To this aim we use first the Pontryagin Maximum Principle, but also second order conditions, namely the Goh and the Legendre-Clebsch conditions.

Let us denote by Δ the subspace generated by the control vectors, that is $\Delta = Span\{Y^1, \ldots, Y^m\}$. Theorem 1 states that if there exists an ideal $\mathfrak{a}$ of the Lie algebra of the group such that $\Delta \subset \mathfrak{a} \subset \Delta + [\Delta, \Delta] + D\Delta$ (see Section 2 for the definition of D) then no time optimal extremal can exist for unbounded inputs.

It is also proven that a generic system on a semi-simple Lie group, with $m \geq 2$, has no time minimizers for unbounded inputs (Theorem 2).

These results are illustrated by examples on different kind of Lie groups, nilpotent, solvable, semi-simple compact and noncompact. Among other things these examples allow to show that in the bounded case the minimizers are not bang-bang in general, due to the existence of singular extremals, and that it may happen that the minimum time is not reached in the unbounded case.

In Section 2 we state the basic definitions needed to define a linear control system on G. We characterize the notion of linear vector field through the normalizer of $\mathfrak{g}$, and its associated derivation. We then recall the Pontryagin Maximun Principle for time optimal problems.

The general results are stated in Section 3. We first follow [2] to translate the system to the tangent space at the identity and write the associated Hamiltonian of Σ in $g^* \times G$ as follows

$$\mathcal{H}(\lambda, g, u) = \langle \lambda, F(g) \rangle + \sum_{j=1}^{m} u_j \langle \lambda, Y^j \rangle, \quad \text{where} \quad F(g) = TL_{g^{-1}} \mathcal{X}_g.$$

Theorems 1 and 2 are stated and proven in this section.

Finally Section 4 is devoted to examples: in the two dimensional affine group $Aff_+(2)$, in the nilpotent Heisenberg group, and in the semisimple groups $SO(3, \mathbb{R})$ and $SL(2, \mathbb{R})$.

2 Basic definitions and notations

More details about linear vector fields and linear systems can be found in [7] and [8].

2.1 Linear vector fields

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra (the set of left-invariant vector fields, identified with the tangent space at the identity). A vector field on G is said to be *linear* if its flow is a one-parameter group of automorphisms. Actually the linear vector fields are nothing else than the so-called infinitesimal automorphisms in the Lie group literature (see [5] for instance). The following characterization is fundamental for our purpose.

A vector field $\mathcal{X}$ on a connected Lie group G is linear if and only if it belongs to the normalizer of $\mathfrak{g}$ in the algebra of analytic vector fields of G, that is

$$\forall Y \in \mathfrak{g} \qquad [\mathcal{X}, Y] \in \mathfrak{g},$$

and verifies $\mathcal{X}(e) = 0$.

On account of this characterization, one can associate to a linear vector field $\mathcal{X}$ the derivation $D = -ad(\mathcal{X})$ of the Lie algebra $\mathfrak{g}$ of G.

Inner derivation. In case where the derivation is inner, that is $\text{ad}(\mathcal{X}) = \text{ad}(X)$ for some $X \in \mathfrak{g}$, the derivation is $D = -\text{ad}(X)$ and the linear field is:

$$\mathcal{X}_g = TL_g.X - TR_g.X,$$

where L_g (resp. R_g) stands for the left (resp. right) translation by g.

In order to simplify the theoretical computations we will have to translate vector fields to T_eG, the tangent space at the identity, by left translation. All along the paper the translation of a linear field $\mathcal{X}$ will be denoted by:

$$F_g = TL_{g^{-1}}.\mathcal{X}_g \in T_eG.$$

In the same way consider the left-invariant vector field Y (resp. the left-invariant one-form λ) identified with its value at the identity $Y = Y_e \in T_eG$ (resp. $\lambda = \lambda_e \in T_e^*G$). Then at the point g we have $Y_g = TL_g.Y$ (resp. $\lambda_g = \lambda \circ TL_{g^{-1}}$).

2.2 Linear systems

Definition 1 *A* **linear system** *on a connected n-dimensional Lie group G is a controlled system*

$$(\Sigma) \qquad \dot{g} = \mathcal{X}_g + \sum_{j=1}^{m} u_j Y_g^j$$

where $\mathcal{X}$ is a linear vector field and the Y^j's are left-invariant ones. The control $u = (u_1, \dots, u_m)$ takes its values in a subset U of $\mathbb{R}^m$.

An input u being given (measurable and locally bounded), the corresponding trajectory of (Σ) starting from the identity e will be denoted by $e_u(t)$, and the one starting from the point g by $g_u(t)$. A straightforward computation shows that

$$g_u(t) = \varphi_t(g) e_u(t),$$

where $(\varphi_t)_{t\in\mathbb{R}}$ stands for the flow of the linear vector field $\mathcal{X}$ (see [8]).

2.3 The system Lie algebra and the rank condition

Let $\mathcal{L}_0$ be the smallest subalgebra of $\mathfrak{g}$ that contains $\{Y^1, \dots, Y^m\}$ and is D-invariant.

Proposition 1 ([8]) *The system Lie algebra is*

$$\mathcal{L} = \mathbb{R}\mathcal{X} \oplus \mathcal{L}_0.$$

The Lie Algebra Rank Condition (LARC) is satisfied by (Σ) if and only if $\mathcal{L}_0 = \mathfrak{g}$.

Remark. The subalgebra $\mathcal{L}_0$ is actually an ideal of $\mathcal{L}$, called the zero-time ideal in the literature (see [10] for instance).

About linear systems, it may be also convenient to consider the so-called *ad-rank condition*. Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ generated by $\Delta = \text{Span}\{Y^1, \dots, Y^m\}$. The ad-rank condition is:

$$\mathfrak{h} + D\mathfrak{h} + D^2\mathfrak{h} + \cdots + D^{n-1}\mathfrak{h} = \mathfrak{g}.$$

In case where $\mathfrak{h} = \Delta$, in particular when $m = 1$, the ad-rank condition amounts to say that the linearized system at the identity is controllable. More generally, the algebra $\mathfrak{h}$ is included in the strong Lie saturate of the system (see [10] for these notions), and under the ad-rank condition the linearization of the extended system is controllable.

2.4 The Pontryagin Maximum Principle for time optimal problems

We recall here the application of PMP to time optimal problems. More details can be found in [12] (see also [1], [10]).

Consider the control system (Σ): $\dot{x} = f(x) + \sum_{j=1}^{m} u_j g_j(x)$ where f and the g_i's are smooth vector fields on a manifold M and the control $u = (u_1, \dots, u_m)$ belongs to some subset U of $\mathbb{R}^m$. The associated Hamiltonian is

$$\mathcal{H}(\lambda_x, x, u) = \langle \lambda_x, f(x) + \sum_{j=1}^{m} u_j g_j(x) \rangle \quad \text{where} \quad \lambda_x \in T_x^* M.$$

If $\widetilde{u}(t)$, $t \in [0, T]$, is a control such that the associated solution $x(t)$ to (Σ) minimizes the time among all admissible curves steering $x(0)$ to $x(T)$, then there exists a Lipschitzian curve $(\lambda(t), x(t))$ in the cotangent space T^*M of M (here $\lambda(t) \in T^*_{x(t)}M$) such that

1. $\lambda(t) \neq 0$ for all $t \in [0, T]$;
2. $\mathcal{H}(\lambda(t), x(t), \widetilde{u}(t)) = \max_{u \in U} \mathcal{H}(\lambda(t), x(t), u)$ for almost all $t \in [0, T]$;
3. $\mathcal{H}(\lambda(t), x(t), \widetilde{u}(t)) \geq 0$ for almost all $t \in [0, T]$;
4. $(\lambda(t), x(t))$ satisfies the equations

$$\begin{cases} \dfrac{d}{dt} x(t) &= \dfrac{\partial}{\partial \lambda} \mathcal{H}(\lambda(t), x(t), \widetilde{u}(t)) = f(x(t)) + \displaystyle\sum_{j=1}^{m} \widetilde{u}_j(t) g_j(x(t)) \\ \dfrac{d}{dt} \lambda(t) &= -\dfrac{\partial}{\partial x} \mathcal{H}(\lambda(t), x(t), \widetilde{u}(t)) \end{cases}$$

3 General results

We consider the linear system

$$(\Sigma) \qquad \dot{g} = \mathcal{X}_g + \sum_{j=1}^{m} u_j Y_g^j,$$

in the connected n-dimensional Lie group G. The subset U of $\mathbb{R}^m$ where the control $u = (u_1, \dots, u_m)$ takes its values will be either $\mathbb{R}^m$ (unbounded case), or $[-B, B]^m$ for some $B > 0$ (bounded case), and the admissible controls will be taken in $L^\infty(\mathbb{R}_+; U)$.

In order to apply the Pontryagin Maximum Principle we should first write the Hamiltonian of the system, that is

$$\mathcal{H}(\lambda_g, g, u) = \langle \lambda_g, \mathcal{X}_g + \sum_{j=1}^{m} u_j Y_g^j \rangle.$$

Since $Y_g^j = TL_g.Y^j$, $\mathcal{X}_g = TL_g.F_g$ and $\lambda_g = \lambda \circ TL_{g^{-1}}$, we can translate $\mathcal{H}$ to the tangent space at the identity, that is:

$$\mathcal{H}(\lambda, g, u) = \langle \lambda, F(g) \rangle + \sum_{j=1}^{m} u_j \langle \lambda, Y^j \rangle$$

Notice that $\mathcal{H}$ is no longer written in the cotangeant space of G but in $\mathfrak{g}^* \times G$. Thanks to the computations of [2] we know that the associated Hamiltonian equations are in $\mathfrak{g}^* \times G$:

$$\begin{cases} \dot{g} &= \mathcal{X}_g + \sum_{j=1}^{m} u_j Y^j(g) \\ \\ \dot{\lambda} &= (-D + \sum_{j=1}^{m} u_j \mathrm{ad}(Y^j))^* \lambda \end{cases}$$

where D is the derivation of $\mathfrak{g}$ associated to $\mathcal{X}$.

3.1 The unbounded case

In the unbounded case the maximization of $\mathcal{H}$ implies

$$\langle \lambda, Y^i \rangle = 0 \qquad \text{for } i = 1, \dots, m. \tag{1}$$

If $(\lambda(t), g(t))$ is an extremal then we have $\langle \lambda(t), Y^i \rangle = 0$ for all t for which the extremal is defined, hence also $\langle \dot{\lambda}(t), Y^i \rangle = 0$ almost everywhere. By the Hamiltonian equations, this last equality is equivalent to

$$\langle \lambda(t), -DY^i + \sum_{j=1}^{m} u_j(t)[Y^j, Y^i] \rangle = 0 \quad a.e., \quad \forall i = 1, \dots, m. \tag{2}$$

Such extremals are called singular (see [4] or [1]) and are known to satisfy also the Goh condition:

$$\langle \lambda(t), [Y^i, Y^j] \rangle = 0, \quad \forall i, j = 1, \dots, m \quad \text{along singular extremals.} \tag{3}$$

Applying the Goh condition to (2) we get $\langle \lambda(t), -DY^i \rangle = 0$, for $i = 1, \dots m$.

Let us denote by $\mathcal{K}$ the set of $X \in \mathfrak{g}$ that satisfy $\langle \lambda(t), X \rangle = 0$ along all extremals. According to the previous considerations it is a subspace of $\mathfrak{g}$ that contains $\Delta + [\Delta, \Delta] + D\Delta$.

Clearly no extremal exists if $\mathcal{K} = \mathfrak{g}$. Indeed this would imply $\lambda(t) = 0$ in contradiction with the PMP.

In order to characterize the cases where $\mathcal{K} = \mathfrak{g}$ we first look for a sufficient condition for $\mathcal{K}$ to be D-invariant. Let $X \in \mathcal{K}$. As previously the differentiation of $\langle \lambda(t), X \rangle = 0$ implies

$$\langle \lambda(t), -DX + \sum_{i=j}^{m} u_j(t)[Y^j, X] \rangle = 0 \quad a.e.$$

and DX belongs to $\mathcal{K}$ as soon as $[Y^j, X] = 0$ for $j = 1, \dots, m$. A natural sufficient condition for $\mathcal{K}$ to be D-invariant is therefore $[\Delta, \mathcal{K}] \subset \mathcal{K}$. This condition is not directly checkable without knowing $\mathcal{K}$, and is useless in that case, but it has nice consequences and can be deduced from more convenient conditions.

Lemma 1 *The rank condition is assumed to hold. If $[\Delta, \mathcal{K}] \subset \mathcal{K}$ then $\mathcal{K} = \mathfrak{g}$.*

Proof.

1. We have already seen that $\mathcal{K}$ is D-invariant under the condition of the lemma. Consequently:

$$\Delta + D\Delta + D^2\Delta + \dots D^{n-1}\Delta \subset \mathcal{K}.$$

2. Let $V = \Delta + D\Delta + D^2\Delta + \dots D^{n-1}\Delta$. This subspace of $\mathcal{K}$ is D-invariant. The assumption of the lemma implies $[\Delta, V] \subset \mathcal{K}$ hence also $D[\Delta, V] = [D\Delta, V] + [\Delta, DV] \subset \mathcal{K}$, and finally $[D\Delta, V] \subset \mathcal{K}$. By induction we get $[D^k\Delta, V] \subset \mathcal{K}$ for all $k \geq 0$ and $[V, V] \subset \mathcal{K}$.

   Since $[V, V]$ is again a D-invariant subspace of $\mathcal{K}$ the same argument shows that $[V, [V, V]] \subset \mathcal{K}$. By induction it turns out that the Lie algebra $\mathcal{L}ie(V)$ generated by V is included in $\mathcal{K}$.

3. The Lie algebra $\mathcal{L}ie(V)$ containing Δ and being D-invariant, the Lie algebra of the system is equal to $\mathbb{R}\mathcal{X} \oplus \mathcal{L}ie(V)$. But the rank condition is then equivalent to $\mathcal{L}ie(V) = \mathfrak{g}$ (see Section 2.3), and we conclude $\mathfrak{g} = \mathcal{K}$.

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The following theorem, whose assumption is easily checkable, can be deduced at once from Lemma 1.

Theorem 1 *Assume* (Σ) *to satisfy the Lie algebra rank condition and the controls to be unbounded. If there exists an ideal* $\mathfrak{a}$ *of* $\mathfrak{g}$ *such that*

$$\Delta \subset \mathfrak{a} \subset \Delta + [\Delta, \Delta] + D\Delta,$$

then no time optimal extremal exists.

Proof.
Since $\Delta + [\Delta, \Delta] + D\Delta \subset \mathcal{K}$, we have $[\Delta, \mathcal{K}] \subset [\mathfrak{a}, \mathcal{K}] \subset \mathfrak{a} \subset \mathcal{K}$.
By Lemma 1 this implies $\mathfrak{g} = \mathcal{K}$.

■

Remarks.

1. If the Lie algebra $\mathfrak{g}$ is Abelian, then any subspace Δ is an ideal and we retrieve the well known fact that no minimum time extremal exists in the Abelian case (for unbounded inputs of course).

2. One might think that the ad-rank condition (see Section 2.3) could be sufficient to assert that $\mathcal{K} = \mathfrak{g}$. It is not so, as shown by Example 4.2. In the single input case the ad-rank condition is $\text{Span}(Y + DY + \cdots + D^{n-1}Y) = \mathfrak{g}$, and if k is the greater integer such that $D^k Y = 0$, the singular control should be

$$u(t) = \frac{\langle \lambda(t), D^{k+1}Y \rangle}{\langle \lambda(t), [Y, D^k Y] \rangle}$$

   It is the way the singular control is computed in Example 4.2.

3.1.1 The unbounded semi-simple case

As in Section 2.3 let us denote by $\mathfrak{h}$ the subalgebra of $\mathfrak{g}$ generated by $\Delta = \text{span}\{Y^1, \ldots, Y^m\}$. According to the results of [9], that can be found in [10], we know that $\mathfrak{h}$ is included in the strong Lie saturate of the system, that is the set of vector fields that can be added to (Σ), viewed as a polysystem, without modifying the closures $\overline{\mathcal{A}_{\leq t}}(g)$ of the attainable sets for all $t > 0$ and $g \in G$. This means that all the left-invariant vector fields belonging to $\mathfrak{h}$ can be added to the set of controlled vector fields without modifying the sets $\overline{\mathcal{A}_{\leq t}}(g)$ for all $t > 0$ and $g \in G$.

This adaptation of extension technics to linear systems on Lie groups is made in [8] and used in [6].

In the particular case where $\mathfrak{h}$ is equal to $\mathfrak{g}$ this implies that $\overline{\mathcal{A}_{\leq t}}(g) = G$, hence $\mathcal{A}_{\leq t}(g) = G$ because the system is Lie determined (see [10]), for all $t > 0$ and $g \in G$. The conclusion is that the minimum time is 0 and is of course never reached.

The condition $\mathfrak{h} = \mathfrak{g}$ may appear very strong (and it is in general), but it is of particular importance when the Lie algebra is semi-simple. Indeed it is proven by Kuranishi in [11] that generically the subalgebra generated by a pair $\{Y^1, Y^2\}$ of elements of a semi-simple Lie algebra $\mathfrak{g}$ is equal to $\mathfrak{g}$. Kuranishi's result implies the following statement.

Theorem 2 *The connected group G is assumed to be semi-simple, the Lie algebra rank condition to hold and the controls to be unbounded.*

A generic system

$$(\Sigma) \qquad \dot{g} = \mathcal{X}_g + \sum_{j=1}^{m} u_j Y_g^j$$

with $m \geq 2$ has no time extremals. Moreover the minimum time between any two points is 0.

In the forthcoming examples 4.3.1 and 4.3.2, that live respectively in the 3-dimensional semi-simple Lie groups SO_3 and SL_2, the control is one-dimensional. If these systems were controlled by two linearly independant left-invariant vector fields then no time extremals could exist.

3.1.2 Minimal time and existence of extremals

It is worth noticing that the lack of extremals does not imply that the minimal time is zero as in the classical (Abelian) case. Indeed it may happen that the minimum time from a given point g_0 to another given point g_1 be positive, say $T > 0$, but that no admissible curve steers g_0 to g_1 in time T. This is the case of Example 4.1 where no extremal exists but where some points cannot be joined in arbitrary small time.

In that case the minimal time may be computed as the limit when B tends to $+\infty$ of the minimal time T_B obtained for inputs bounded by B.

3.2 The bounded case

We consider now the case where $-B \leq u_j \leq B$ for $j = 1, \ldots, m$. Thanks to Filippov's Theorem we know that time minimizers exist (see [1]).

The maximization of $\mathcal{H}$ implies

1. $u_j = \epsilon_j B$, where $\epsilon_j = \text{sign}\langle \lambda, Y^j \rangle$, if $\langle \lambda, Y^j \rangle \neq 0$;
2. u_j is not determined if $\langle \lambda, Y^j \rangle = 0$.

Thus we get 2^m different dynamics, to which must possibly be added the dynamics related to $\langle \lambda, Y^j \rangle = 0$ for some j's.

3.2.1 The bang-bang problem

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It is well known that in the Abelian simply connected case, that is when $G = \mathbb{R}^n$, and when the set U is a polytope, a time optimal control takes values at the vertices of U and switches a finite number of times between these values in a compact interval of time. The proof uses the fact that the covector $\lambda(t)$ is an analytic solution of a differential equation that does not depend on the control. Indeed the general equation of λ is:

$$\dot{\lambda} = (-D + \sum_{j=1}^{m} u_j \text{ad}(Y^j))^* \lambda$$

and the terms depending on the u_j's disappear in the Abelian case. The picture is different in the non Abelian one and a general bang-bang theorem cannot be proven. Some extremals that are not bang-bang are exhibited in the examples.

4 Examples

4.1 An Example in $Aff_+(2)$

The 2-dimensional connected affine group is the Lie group:

$$G = Aff_+(2) = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}; \quad (x, y) \in \mathbb{R}_+^* \times \mathbb{R} \right\}.$$

Its underlying manifold can be identified with $\mathbb{R}_+^* \times \mathbb{R}$, and its Lie algebra $\mathfrak{g} = \mathfrak{aff}(2)$ is generated by the left-invariant vector fields $X = x\dfrac{\partial}{\partial x}$ and $Y = x\dfrac{\partial}{\partial y}$ in natural coordinates. Since $[X, Y] = Y$ this 2-dimensional algebra is solvable[1].

[1] Define the derived series of $\mathfrak{g}$ by $\mathcal{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and by induction $\mathcal{D}^{n+1}\mathfrak{g} = [\mathcal{D}^n\mathfrak{g}, \mathcal{D}^n\mathfrak{g}]$, then $\mathfrak{g}$ is solvable if $\mathcal{D}^n\mathfrak{g}$ vanishes for some integer n.

It is easy to see that in the basis (X, Y) all derivations D of $\mathfrak{aff}(2)$ have the form $D = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$ where a and b are real numbers, and that the linear vector field $\mathcal{X}$ associated to such a derivation is $\mathcal{X}(g) = (a(x-1) + by)\frac{\partial}{\partial y}$ for $g = (x, y)$ (See [6] for more details).

Recall also from [6] that up to a group automorphism a linear system in $Aff_+(2)$ that satisfies the rank condition has the following form

$$(\Sigma_b) \qquad \begin{cases} \dot{x} = & u\alpha x \\ \dot{y} = & x - 1 + by \qquad \text{with} \qquad \alpha \neq 0. \end{cases}$$

and that it is globally controllable if and only if $b = 0$. In that case the system is

$$(\Sigma) \qquad \begin{cases} \dot{x} = & u\alpha x \\ \dot{y} = & x - 1 \qquad \text{with} \qquad \alpha \neq 0. \end{cases}$$

Here the linear vector field is associated to the derivation $D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in the canonical basis (X, Y), and the controlled vector is αX. The Hamiltonian is

$$\mathcal{H}(\lambda_g, g, u) = \langle \lambda_g, \mathcal{X}_g \rangle + u\langle \lambda_g, \alpha X_g \rangle = \langle \lambda, F_g \rangle + u\langle \lambda, \alpha X \rangle.$$

Let us first look at the unbounded case. The maximization condition implies $\langle \lambda(t), X \rangle = 0$, and then

$$0 = \langle \dot{\lambda}(t), X \rangle = \langle \lambda(t), -DX + u\alpha[X, X] \rangle = \langle \lambda(t), -Y \rangle.$$

This implies $\lambda = 0$ so that no extremal trajectory exists. Actually $\text{Span}(X, Y) = \Delta + \mathcal{D}\Delta$ and we find the results of Section 3.1, that is $\Delta + \mathcal{D}\Delta \subset \mathcal{K}$.

However the equation $\dot{y} = x - 1$ and the fact that x be positive imply that an admissible curve cannot steer $(1, 0)$ to $(1, -d)$ in time less than d.

Let us now compute the time minimizing curve between these points for bounded controls, $u \in [-B, B]$. We can assume $\alpha > 0$ without loss of generality. In coordinates the Hamiltonian is (with $\lambda_g = (p, q)$):

$$\mathcal{H}(\lambda_g, g, u) = \langle \lambda_g, \mathcal{X}_g \rangle + u\langle \lambda_g, \alpha X_g \rangle = up\alpha x + q(x - 1)$$

and the Hamiltonian equations are:

$$\begin{cases} \dot{x} = & u\alpha x \\ \dot{y} = & x - 1 \end{cases} \qquad \begin{cases} \dot{p} = & -u\alpha p - q \\ \dot{q} = & 0 \end{cases}$$

Assume $p(t_0) = 0$ for some t_0. Then $q = q_0 \neq 0$ because the pair $(p(t), q(t) = q_0)$ vanishes nowhere. This implies $\dot{p}(t_0) = -q \neq 0$, hence that p can vanish at most once.

The consequence is that an optimal control takes the constant value B or $-B$ and changes at most once. Further the only possibility from $(1,0)$ to $(1,-d)$ is first $u = -B$ on $[0, \frac{T}{2}]$ (in order that $x(t) \leq 1$ and $\dot{y} \leq 0$), and then $u = B$ on $[\frac{T}{2}, T]$, in order that $x(T) = 1$. A straightforward computation gives

$$\begin{cases} x(t) = e^{-B\alpha t} & t \in [0, \frac{T}{2}] \\ x(t) = e^{-B\alpha T} e^{B\alpha t} & t \in [\frac{T}{2}, T] \end{cases} \quad \text{and} \quad y(T) = \frac{2}{B\alpha}(1 - e^{-B\alpha \frac{T}{2}}) - T.$$

To finish $y(T) = -d \Longleftrightarrow T = d + \frac{2}{B\alpha}(1 - e^{-B\alpha\frac{T}{2}}) > d$. But $T \longmapsto_{B \mapsto +\infty} d$ and the minimal time to steer $(1,0)$ to $(1,-d)$ with unbounded controls is indeed d and is not reached.

4.2 An Example in the Heisenberg group

The Heisenberg group is the matrix group

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix};\quad (x,y,z) \in \mathbb{R}^3 \right\}.$$

Its Lie algebra $\mathfrak{g}$ is generated by the following left-invariant vector fields, here written in natural coordinates (x,y,z):

$$X = \frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad Z = \frac{\partial}{\partial z}.$$

The only Lie bracket that does not vanish is $[X,Y] = Z$. A straighforward computation using the equality $DZ = [DX, Y] + [X, DY]$ shows that the derivations of $\mathfrak{g}$ are the endomorphisms whose matrix in the basis (X,Y,Z) writes

$$D = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & a+d \end{pmatrix}.$$

Further the linear vector field $\mathcal{X}$ associated to this derivation writes in natural coordinates

$$\mathcal{X} = (ax + by)\frac{\partial}{\partial x} + (cx + dy)\frac{\partial}{\partial y} + (ex + fy + (a+d)z + \frac{1}{2}cx^2 + \frac{1}{2}by^2)\frac{\partial}{\partial z}.$$

It is shown in [6] that a single-input system that satisfies the Lie algebra rank condition is up to a group automorphism equal to:

$$(\Sigma) \qquad \dot{g} = \mathcal{X}_g + uX_g \qquad \text{where} \qquad D = \begin{pmatrix} 0 & b & 0 \\ 1 & d & 0 \\ 0 & f & d \end{pmatrix}$$

is the derivation associated to $\mathcal{X}$ in the basis (X, Y, Z).

Let us look at the time optimal problem for unbounded inputs. The Hamiltonian is

$$\mathcal{H}(\lambda, g, u) = \langle \lambda, F_g \rangle + u\langle \lambda, X \rangle.$$

The maximization condition implies $\langle \lambda(t), X \rangle = 0$, and then

$$0 = \langle \dot{\lambda}(t), X \rangle = \langle \lambda(t), -DX + u[X, X] \rangle = \langle \lambda(t), -Y \rangle,$$

since $DX = Y$. The condition $\langle \lambda(t), Y \rangle = 0$ implies in turn

$$\begin{aligned} 0 &= \langle \dot{\lambda}(t), Y \rangle = \langle \lambda(t), -DY + u[X, Y] \rangle \\ &= \langle \lambda(t), -(bX + dY + fZ) + uZ \rangle = \langle \lambda(t), -fZ + uZ \rangle. \end{aligned}$$

Since $\langle \lambda(t), X \rangle = \langle \lambda(t), Y \rangle = 0$ but $\lambda(t) \neq 0$, we get $u = f$. This example illustrate the fact that some of the maximization equations may provide the values of the optimal controls. Here the only possibility is u constant equal to f.

An extremal trajectory has to satisfy the condition $\mathcal{H} \geq 0$ of the PMP, but also the Legendre-Clebsch condition because the system is single-input. For a general single-input system

$$\dot{x} = X(x) + uY(x) \quad \text{with} \quad u \in \mathbb{R},$$

the Legendre-Clebsch condition is $\langle \lambda(t), [Y, [Y, X]](x(t)) \rangle \leq 0$ along any singular extremal $(\lambda(t), x(t))$ (see [1] or [4]).

Let us consider the particular case where $b = d$, but $f \neq 0$. Here $D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & f & 0 \end{pmatrix}$ and $\begin{pmatrix} X & DX & D^2X \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & f \end{pmatrix}$ so that the ad-rank condition holds, thanks to $f \neq 0$ (and the system is controllable, see [6]). Nevertheless the condition $\mathcal{H} \geq 0$ and the Legendre-Clebsch one are satisfied for a suitable choice of λ. Indeed in the coordinates $g = (x, y, z)$ and $\lambda = (p, q, r)$ the conditions $\langle \lambda, X \rangle = \langle \lambda, Y \rangle = 0$ become $p = 0$ and $q + rx = 0$.

For the control $u(t) = f$, and taking into account the above equalities, the Hamiltonian is in coordinates: $\mathcal{H}(\lambda, g, u = f) = qx + r(fy + \frac{1}{2}x^2) = \frac{1}{2}qx + rfy$.

On the other hand $[X, [X, \mathcal{X}]] = [X, DX] = [X, Y] = Z$ and $\langle \lambda(t), Z \rangle = r$. The Legendre-Clebsch condition is therefore $r \leq 0$. From the Hamiltonian we know that r is constant and cannot vanish ($r = 0$ would imply $p = q = r = 0$). Finally we get

$$r < 0 \quad \text{and} \quad \frac{1}{2}qx + rfy \geq 0,$$

which is always possible by a suitable choice of $q(0)$ except if $x(0) = 0$ in which case the additional condition $rty \geq 0$ is necessary.

Let us turn our attention to the bounded case $-B \leq u \leq +B$. According to the above there are two different cases depending on $|f| > B$ or not.

If $|f| > B$ the control can only take the values $-B$ and $+B$. More accurately $u(t) = \epsilon B$, where $\epsilon = \text{sign}\langle \lambda(t), X \rangle$. On the contrary if $|f| \leq B$ a time minimizer may contain arcs of singular extremals.

4.3 Examples on SO(3,$\mathbb{R}$) and SL(2,$\mathbb{R}$)

4.3.1 The compact case SO_3

Let $G = SO(3, \mathbb{R})$ be the rotational group. Its Lie algebra $\mathfrak{so}(3,\mathbb{R})$ is the set of skew-symmetric 3×3 real matrices. Its canonical basis $\{X, Y, Z\}$ satisfies

$$[X, Y] = Z, \quad [Z, X] = Y, \quad \text{and } [Y, Z] = X$$

Consider the unbounded linear system

$$(\Sigma) \qquad \dot{g} = \mathcal{X}_g + uY_g, \text{ where } D = -\text{ad}(X) \text{ and } U = \mathbb{R}.$$

Since $Span_{LA}\{\mathcal{X}, Y\} = \mathfrak{g}$ the system satisfies the LARC condition. Furthermore, Σ is controllable, [3] .

The associated Hamiltonian function reads as

$$\mathcal{H}(\lambda_g, g, u) = \langle \lambda, F_g \rangle + u\langle \lambda, Y \rangle,$$

and the maximization condition implies $\langle \lambda(t), Y \rangle = 0$. By differentiation

$$0 = \langle \dot{\lambda}(t), Y \rangle = \langle \lambda(t), (-D + u \text{ ad}(Y))Y \rangle = \langle \lambda(t), Z \rangle.$$

A second derivation yields

$$0 = \langle \dot{\lambda}(t), Z \rangle = \langle \lambda(t), -DZ + u \text{ ad}(Y)Z \rangle = u\langle \lambda(t), X \rangle.$$

Since $\langle \lambda(t), Y \rangle = \langle \lambda(t), Z \rangle = 0$ it is not possible that $\langle \lambda(t), X \rangle$ vanishes and we finally obtain $u(t) = 0$ almost everywhere. In conclusion the only possible minimizers, that is the only projection of extremal curves are the integral curves of:

$$\dot{g} = \mathcal{X}_g.$$

Can the conditions $\mathcal{H} \geq 0$ and the Legendre-Clebsch one be satisfied? Since $[Y, [Y, \mathcal{X}]] = [Y, DY] = [Y, -Z] = -X$ the Legendre-Clebsch condition is here:

$$-\langle \lambda, X \rangle \leq 0 \qquad \text{(Legendre-Clebsch)}$$

Since $\langle \lambda, X \rangle \neq 0$ this condition is satisfied, even if it means changing the sign of λ.

On the other hand the condition $\mathcal{H} \geq 0$ with $u = 0$ is simply $\langle \lambda, F_g \rangle$. But the derivation $D = -\mathrm{ad}(X)$ is here inner and the vector field $\mathcal{X}$ is defined by $\mathcal{X}_g = TL_g.X - TR_g.X$ (see [7]).

Therefore $F_g = TL_{g^{-1}}\mathcal{X}_g = X - TL_{g^{-1}}TR_g.X$, and the condition $\mathcal{H} \geq 0$ turns out to be:

$$\mathcal{H} \geq 0 \Longleftrightarrow \langle \lambda, X \rangle \geq \left\langle \lambda, \mathrm{Ad}(g^{-1})X \right\rangle.$$

This condition is satisfied for g in a non-empty subset of G.

Since the singular extremals are obtained for $u = 0$ they must be taken into account in the bounded case $-B \leq u \leq +B$, and a time minimizer may contain arcs of singular extremals.

4.3.2 The non compact case SL_2

It appears to be very similar to the previous one.

Let $G = SL(2, \mathbb{R})$ be the group of order two real matrices with determinant 1. Its Lie algebra is given by the real matrices of trace zero and order two,

$$\mathfrak{sl}(2, \mathbb{R}) = Span\{H, S, A\}, \text{ where } [H, S] = 2A, \ [H, A] = 2S, \ [S, A] = -2H.$$

Consider the unbounded linear system

$$(\Sigma) \qquad \dot{g} = \mathcal{X}_g + uH_g, \text{ where } D = -ad(A) \text{ and } U = \mathbb{R}.$$

As before, we conclude that a necessary condition for the existence of a minimizer is $u = 0$. Thus the only possible minimizers connect points of the same integral curve of the linear vector field

$$\dot{g} = \mathcal{X}_g.$$

5 Acknowledgements

Se agradece a la Universidad Nacional de San Agustín de Arequipa, UNSA, por el financiamiento del proyecto de investigación según Contrato N◦ IAI-014-2018-UNSA.

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