Toward Applications of Linear Control Systems on the Real World and Theoretical Challenges

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Abstract

Many significant real world challenges arise as optimization problems on different classes of control systems. In particular, ordinary differential equations with symmetries. The purpose of this article is twofold. First, give the information we have about the class of Linear Control Systems Σ_G on a low dimension matrix Lie group G. Second, invite the Mathematical community to consider possible applications through the Pontryagin Maximum Principle for Σ_G. And also to challenge some theoretical open problems. The class Σ_G is a perfect generalization of the classical Linear Control System on the Abelian group $\mathbb{R}^n$.

Let G be a Lie group of dimension two or three. Related to Σ_G, this review describes the actual results about controllability, the time-optimal Hamiltonian equations and, the Pontryagin Maximum Principle. We show how to build Σ_G, through several examples on low dimensional matrix groups.

Key words: Linear Control Systems, Matrix Lie groups, Controllability, Pontryagin Maximum Principle

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1 Introduction

Many significant real world challenges arise as optimization problems on different classes of control systems. In particular, ordinary differential equations with symmetries. The purpose of this article is twofold. First, give the information we have about the class of Linear Control Systems Σ_G on a low dimension matrix Lie group G. Second, invite the Mathematical community to consider possible applications through the Pontryagin Maximum Principle for Σ_G. And also to challenge some theoretical open problems. The class Σ_G is a perfect generalization of the classical Linear Control System on the Abelian group $\mathbb{R}^n$.

In this review, we explicitly show the Lie algebras of dimension 2 and 3, the face of the corresponding simply connected Lie groups, the linear and invariant vector fields. They are the main ingredients to build the dynamic of Σ_G. Furthermore, for both dimensions we characterize the Linear Algebra Rank Condition $LARC$, the controllability property, the existence, uniqueness and the shape of control sets in dimension 2. There is no information about control sets for Σ_G on dimension 3. From the applied point of view, we give explicitly the Hamiltonian equations of a time optimal problem for Σ_G, as appear in [9] and [21]. The quadratic optimal problem is an open problem for this class.

It is very well known that control systems, in particular the classical class $\Sigma_{\mathbb{R}^n}$, had been used as a model for beautiful and concrete applications. For instance, in Robotic, Engineering, Medicine, Biology, Physics, Chemistry, Industry, etc., [23],[24],[27],[28],[30],[35],[37], [40],[44],[47] and [50]. And, we are confident that the same is true for Σ_G. There are some reasons to believe that. At first place, the notion of Lie group allows to discover the symmetries of analytical structures. And, also to discover the symmetries of classes of differential equations, [38]. Examples of these manifolds are the Abelian group $\mathbb{R}^n$, the spheres $S^n \subset \mathbb{R}^{n+1}$ for $n = 1, 3$ and 7, the set $GL(n, \mathbb{R})$ of the invertible real matrices of order n , and its relevant subgroups $SL(n, \mathbb{R})$ of matrices with determinant 1, the orthogonal group $O(n, \mathbb{R})$, the spinor group $Spin(n, \mathbb{R})$, the unitary group $U(n, \mathbb{R})$ and many others. Not just for real, but also complex coefficients. However, the main reason comes from the Jouan Equivalence Theorem, [34] which roughly says that: "Any affine control systems on an arbitrary differentiable manifold is equivalent to a linear control system on a Lie group, or on a homogeneous space if and only if the Lie algebra generated by the vector fields of the system is finite-dimensional"

To understand the meaning of the Equivalence Theorem , and also as a general motivation, consider a tumor growth with initial condition x_0 and dynamics determined by a vector field f on the space state M, as the solution $x(t)$ of the associated differential equation,

$$\dot{x(t)} = f(x(t)),\ x(t) \in M, \text{ with } x(0) = x_0.$$

Deciding which combination of treatments is right for a patient is critical. The introduction of the treatments $g^1, ..., g^m$, and the control function $u = (u_1, \ldots, u_m) \in \mathcal{U}$ define an affine control system which changes the behavior of the tumor, as the solutions of the controlled family of differential equations

$$\Sigma_M : \dot{x(t)} = f(x(t)) + \sum_{j=1}^{m} u_j(t) g^j(x(t)),\ x(t) \in M,\ u \in \mathcal{U},\ x(0) = x_0.$$

Here, $\mathcal{U}$ is the admissible class of control functions to be chosen.

This process gives you a way to combine a global tumor treatment in time. Two theoretical-practical problems appear:

- To compute $\mathcal{A}(x_0) \subset M$, the reachable set from x_0 through the controls $u \in \mathcal{U}$, in positive time

- Assume $x_1 \in \mathcal{A}(x_0)$. Starting at x_0, is it possible to reach x_1 with the minimum time?, or with the minimum collateral damage?

The system Σ_M is said to be controllable from x_0 if $\mathcal{A}(x_0) = M$. And, is said to be controllable if it is controllable from any element of M. Let $t > 0$, for technical reasons sometimes it is necessary to consider the set $\mathcal{A}(x_0, t)$, i.e., the reachable set from x_0 through the controls $u \in \mathcal{U}$ up to the time t. And also, the set $\mathcal{A}^*(x_0)$ which are the elements that can be carried to the state x_0 in positive time through the system.

Like the tumor treatment problem, similar questions can be asked in the real world for many situations. From a practical point of view, given a manifold M, the drift f to be controlled and the control vectors $g^1, ..., g^m$, the admissible class of control $\mathcal{U}$ must be properly chosen, according to any real situation.

To establish the Equivalence Theorem, we need to introduce the notion of Lie brackets between vector fields, given by the formula,

$$[X,Y] = X.Y - Y.X, \text{ where } X.Y = \frac{\partial}{\partial X} Y.$$

The Lie algebra $Span_{\mathcal{LA}}\{f, g^1, ..., g^m\}$ denotes the small vector space generated by $f, g^1, ..., g^m$ and closed by the bracket $[,]$.

Theorem 1 *(Jouan Equivalence Theorem,[34]) If for* Σ_M

$$\dim(Span_{\mathcal{LA}}\{f, g^1, ..., g^m\}) < \infty,$$

then, there exits a Lie group G *such that* Σ_M *is equivalent to* Σ_G *or, it is equivalent to* $\Sigma_{G/H}$*, where* G/H *is a homogeneous space.*

Equivalent systems share the same topological, dynamic and algebraic properties. Therefore, it is possible to get information of any arbitrary system Σ_M which satisfy the Jouan condition through a linear system Σ_G or via a homogeneous system $\Sigma_{G/H}$. Here, H is a closed subgroup of G. This is one of the main reason why it is necessary to classify Σ_G for different classes of groups: compact, non-compact, Abelian, nilpotent, solvable, simple, semi-simple, and the direct and semi-direct product between them. Since 1999, a group of mathematician had been working in the structure of Σ_G. Results about controllability, observability , and the *existence, uniqueness, and topological properties of control sets, (maximal regions of controllability), were established for several classes of groups. We refer the readers to the following [5],[6], [7],[8],[11],[14],[15],[16],[17], [18],[19],[20],[26],[31],[32],[33],[34] and [36].

As mentioned before, the main goal of this review is to invite the community to search for real world applications of Σ_G. However, there are also several theoretical open problems to be challenging. For instance, we mention that except for the Euclidean case, any other classical optimal issue for Σ_G, like the quadratic one, is an open problem for Σ_G.

According to the aim of this review article we concentrate on matrix groups. Precisely, for a group G of dimension 2 and 3 we describe the main ingredients to build Σ_G by showing a basis of the Lie algebra $\mathfrak{g}$ of G, the face of $\mathfrak{g}$-derivations and all possible associated linear vector fields. We also mention the existent controllability results and the Hamiltonian equations to apply the Pontryagin Maximum Principle in these cases.

For general view of the theory of control systems we refer the readers to [3],[25],[35] and [49]. For the Lie theory, we mention [29],[39],[46],[48]. For the connection with Sub-Riemannian, almost-Riemmanian geometry we suggest the references [1],[2],[4].

This article is organized as follows. Section 2 introduces the algebraic structures to define Σ_G. Section 3 contains an explicit description of linear control systems and their main properties on groups of dimensions 2 and 3. In Section four we establish the Pontryagin Maximum Principle for Σ_G. Finally, in Section 5 we end with several examples. We include a classical optimal problem on the Euclidean plane, several examples on the 2-dimensional solvable group, and a time optimal theoretical problems on 3-dimensional groups.

2 Matrix groups dynamics and systems

In the sequel, $\mathfrak{gl}(n,\mathbb{R})$ will denote the vector space of the n by n matrices with real coefficients. The open set $GL(n,\mathbb{R}) = [\det(A)^{-1}(0)]^c \subset \mathfrak{gl}(n,\mathbb{R}),.$ is the group of invertible matrices. In fact, $\mathfrak{gl}(n,\mathbb{R})$ is canonically isomorphic to $\mathbb{R}^{n^2}$, the determinant map $\det : \mathfrak{gl}(n,\mathbb{R}) \to \mathbb{R}$ is continuous , and $\det(A)^{-1}(0)$ is closed. We also denote by $GL^+(n,\mathbb{R})$ the connected component of $GL(n,\mathbb{R})$ which contains the identity element $Id = e$. In particular, for any $A \in GL^+(n,\mathbb{R})$, $\det(A) > 0$. It turns out that the topology and the differentiable structure of $GL^+(n,\mathbb{R})$ are the induced ones, coming directly from the Euclidean space $\mathbb{R}^{n^2}$. Here, we just consider closed or path connected subgroups G of $GL^+(n,\mathbb{R})$. Thus, G is a matrix Lie group. In particular, it is possible to define appropriate control systems whenever the dynamic determining the system is well defined on G. The notion of a linear control system on a connected Lie group G depends on two different classes of dynamics: linear and invariant vector fields. For that, we need to introduce the notion of the Lie algebra $\mathfrak{g}$ of G, which is isomorphic to the tangent space T_eG of G at the identity element e. The tangent space T_gG of G at g reads

$$T_gG = \{Y_g : \exists\gamma : (-\varepsilon,\varepsilon) \stackrel{continuous}{\longrightarrow} G, \gamma(0) = g \text{ and } \exists\dot{\gamma}(0) = Y_g\},$$

and $TG = \cup_{g\in G} T_gG$ denotes the tangent bundle.

A vector field P on G is defined by the selection of $P_g \in T_gG$ for any $g \in G$. Since we consider just subgroups G of $GL^+(n,\mathbb{R})$, it follows that

$$T_gG \subset T_gGL^+(n,\mathbb{R}) = g + T_eGL^+(n,\mathbb{R}) = g + \mathfrak{g}l(n,\mathbb{R}).$$

For matrices, the bracket $[X,Y] = XY - YX$, is the commutator.

By definition, the drift $\mathcal{X}$ is a *linear vector field* if its flows $\{\mathcal{X}_t : t \in \mathbb{R}\}$ is a 1-parameter group of $Aut(G)$, the group of G-automorphisms. Moreover, $\mathcal{X}$ induced a derivation

$$\mathcal{D}(\mathcal{X}) = \mathcal{D} : \mathfrak{g} \longrightarrow \mathfrak{g} : \mathcal{D}[X,Y] = [\mathcal{D}X, Y] + [X, \mathcal{D}Y], \; X, Y \in \mathfrak{g},$$

i.e., a linear transformation which respects the Leibnitz rule for the Lie bracket. Furthermore, $\mathcal{X}$ is computed through the following identities

$$\mathcal{X}_g = \left(\frac{d}{dt}\right)_{t=0} \mathcal{X}_t(g), \; \mathcal{X}_t(\exp Y) = e^{t\mathcal{D}}Y, \text{ and } \mathcal{X}_t(gh) = \mathcal{X}_t(g)\mathcal{X}_t(h). \quad (1)$$

In fact, since we just consider connected groups, any element $g \in G$ is a finite product of exponential members of $\mathfrak{g}$.

A special situation happens when the Lie algebra $\mathfrak{g}$ is semi-simple, i.e., when the solvable radical $r(\mathfrak{g})$ of $\mathfrak{g}$ is trivial. Here, $r(\mathfrak{g})$ is defined as the biggest solvable subalgebra of $\mathfrak{g}$. In this case, any derivation $\mathcal{D}$ is inner. It turns out that there exists an invariant vector field $Y = Y(\mathcal{D})$, such that $\mathcal{D} = [, Y]$. Therefore, $\mathcal{D}$ is easily computed by

$$\mathcal{D} = -ad(Y) \Longrightarrow \mathcal{X}_g = gY - Yg, \; g \in G.$$

Denote by $T^n = S^1 \times ... \times S^1$, ($n$-times) the Torus. Therefore, $Aut(T^n) = SL(n, Z)$ is the discrete group of determinant 1 matrices of order n with integer coefficients. It turns out that any linear vector field $\mathcal{X}$ on T^n is trivial, i.e., $\mathcal{X}_g = 0$. In fact, $\{\mathcal{X}_t : t \in \mathbb{R}\}$ should be discrete. Because of that, we do not consider T^n in this analysis.

An *invariant vector field* is determined by G-translations. Precisely, to define a left invariant vector field Y on G we just need to determine a tangent vector at the identity element $Y = Y_e$. Thus, for any $g \in G$ its value $Y_g \in T_gG$ is determined by the derivative at e of the corresponding left translation L_g. Precisely,

$$Y_g = (dL_g)_e(Y_e), \text{ where } (dL_g)_e : T_eG \to T_gG$$

Since we work with matrices, it turns out that Y_g is obtained by the derivative of the curve $\gamma(t) = g\exp(tY)$.Therefore, $Y_g = gY$, with $Y_e = Y$. Thus, $\mathfrak{g}$ is isomorphic to the tangent space T_eG. Right invariant vector fields are defined in a similarly way. The vector space $\mathfrak{g}$ of the left invariant vector fields of G, is a Lie algebra,[46],[48]. In fact, the skew-symmetric bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfy

1. $[P, Q] = -[Q, P]$ skew-symmetric , and the Jacobi identity

2. $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$, for any $X, Y, Z \in \mathfrak{g}$.

Recall that the Lie algebra $\mathfrak{g}$ is said to be:
Abelian, if for any $X, Y \in \mathfrak{g}$, $[X, Y] = 0$.
Nilpotent, if $\exists k \geq 1 : \mathbf{ad}^1 = [\mathfrak{g}, \mathfrak{g}] \supset ... \supset \mathbf{ad}^{k+1} = \left[\mathbf{ad}^k, \mathfrak{g}\right] = 0$.
Solvable, if $\exists k \geq 1 : \mathbf{ad}^1 \supset ... \supset \mathbf{ad}^{(k)} = \left[\mathbf{ad}^{(k-1)}, \mathbf{ad}^{(k-1)}\right] = 0$.
Semisimple, if the largest solvable subalgebra $\mathfrak{r}(\mathfrak{g})$ of $\mathfrak{g}$ is trivial.
Finite semi-simple center if any semi-simple subalgebra has trivial center.
Finally, the group G is said to be Abelian, nilpotent, solvable, semi-simple, and finite semi-simple center, if $\mathfrak{g}$ satisfy the corresponding property .

2.1 The notion of linear control system on G

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. By definition, see [17], [36], a linear control systems Σ_G is determined by the differential equations

$$\Sigma_G : \ \dot{g}(t) \ = \mathcal{X}(g(t)) + \sum_{j=1}^{m} u_j(t) \ Y^j(g(t)), \ g(t) \in G, \ u \in \mathcal{U},$$

parametrized by $u \in \mathcal{U}$. Here, $\mathcal{X}$ is a linear vector field which means that its flows $\{\mathcal{X}_t : t \in \mathbb{R}\} \subset Aut(G)$ is a 1-parameter group of G-automorphisms. And, for any $j = 1, ..., m$, the control vector $Y^j \in \mathfrak{g}$ is a left invariant vector field. We consider a large class of admissible control functions

$$\mathcal{U} \ = \{u : [0, T_u] \longrightarrow \Omega \subset \mathbb{R}^m : \text{u is locally integrable}\},$$

where Ω is a closed and convex subset with $0 \in int(\Omega)$. Just observe that $\mathcal{U}$ contains the constant, continuous and also differentiable functions.

Because of that, given any initial condition $g \in G$ and each control $u \in \mathcal{U}$ there exist a solution $\varphi(g, u, t)$ of Σ_G. Precisely,

$$\varphi(g, u, t) = \mathcal{X}_t(g) \ \varphi(e, u, t),$$

where $\varphi(e, u, t)$ is the solution of the system with the same control u through the identity element e, [17]. We denote the reachable set from e by

$$\mathcal{A} = \{\varphi(e, u, t) : u \in \mathcal{U}\}, \ \text{and the set,}$$

$$\mathcal{A}^* = \{g \in G : \exists u \in \mathcal{U}, \text{ and } t > 0, \text{ with } \varphi(g,u,t) = e\},$$

of state of the group that can be transferred to the identity in a positive time.

We call the system unbounded if $\Omega = \mathbb{R}^m$, and bounded if Ω is compact. From now $\mathcal{U}_{\mathbb{R}^m}$ will denote the unbounded case, $\mathcal{U}_{\Omega}$ the bounded one and $\mathcal{U}$ when both condition are possible.

Therefore, the notion of Σ_G is a natural extension of the classical linear control system $\Sigma_{\mathbb{R}^n}$ on the Abelian group $\mathbb{R}^n$.

Recall that the system satisfy the Lie algebra rank condition $(LARC)$, if

$$Span_{\mathcal{LA}}\left\{\mathcal{X}, Y^1, \ldots, Y^m\right\} = \mathfrak{g}.$$

More strong, the system satisfy the ad-rank condition $adRC$ if

$$Span\left\{\mathcal{D}^k(Y^j) : j = 1, ..., m, \; k \geq 0\right\} = \mathfrak{g}.$$

We denote by Δ the Lie algebra generated by the control vectors, i.e.,

$$\Delta = Span_{LA}\left\{Y^1, ..., Y^m\right\}.$$

Finally, we introduce the notion of a control set, which means a region where controllability holds in its interior. Let Σ_G be a linear control system. A subset $\mathcal{C}$ is said to be a control set if

1. for any $g \in G$, $\exists u$ such that $\varphi(g,u,t) \in \mathcal{C}$, $t \geq 0$
2. $\mathcal{C} \subset cl(\mathcal{A}(g))$, for any $g \in \mathcal{C}$
3. $\mathcal{C}$ is maximal with respect 1 and 2.

Furthermore, the control set $\mathcal{C}$ is said to be positive invariant if,

$$g \in \mathcal{C} \Rightarrow \varphi(g,u,t) \in \mathcal{C}, \text{ for any } u \in \mathcal{U}, \; t > 0,$$

Here, cl denotes the topological closure.

2.2 The classical Linear control system on $\mathbb{R}^n$

In the Euclidean Abelian group

$$\mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, \; i = 1, \ldots, n\}$$

the classical linear control system is as follows

$$\Sigma_{\mathbb{R}^n} : \dot{x}(t) = Ax(t) + Bu, \; u \in \mathcal{U}.$$

In this case, $\mathcal{D}$ corresponds to the matrix A of order n, which flows $\left\{e^{tA} : t \in \mathbb{R}\right\} \subset Aut(\mathbb{R}^n)$ is computed from the exponential map. On the other hand, $Bu \in \mathbb{R}^n$ can be written as $\sum_{j=1}^{m} u_j b^j$, where $b^1, ..., b^m$, are the column vectors of the n by m matrix B . Since, any vector $b \in \mathbb{R}^n$ induces by translation an invariant vector field on $\mathbb{R}^n$, it is now clear that Σ_G is a perfect extension of $\Sigma_{\mathbb{R}^n}$.

In [49] the solution with initial condition and control u is given by,

$$\varphi(x_0, u, t) = e^{tA} x_0 + \int_0^t e^{-\tau A} Bu(\tau) d\tau.$$

2.3 The $\mathcal{D}$-decomposition of $\mathfrak{g}$

The dynamic behavior of Σ_G strongly depends on the spectrum of $\mathcal{D}$ associated to $\mathcal{X}$. We consider the Lie algebra decomposition of $\mathfrak{g}$ induced by $\mathcal{D}$. For any $\alpha \in Spec(\mathcal{D})$, the $\mathcal{D}$-generalized eigenspaces reads

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} : (\mathcal{D} - \alpha Id)^n Y = 0, \text{ for some } n \geq 1.$$

It turns out that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta \in Spec(\mathcal{D})$ and 0 otherwise. And, $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$, where

$$\mathfrak{g}^+ = \bigoplus_{\mathrm{Re}(\alpha) > 0} \mathfrak{g}_\alpha, \; \mathfrak{g}^0 = \bigoplus_{\mathrm{Re}(\alpha) = 0} \mathfrak{g}_\alpha \; and \; \mathfrak{g}^- = \bigoplus_{\mathrm{Re}(\alpha) < 0} \mathfrak{g}_\alpha.$$

$\mathfrak{g}^+, \mathfrak{g}^0, \mathfrak{g}^-$ are Lie subalgebras and $\mathfrak{g}^+$, $\mathfrak{g}^-$ are nilpotent, [26] and [46].

We finish this section to mention some differences between Σ_G and the well known class of invariant control system Inv_G on G, [42], [43]. For Inv_G the drift is also an element of the Lie algebra $\mathfrak{g}$. One of the main difference comes from the fact that the reachable set from the identity is a semigroup for Inv_G but not for Σ_G. This difference has important consequences. For example, the controllability property of Inv_G turns a local property, which is no the case for Σ_G, [15]. On the other hand, the Hamiltonian lifting of the linear vector field is more complicated because the invariant field does not depends on the state.

3 Linear control systems and controllability

For a group G of dimension 2 and 3, in the sequel, we describe the main ingredients to build Σ_G. We show a basis of the Lie algebra $\mathfrak{g}$ of G, the face of all $\mathfrak{g}$-derivations, and the associated linear vector fields. We also mention some normal form of Σ_G associated with the $LARC$ property, which we always assume, without loss of generality. Finally, we characterize the controllability property of each system, [1]

3.1 The dimension 2

In dimension 2 there are just two Lie algebras: the Abelian and solvable.

3.1.1 The Abelian structure

The Lie algebra $\mathbb{R}^2$ is generated by the basis $X = e_1, Y = e_2$, with $[X, Y] = 0$. The associated group is also $\mathbb{R}^2$ with the Abelian structure

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

induced by its Lie algebra. The Left invariant vector fields are given by

$$X_g = \frac{\partial}{\partial x}, Y_g = \frac{\partial}{\partial y}, \text{ where } g = xe_1 + ye_2.$$

Any matrix $\mathcal{D} = A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $a_{i,j} \in \mathbb{R}$ where $i, j = 1, 2$, is a derivation, determining the linear vector field $\dot{x}(t) = Ax(t)$. Therefore, any classical linear control system on the Abelian group has the face

$$\Sigma_{\mathbb{R}^2} : \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u, \ u \in \mathcal{U}_{\mathbb{R}}, \text{ or } u \in \mathcal{U}_{\Omega}.$$

[1] A few words about assuming $LARC$. For a regular (constant dimension) and involutive (closed by Lie brackets) distribution Ψ generated by a family of vector fields Γ on a manifold M, the Sussmann Orbit Theorem implies that the integral manifold $M(\Psi)$ of Ψ, a submanifold of M is built by the concatenation of the integral curves of the vector fields in Γ, in positive and negative time. In particular, if $M(\Psi) \subsetneq M$ we just consider Γ on $M(\Psi)$. Therefore, by assuming that Σ_G satisfy the $LARC$ property, it turns out that $G(\Psi) = G$, when

$$\Psi = Span_{\mathcal{L}A} \left\{ \mathcal{X}, Y^1, ..., Y^m \right\}.$$

In reference [25] it is possible to find the controllability results for the unbounded and bounded admissible control functions, see also [49]. If $\mathcal{U} = \mathcal{U}_{\mathbb{R}^2}$, the $LARC$ property is equivalent to the controllability property, which is equivalent to the Kalman rank condition. Precisely,

Theorem 2 *Let $\Sigma_{\mathbb{R}^2}$ be a linear control system with $\mathcal{U} = \mathcal{U}_{\mathbb{R}^2}$. Then,*

$$\Sigma_{\mathbb{R}^2} \text{ is controllable } \Leftrightarrow \ \operatorname{rank}(b\ Ab) = 2,\ b = (b_1 b_2)^T.$$

Furthermore, consider $u_* < 0 < u^*$ and $\Omega = [\ u_*, u^*]$. If $\mathcal{U} = \mathcal{U}_\Omega$, it holds

1. $\Sigma_{\mathbb{R}^2}$ is controllable $\Leftrightarrow$ Kalman rank condition and $Spec_{Ly}(\mathcal{D}) = \{0\}$.
2. Around the identity element 0, there exists an unique control set with non-empty interior, given by
$$\mathcal{C} = cl(\mathcal{A}) \cap \mathcal{A}^*.$$

3.1.2 The Solvable structure

Here, we follow the reference [33]. The Lie algebra $\mathfrak{a}ff_+(\mathbb{R})$ is realized as the vector space of the real matrices of order 2 generated by $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

$Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, with $[X, Y] = Y$. The simply connected Lie group reads as

$$G = \left\{ g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0,\ y \in \mathbb{R} \right\} \cong Aff_+(\mathbb{R}) = \mathbb{R}_+ \times \mathbb{R},$$

with the solvable algebraic structure

$$(x_1, y_1) \times (x_2, y_2) = (x_1 x_2, x_1 y_2 + x_1)$$

The left invariant vector fields are given by $X_g = gX = x\frac{\partial}{\partial x}$ and $Y_g = gY = x\frac{\partial}{\partial y}$. Any derivation

$$\mathcal{D} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, a, b \in \mathbb{R}$$

is inner, and induces the linear vector field

$$\mathcal{X}_g = \begin{pmatrix} 0 & a(x-1) + by \\ 0 & 0 \end{pmatrix},\ \ a, b \in \mathbb{R}.$$

Next, we follow reference Jouan. The linear system Σ_G

$$\dot{g} = \mathcal{X}_g + ug(cX + dY)$$

satisfies $LARC$ if and only if

$$\Sigma_G : \begin{cases} \dot{x} = & ucx, c \neq 0 \\ \dot{y} = & x - 1 + by \end{cases}, \; u \in \mathcal{U}.$$

In this case, the linear system Σ_G is said to be in normal form.

Theorem 3 *[33] Assume, Σ_G is in normal form, therefore*

$$\Sigma_G \text{ is controllable } \Leftrightarrow b = 0.$$

In [5], the authors compute the control set of any linear control system Σ_G on G, with and without non-empty interior. There are six classes. Here, we just show three.

1, Let us consider

$$\Sigma_G : \begin{cases} \dot{x} = ux \\ \dot{y} = by \end{cases}, \quad \text{where} \quad u \in \Omega = [\, u_*, u^*] \,, \tag{2}$$

whose solutions starting at $(x, y) \in G$ are given by concatenations of the flows

$$\varphi(t, (x, y), u) = (\mathrm{e}^{tu}x, \mathrm{e}^{tb}y), \;\; t \in \mathbb{R}.$$

As a matter of fact, the only control set is given by $\mathcal{C} = \mathbb{R}_+ \times \{0\}$.

2. Let us consider the linear system

$$\Sigma_G : \begin{cases} \dot{x} = 0 \\ \dot{y} = a(x-1) + ux \end{cases}, \quad \text{where} \quad u \in \Omega = [\, u_*, u^*] \,, \tag{3}$$

whose solution with initial condition (x, y) and control u is given by

$$\varphi((x, y), u, t) = (x, (a(x-1) + ux)t + y), \; t \in \mathbb{R}.$$

Here, there are an infinity number of control sets. In fact, for any $x \in \mathbb{R}$

$$\mathcal{C}_x = \{x\} \times \mathbb{R}, \text{ when } x \in (1 - \varepsilon, 1 + \varepsilon), \text{ and any } y \in \mathbb{R}$$

for an existent positive number ε.

3. Consider the linear system

$$\Sigma_G : \left\{ \begin{array}{l} \dot{x} = ux \\ \dot{y} = by + ux \end{array} \right. , \quad \text{where } , b < 0, \;\; u \in \Omega = [\, u_*, u^*] \,, \tag{4}$$

In this case it is possible to prove that the only one control set with non-empty interior is given by

$$\mathcal{C} = cl\mathcal{A}(x,y) \text{ for any } (x,y) \in \mathcal{C}$$

3.2 The dimension 3

In this Section, we describe all the ingredients to build a linear control system on any 3-dimensional matrix Lie groups. In particular, in the Abelian, nilpotent, solvable, finite center, semi-simple compact and non-compact case.

3.2.1 The Abelian structure

Here, $\mathbb{R}^3$ is generated by the basis $X = e_1, Y = e_2$, $Z = e_3$. And, all he brackets are null. The associated group is also $\mathbb{R}^3$ with the Abelian structure

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2),$$

induced by its Lie algebra. The Left invariant vector fields are given by

$$X_g = \frac{\partial}{\partial x},\; Y_g = \frac{\partial}{\partial y},\; Z_g = \frac{\partial}{\partial z} \text{ where } g = xe_1 + ye_2 + ze_3.$$

Any matrix $\mathcal{D} = A = (a_{i,j})$, $a_{i,j} \in \mathbb{R}$ $i,j = 1,2,3$, is a derivation, determining the linear vector field $\dot{x}(t) = Ax(t)$. Therefore, any classical linear control system on the Abelian group has the face

$$\Sigma_{\mathbb{R}^3} : \dot{x}(t) = Ax(t) + Bu(t),\;\; u \in \mathcal{U}.$$

Here, the matrix $B = (b_{i,j})$, with the possibilities: $i = 1,2,3$; $j = 1$, and $u_1 \in \mathcal{U}$, or, $i = 1,2,3$; $j = 1$, 2 and $u_1 \in \mathcal{U}$, $u_2 \in \mathcal{U}$.

In reference [25] it is possible to find the controllability results for the unbounded and bounded admissible control functions, see also [49]. If $\mathcal{U} = \mathcal{U}_{\mathbb{R}^3}$, the $LARC$ property is equivalent to the controllability property which is also equivalent to the Kalman rank condition. Precisely,

Theorem 4 *Let $\Sigma_{\mathbb{R}^3}$ be a linear control system with $\mathcal{U} = \mathcal{U}_{\mathbb{R}^3}$. Then,*

$$\Sigma_{\mathbb{R}^3} \text{ is controllable } \Leftrightarrow \operatorname{rank}(B\ AB\ A^2B) = 3.$$

Furthermore, consider $u_* < 0 < u^*$ and $\Omega = [\ u_*, u^*]$. If $\mathcal{U} = \mathcal{U}_\Omega$, it holds

1. $\Sigma_{\mathbb{R}^3}$ is controllable $\Leftrightarrow$ Kalman rank condition and $Spec_{Ly}(\mathcal{D}) = \{0\}$.
2. Around the identity element 0, there exists an unique control set with non-empty interior, given by

$$\mathcal{C} = cl(\mathcal{A}) \cap \mathcal{A}^*.$$

In what follows we denote by $M(i,j)$, the matrix of order three with all coefficients 0, except 1 at the position (i,j).

3.2.2 The nilpotent structure

The Heisenberg Lie algebra is the only nilpotent Lie algebra of dimension three. It is generated by $X = M(1,2)$, $Y = M(2,3)$, and $Z = M(1,3)$. The simply connected Heisenberg Lie group reads as

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \longleftrightarrow (x, y, z) \in \mathbb{R}^3.$$

The left invariant vector fields and its brackets are given by

$$X_g = \frac{\partial}{\partial x}, Y_g = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, Z_g = \frac{\partial}{\partial z}; \ \ [X,Y] = Z, [X,Z] = [Y,Z] = 0.$$

An arbitrary derivation of the Heisenberg has the form,

$$\mathcal{D} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & a+d \end{pmatrix} : a, b, c, d, e, f \in \mathbb{R},$$

and induces the general linear vector field on G,

$$\mathcal{X}(g) = (ax + by)\frac{\partial}{\partial x} + (cx + dy)\frac{\partial}{\partial y} + (ex + fy + (a+d)z + \frac{1}{2}cx^2 + \frac{1}{2}by^2)\frac{\partial}{\partial z}.$$

Next, we follow reference [33]. The authors consider two different situations: one and two control vectors. On the Heisenberg Lie group, $\mathcal{U} = \mathcal{U}_{\mathbb{R}^m}$.

The single control vector

Any linear control system on the Heisenberg group satisfy the *LARC* property if and only if it is written in normal form

$$\Sigma_G : \dot{g} = \mathcal{X}_g + uX_g,\ u \in \mathcal{U},$$

where, $\mathcal{X}$ comes from the derivation $\begin{pmatrix} 0 & b & 0 \\ 1 & d & 0 \\ 0 & f & d \end{pmatrix}$, and $X = M(1,2)$.

Theorem 5 *([33]) If Σ_G is in normal form, then*

$$\dot{g} = \mathcal{X}_g + uX_g \text{ is controllable} \Longleftrightarrow b < -\tfrac{d^2}{4} \Leftrightarrow d = 0,\ f \neq 0.$$

It is possible to obtain algebraic conditions for a 1-input linear system on the Heisenberg group when the system is not written in normal form, Jouan.

The two control vectors

Any linear control system which satisfy *LARC* reads as follows,

$$\Sigma_G : \dot{g} = \mathcal{X}_g + u_1 P_g + u_2 Q_g,\ u \in \mathcal{U}$$

where $P, Q \in \mathfrak{g}$ are linear independent invariant vector fields on G.

Theorem 6 *([33]) Let $\Sigma_{\mathbb{G}}$ be a linear control system with two control vectors and $\mathcal{U} = \mathcal{U}_{\mathbb{R}^3}$. Then,*

$$\Sigma_G \textit{ is controllable} \Leftrightarrow \Sigma_G \textit{ satisfy LARC.}$$

3.2.3 The solvable structure

According to Onishchik and Vinberg, [39], there are five nonisomorphic, non-nilpotent solvable Lie algebras of dimension three. On the other hand, for these five classes of Lie algebras, the authors in [6], describe general formulas through the semi-direct product to compute the simply connected Lie groups associated to the Lie algebras, the derivations and its associated linear vector fields in any case. Furthermore, they characterize the controllability property on any linear control systems which satisfy the *LARC* property for one and two control vectors. We also inform the redears that we do not have information about control sets.

Let us consider the canonical basis,

$$\mathfrak{g} = Span\left\{X = (1,0,0), Y = (0,1,0), Z = (0,0,1)\right\}.$$

The classification goes through the notion of the semi-direct product between the Abelian Lie algebras $\mathbb{R}$ and $\mathbb{R}^2$. Precisely,

$$\mathfrak{g}_\theta = \mathbb{R} \otimes_\theta \mathbb{R}^2 : [(z_1, v_1), (z_2, v_2)] = (0, z_1\theta v_2 - z_2\theta v_1) \in \mathfrak{g}.$$

Up to automorphisms, these solvable Lie algebras depend on a matrix θ.

The Lie algebra $\mathfrak{r}_2 : \theta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $[X, Z] = Z, [X, Y] = 0$.

The Lie algebra $\mathfrak{r}_3 : \theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $[X, Y] = Y, [X, Z] = Y + Z$.

The Lie algebra $\mathfrak{r}_{3,\lambda} : \theta = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$, $\lambda \in \mathbb{R}, \lambda \neq 0$, $[X, Y] = Z$,
$[X, Z] = -Y$.

The Lie algebra $\mathfrak{r}^*_{3,\lambda} : \theta = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$, $\lambda \in \mathbb{R}, \lambda \neq 0$, $[X, Y] = \lambda Y$,
$[X, Z] = -Y + \lambda Z$.

The Lie algebra $\mathfrak{e} : \theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $[X, Y] = Z$, $[X, Z] = -Y$.

Let us show how it works this special product. By the own definition, for every solvable class, $[Y = (0, 1, 0), Z = (0, 0, 1)] = (0, 0, 0)$. For $\mathfrak{r}_2$, the bracket $[X, Z]$ is computed as follows

$$[X, Z] = (0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}) = Z.$$

The solvable Lie groups

The simply connected Lie groups $R_2, R_3, R_{3,\lambda}, R^*_{3,\lambda}$ and E with Lie algebras $\mathfrak{r}_2$,$\mathfrak{r}_3$, $\mathfrak{r}_{3,\lambda}$, $\mathfrak{r}^*_{3,\lambda}$ and $\mathfrak{e}$ respectively, are computed by the semi-direct product $G_\theta = \mathbb{R} \otimes_\rho \mathbb{R}^2$ through the representation ρ and the formula

$$\rho_\theta(t) = e^{t\theta}, \text{ where } e^{t\theta} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\theta^n,\ \theta^0 = Id,\ t \in \mathbb{R}.$$

Furthermore, any connected Lie group G which solvable Lie algebra is determined by the matrix θ, is given by a homogeneous space G_θ/P, where P is a discrete subgroup of G_θ. For more details, see [6].

Linear and left invariant vector fields

Next, we show the face of the dynamic involved in Σ_G on any three dimensional solvable non-nilpotent simply connected Lie group G.

Algebra, Group	Left invariant vector fields	Linear vector fields
$\mathfrak{g}_\theta = \mathbb{R} \otimes_\theta \mathbb{R}^2$	$Y = (a, w) \in \mathfrak{g}_\theta$	$\mathcal{X}(t, 0) = (0, \Lambda_t \xi)$
$G_\theta = \mathbb{R} \otimes_\rho \mathbb{R}^2$	$Y(t, v) = (a, \rho_t w)$	$\mathcal{X}(t, v) = (0, \mathcal{D}^* v + \Lambda_t \xi)$

where $\mathcal{D}^*$ is defined through the formula $\mathcal{D}(0, v) = (0, \mathcal{D}^* v)$, and

$$\Lambda_t = \left\{ \begin{array}{cc} (\rho_s - 1)\theta^{-1} & \det(\theta) \neq 0 \\ \begin{pmatrix} s & 0 \\ 0 & e^s - 1 \end{pmatrix} & \det(\theta) = 0. \end{array} \right\}.$$

Any linear control system Σ_G on a three-dimensional, solvable, connected, nonnilpotent Lie group G that satisfies $LARC$ is equivalent to one of the following linear systems:

1. The one control vector,

$$\dot{g} = \mathcal{X}(g) + uY^1(g), \text{ where } Y_1 = (1, 0).$$

2. The two control vectors,

$$\dot{g} = \mathcal{X}(g) + u_1 Y^1(g) + u_2 Y^2(g), \text{with } Y_1 = (1, 0), Y_2 = (0, w), \text{some } w \in \mathbb{R}^2$$

Next, we present a characterization of the controllability property in both cases.

The single control vector

As always, we assume the $LARC$ property.

Theorem 7 *([6]) Let $\Sigma_G : \dot{g} = \mathcal{X}(g) + uY^1(g)$, $Y_1 = (1, 0)$. Then,*

1. *If $G = R_2$: Σ_G is controllable $\Leftrightarrow \mathfrak{g}^0 \simeq \mathfrak{aff}(\mathbb{R})$*

2. *If $G = E_2$ or R_3: Σ_G is controllable $\Leftrightarrow \mathfrak{g} = \mathfrak{g}^0$ and $\mathcal{D}^* \not\equiv 0$*

3. *If $G = R_{3,\lambda}$: Σ_G is controllable $\Leftrightarrow \lambda = 1$ and $\mathcal{D}^*$ has a pair of complex eigenvalues*

4. *If $G = R'_{3,\lambda}$: Σ_G is controllable.*

The two control vectors

Let Σ_G be a linear control system with two control vectors which satisfy the $LARC$ property. For the controllability property it holds,

Theorem 8 *([6]) Let* $\Sigma_G : \dot{g} = \mathcal{X}(g) + u_1 Y^1(g) + u_2 Y^2(g)$,

$Y_1 = (1, 0)$ and $Y_2 = (0, w)$, for some $w \in \mathbb{R}^2$. Then,

Theorem 9

1. *If* $G = R_2$: Σ_G *is controllable* $\Leftrightarrow \dim \mathfrak{g}^0 > 1$ *or* $\dim \mathfrak{g}^0 = 1$ *and* $\Delta \simeq \mathfrak{aff}(\mathbb{R})$
2. *If* $G = E_2$*,or* $R'_{3,\lambda}$*:* Σ_G *is controllable*
3. *If* $G = R_3$*:* Σ_G *is controllable* $\Leftrightarrow \mathfrak{g} = \mathfrak{g}^0$
4. *If* $G = R_{3,\lambda}$*:* Σ_G *is controllable* $\Leftrightarrow \ker \mathcal{D}^* \not\subset \Delta$ *or* $\mathcal{D}$ *has a pair of complex eigenvalues.*

3.2.4 Finite semi-simple center structure

In [8] the authors introduce the notion of a finite semi-simple Lie algebra $\mathfrak{g}$,which means that any semi-simple subalgebra of $\mathfrak{g}$ has a finite center. In particular, any solvable Lie algebra has this property. But also, any semi-simple Lie algebra with a finite center like the skew-symmetric algebra $\mathfrak{s}o(n, \mathbb{R})$ and $\mathfrak{sl}(n, \mathbb{R})$, the trace zero matrices of order n. In fact, in both cases the center $\mathfrak{z}$ is discrete: the identity Id , and $\pm Id$, respectively. In the sequel, we consider two topological different cases.

3.2.5 The compact semi-simple structure

The vector space $\mathfrak{g} = Span\{X, Y, Z\}$ is the Lie algebra of skew-symmetric matrices of order three, generated by the basis

$$X = M(1, 2) - M(2, 1), Y = M(2, 3) - M(3, 2), M(1, 3) - M(3, 1).$$

The brackets rules are $[X, Y] = Z$, $[X, Z] = -Y, [Y, Z] = X$.

Since $\mathfrak{g}$ is semi-simple, any derivation is inner. It turns out that $\mathcal{D} = -ad(U)$, for $U \in \mathfrak{so}(3, \mathbb{R})$. Thus, the linear vector field associated to $\mathcal{D}$ reads

$$\mathcal{X}_g = gU - Ug, \ U \in \mathfrak{g}.$$

The connected Lie group with Lie algebra $\mathfrak{so}$(3,$\mathbb{R}$) is the orthogonal group with positive determinant $SO(3, \mathbb{R})$, i.e., it is the rotational group of $\mathbb{R}^3$.

Any linear control system on the compact group $SO(3,\mathbb{R})$, reads as

$$\Sigma_G : \dot{g} = \mathcal{X}_g + u_1 P_g + u_2 Q_g,\ \ P, Q \in \mathfrak{g},\ \ u_1, u_2 \in \mathcal{U}.$$

The single control vector case comes from $u_2 = 0$.

Since $SO(3,\mathbb{R})$ is a topological compact manifold, it turns out that $LARC$ is equivalent to the controllability property, [15]. Thus, in both cases, we have

$$\Sigma_G \text{ is controllable } \Leftrightarrow Span_{\mathcal{LA}}\{\mathcal{X}, P, Q\} = \mathfrak{so}(3,\mathbb{R}).$$

3.2.6 The non-compact semi-simple structure

Here, we follow the reference [7]. The Iwasawa decomposition of

$$\mathfrak{sl}(2;\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

is given by

$$\mathfrak{k} = Span\left\{X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}, \mathfrak{a} = Span\left\{Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}, \mathfrak{n} = Span\left\{Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\},$$

with brackets $[X, Y] = 2Z - 4X$, $[X, Z] = Y, [Y, Z] = 2Z$.

According to this data, and through the exponential map the Lie group decomposes as the product $G = KAN$, where

$$K = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : 0 \leq \theta < 2\pi\right\}, A = \left\{\begin{pmatrix} x & 0 \\ 0 & \frac{1}{x} \end{pmatrix} : x > 0\right\}, N = \left\{\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} : y \in \mathbb{R}\right\}$$

is the compact, Abelian and nilpotent part of G, respectively. Furthermore, any element in G has the form $g = (\theta, x, y)$.

Any $U = X, Y, Z$ give raise a linear vector field as follows,

$$\mathcal{D} = -ad(U) \Longrightarrow \mathcal{X}_g = gU - Ug.$$

About the controllability, we have here some results for local controllability on the identity element, some controllability relationship between the linear systems and an associated invariant system, [15]. Finally, we describe a result that appear very recently involving the notion of a control set, which roughly means a region where the system is controllable in its interior. And the linear system will be controllable if this set is positive invariant.

Theorem 10 *([7]) For a linear system on a semisimple Lie group G with the LARC property, we have*

1. $\exists \mathcal{C}$, a control set with nonempty interior around e given by,

$$\mathcal{C} = cl(\mathcal{A}) \cap \mathcal{A}^*$$

2. The only possible invariant control set is the whole group, i.e.,

$$\text{if } \mathcal{C} \text{ is positive invariant then } \mathcal{C} = G.$$

Thus, the system Σ_G is controllable.

4 The Pontryagin Maximum Principle

The Pontryagin Maximum Principle is one of the main theoretical and useful optimal results available in the literature. Lev Pontryagin, a theoretical topological Mathematician, publish this result on his book *Theory of Optimal Processes*, [40], in collaboration with several researchers of the Steklov Institute in Russia. For his work during the period 1956-1961, he got the Lenin Price. The principle is fundamental to the control system theory and its applications. But, also is instrumental for the Carnot-Caratheodory geometry, the sub-Riemannian , and the almost-Riemannian geometries with applications in several areas. The mathematical machinery to establish and prove these results strongly depends on the associated Hamiltonian equations, which are obtained from a geometric point of view by lifting the initial system to the co-tangent bundle of the state manifold, [3].

The Principle works on the co-tangent bundle T^*M of the manifold state M. Our case is quite favorable, since T^*G is a trivial bundle. Precisely,

$$T^*G \cong \mathfrak{g}^* \times G,$$

where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of the connected group G. To establish this Principle on Σ_G, we recall the Hamiltonian equations associated to Σ_G, as appear in [9] and [21].

For a given admissible control $u \in \mathcal{U}$, the associated Σ_G-Hamiltonian H_u defined on $\mathfrak{g}^* \times G$ reads as

$$H_u(\lambda, g) = \ < \lambda, F(g) > + \sum_{j=1}^{m} \langle \lambda, Y^j \rangle >, \ g \in G, \tag{5}$$

where, $\lambda : \mathfrak{g} \longrightarrow \mathbb{R}$, $F(g) = (dL_{g^{-1}})_g(\mathcal{X}_g)$, and $Y^j = (dL_g)_{g^{-1}}Y_g^j$.

Just observe that the system was translated from $g \in G$ to the identity.

Next, we introduce the Pontryagin Maximum Principle for the class of linear control systems on a matrix Lie group.

Theorem 11 *[21] Let $u^* : [0,T] \longmapsto \Omega \subset \mathbb{R}^m$ be an admissible control such that the solution $\varphi(, u^*, t)$ of Σ_G minimizes the time among all Σ_G-admissible curves sending $\varphi(, u, 0)$ to $\varphi(, u, T)$. Then, there exists a Lipschitzian curve $(\lambda(t), g(t)) \in T^*G$ such that*

1. $\lambda(t) \neq 0$ for all $t \in [0,T]$. And, for almost all $t \in [0,T]$
2. $H_{u*}(\lambda(t), g(t)) = \max_{u \in \mathcal{U}} H_u(\lambda(t), g(t))$
3. $H_{u*}(\lambda(t), g(t)) \geq 0$.

Furthermore, $(\lambda(t), g(t)) \in \mathfrak{g}^* \times G$ satisfy the Hamiltonian equations

$$\dot{g} = \mathcal{X}_g + \textstyle\sum_{j=1}^m u_j Y^j(g) \quad \text{and} \quad \dot{\lambda} = \lambda \circ (-\mathcal{D} + \textstyle\sum_{j=1}^m u_j \; ad(Y^j). \qquad (6)$$

Here, $-\mathcal{D}$ and $ad(Y^j) = [Y^j,]$ are the lifting vector fields from G to T^*G of the corresponding dynamic $\mathcal{X}$ and Y^j, $j = 1, ..., m$. In the general case of a system Σ_M on a differentiable manifold M, this construction depends on a differentiable non degenerate 2-form which always exists on T^*M. The second relationship is a differential equation on $\mathfrak{g}^*$ induced by $(1+m)$ derivations.

Next, we collect the main results we know about the time optimal problem for Σ_G, as appears in [21]. Let Σ_G be a linear control system, which includes the constant control $u \in \Omega \in \mathbb{R}^m$.

We consider first the unbounded case. Since H_u is maximum, then $\langle \lambda, Y^j \rangle = 0$, for $j = 1, \ldots, m$. Moreover, if $(\lambda(t), g(t))$ satisfy the Pontryagin Maximum Principle, it follows that,

$$\langle \dot{\lambda}(t), Y^j \rangle = 0, \text{ and } \langle \lambda(t), -DY^j + \sum_{i=1}^m u_i [Y^i, Y^j] \rangle = 0, \quad \text{a.e., } j = 1, \ldots, m.$$

For the bounded case, let $B > 0$, and $-B \leq u_j \leq B$ for $j = 1, \ldots, m$. Since H_u is maximum, it turns out that

$$u_j = sign\langle \lambda, Y^j \rangle B, \quad \text{if } \langle \lambda, Y^j \rangle \neq 0,$$

Otherwise, u_j is not determined. However, according to Filippov's Theorem in [3], minimizers exist. Moreover, if $[Y^i, Y^j] = 0$ for $i, j = 1, \ldots, m$, then we get an extra equation $\langle \lambda(t), DY^j \rangle = 0$. In the classical case $\Sigma_{\mathbb{R}^n}$, $ad(b_j) = 0$, since the Lie algebra $\mathbb{R}^n$ is Abelian. But on groups, it is not the case. In particular, the bang bang control theorem for $\Sigma_{\mathbb{R}^n}$, is not longer true for Σ_G, see [21].

5 Examples

In this Section, we develop several examples on different groups of dimensions 2 and 3. We start with the more famous one: *Stop a train in a station in minimum time,* coming from the Pontryagin book, [40].

Example 12 *Let us consider a train Γ of mass one moving on the real line without friction. Denote by $x(t)$ the distance from Γ to the origin (the station) at time t. From the Newton law, we have $\dot{x}(t) = y(t)$ and $\dot{y}(t) = u(t)$. Here, $u \in \mathcal{U}_\Omega$ with $\Omega = [-1.1] \subset \mathbb{R}$. Thus, we obtain a classical control system on the Abelian Lie group $\mathbb{R}^2$, as follows*

$$\Sigma_G : \dot{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + u(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; u \in \mathcal{U}_\Omega.$$

Any control u determine an ordinary differential equation. Geometrically, any initial condition (x, y) should be steer to $(0, 0)$ in minimum time. The system is restricted, satisfy the Kalman rank condition, $\text{rank}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2$ and $Spec(A)_{Ly} = \{0\}$. Hence, according to Theorem 2, the system is controllable.

From the Pontryagin Maximum Principle, we should consider two class of controls: $u \equiv 1$ and $u \equiv -1$ and the minimum time curve is built with at most one change of the control.

The family of parabolas generated by the solutions of the differential equations

$$\dot{x}(t) = y(t), \; \dot{y}(t) = -1 \text{ and } \dot{x}(t) = y(t), \; \dot{y}(t) = 1$$

generates a curve built through two specific parabolas reaching the origin: ϕ^- with control -1 and ϕ^+ with control 1. Hence, starting from any arbitrary initial condition (x_0, y_0) outside this curves, you choice the unique parabola which starting from (x_0, y_0) and moving in positive time hit one of the curves ϕ^- or ϕ^+. After that, you change the control by remaining in the hitting curve up to reach the target. For instance, if you start from (x_0, y_0) in the third quadrant and under ϕ^-, you first take the integral curve $\gamma(t, (x_0, y_0), u \equiv 1)$, which means that you accelerate at maximum up to intersect ϕ^-. After that, you follow the ϕ^- trajectory by breaking at maximum and finishing in the origin. Just observe that the projection in the first variable

$$\pi_1(\gamma(t, (x_0, y_0), u \equiv 1) \cap \phi^-) \in \mathbb{R}$$

give you the distance to the station where you need to change the control.

Example 13 *In the* 2*-dimensional connected affine group* $Aff_+(2)$*, we consider the linear system,*

$$\Sigma_{Aff_+(2)} : \qquad \begin{cases} \dot{x} = & ux \\ \dot{y} = & x - 1 \end{cases}, \quad u \in \mathcal{U}.$$

According to Theorem, the systems is controllable. Here,

$$D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \text{and } \mathcal{X}_g = \begin{pmatrix} 0 & x-1 \\ 0 & 0 \end{pmatrix}.$$

The Hamiltonian reads,

$$\mathcal{H}(\lambda_g, g, u) = \langle \lambda, F_g \rangle + u\langle \lambda, X \rangle.$$

For the unbounded case, we obtain $\langle \lambda(t), X \rangle = 0$. Thus,

$$\langle \dot{\lambda}(t), X \rangle = \langle \lambda(t), -DX + u[X, X] \rangle = \langle \lambda(t), -Y \rangle = 0.$$

Since, λ is null on a basis of the Lie algebra $aff_+(2)$ of the group, it follows that $\lambda = 0$. It turns out that no extremal trajectory exists. Actually, in [21] the authors show that the minimal time to reach $(1, -1)$ from the identity is 1 but there is not a control $u \in \mathcal{U}$ which connect the states in 1 unit of time.

Next, we analyze the bounded case, $u \in [-1, 1]$. In coordinates $\lambda_g = (p, q)$ the Hamiltonian is given by (5) as

$$\mathcal{H}(\lambda_g, g, u) = \langle \lambda_g, \mathcal{X}_g \rangle + u\langle \lambda_g, X_g \rangle = pux + q(x - 1).$$

Therefore, the Hamiltonian equations are:

$$\begin{cases} \dot{x} = & \frac{\partial \mathcal{H}}{\partial p} = ux \\ \dot{y} = & \frac{\partial \mathcal{H}}{\partial q} = x-1 \end{cases} \quad \text{and} \quad \begin{cases} \dot{p} = & -\frac{\partial \mathcal{H}}{\partial x} = -up-q \\ \dot{q} = & -\frac{\partial \mathcal{H}}{\partial y} = 0. \end{cases}$$

First of all, q is constant. Now, if for some $t_0 : p(t_0) = 0$, then $q = q_0 \neq 0$. In fact, by the Pontryagin Maximum Principle $(p(t), q_0)$ vanishes nowhere. Since $\dot{p}(t_0) = -q_0 \neq 0$. It turns out that an optimal control takes the constant value B or $-B$ and changes at most once. In fact, p can not vanish more than once.

In [5] the authors prove that the system is controllable by showing explicitly the trajectories connecting any two sates. So, let us show how to reach the state $(e, 1)$ from the identity element by an optimal path. According to their computations, the solution starting at any initial condition $(x, y) \in G$, is given by concatenation of the following two flows

$$\begin{aligned} \phi((x,y),u,t) &= (e^{ut}x, \frac{(e^{ut}-1)x}{u} - t + y), t \in \mathbb{R}, u \neq 0 \\ \phi((x,y),0,t) &= (x,(x-1)t+y), t \in \mathbb{R}, u = 0. \end{aligned}$$

In order to travel to the right side from $(1, 0)$, we need to consider first the control $u = 1$. We get,

$$\phi((1,0),1,t) = (e^t, (e^t-1)-t), \ t \in \mathbb{R}.$$

When, $t = 1$, the state of the curve is $(e, e-2)$. However, the second coordinate does not coincide with our target. Therefore, the optimal control should change from $u \equiv 1$ to $u \equiv -1$, in some instant t_0 of the curve. Thus, we need to apply the control $u \equiv -1$, to the initial state $(e^{t_0}, (e^{t_0}-1)-t_0)$. We obtain,

$$\phi((e^{t_0}, (e^{t_0}-1)-t_0), -1, t) = (e^{-t}e^{t_0}, (1-e^{-t})e^{t_0} - t + (e^{t_0}-1) - t_0).$$

It turns out that

$$-t + t_0 = 1 \text{ and } e^{t_0} - e - t + e^{t_0} - 1 - t_0 = 1.$$

From that, we get

$$e^{t_0} - t_0 = \frac{1+e}{2}.$$

The continuous function $f(x) = e^x - x$, satisfy $f(\frac{1}{2}) < \frac{1+e}{2} < f(2)$. Since, f is strict increasing the only one switch time t_0 exists. The optimal curve is given by the optimal control $u^*(\zeta) = \left\{ \begin{array}{c} 1 \text{ if } 0 < \zeta \leq t_0 \\ -1 \text{ if } t_0 < \zeta \end{array} \right\}$ and the optimal time is $t = t_0 - 1$.

Example 14 *For a* 3*-dimensional semisimple compact group* $G = SO(3,\mathbb{R})$, *we consider the unbounded linear system*

$$\Sigma_{SO(3,\mathbb{R})}: \qquad \dot{g} = \mathcal{X}_g + u_1 X_g + u_2 Y_g, \text{ where } D = -ad(X).$$

Since $Span_{LA}\{\mathcal{X}, X, Y\} = \mathfrak{g}$ *the system satisfies the LARC condition. As we know,* Σ *is controllable.*

The associated Hamiltonian function reads as

$$\mathcal{H}(\lambda_g, g, u) = \langle \lambda, F_g \rangle + u_1 \langle \lambda, X \rangle + u_2 \langle \lambda, Y \rangle.$$

The maximization condition implies $\langle \lambda(t), X \rangle = \langle \lambda(t), Y \rangle = 0$. According to (6),

$$0 = \langle \dot{\lambda}(t), X \rangle = \langle \lambda(t), (-DX + u_1 ad(X) + u_2 ad(Y))X \rangle = \langle \lambda(t), -u_2 Z \rangle,$$

In the same way,

$$0 = \langle \dot{\lambda}(t), Y \rangle = \langle \lambda(t), -DY + u_1 Z \rangle = \langle \lambda(t), (u_1 - 1)Z \rangle.$$

According to the Pontryagin Maximum Principle, if a time optimal minimizer exists, then $\langle \lambda(t), Z \rangle \neq 0$. It turns out that $u_1 \equiv 1$, and $u_2 \equiv 0$. So, the only existent minimizers connect two points on the same integral curves of the vector field

$$\dot{g} = \mathcal{X}_g + X_g.$$

In order to invite the reader to work out an optimal problem of a linear system Σ_G, on a 3-dimensional Lie group, we give a matrix representation of the solvable Lie algebras $\mathfrak{r}_2$ and $\mathfrak{e}$.

1. For $\mathfrak{r}_2$ consider the basis

$$X = M(1,1), Y = M(2,3), Z = M(1,3).$$

With the change of variable Y by Z, we obtain $\mathfrak{r}_2$ with the same bracket rules: $[X,Y]=Y$, $[X,Z]=0$. The left invariant vector fields are given by

$$X_g = -y\frac{\partial}{\partial x}, Y_g = -y\frac{\partial}{\partial y}, Z_g = \frac{\partial}{\partial z}.$$

the associated simply connected matrix Lie group G reads as

$$G = \{g = \begin{pmatrix} -y & 0 & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x,z \in \mathbb{R}, y<0.\} \longleftrightarrow (x,y,z) \in \mathbb{R}\times\mathbb{R}_{-}\times\mathbb{R}.$$

with the solvable structure

$$(x_1,y_1,z_1)*(x_2,y_2,z_2) = (x_1 - y_1x_2, -y_1y_2, z_1+z_2).$$

The derivation Lie algebra has dimension 4. Precisely, each derivation

$$\mathcal{D} = \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & c & d \end{pmatrix} : a,b,c,d \in \mathbb{R},$$

determines the linear vector field

$$\mathcal{X}_g = (ax+by+b)\frac{\partial}{\partial x} + (dz - c\ln(-y))\frac{\partial}{\partial z}.$$

2. Consider the Lie algebra with a basis

$$X = M(1,3), Y = M(2,3), Z = M(1,2) - M(2,1),$$

and brackets $[X,Y]=Z, [X,Z]=-Y$. With the change of variable X by Y and Y by Z, we recover $\mathfrak{e}$.

The associated group is the so called Euclidean group of the affine transformations of $\mathbb{R}^2$, which preserves the Euclidean metric. It is isomorphic to the order three matrices,

$$E = \left\{ \begin{pmatrix} P & x \\ 0 & 1 \end{pmatrix} : PP^T = Id,\ x \in \mathbb{R}^2 \right\}.$$

This realization is geometrically. In fact, the orthogonal matrix P and the vector $x \in \mathbb{R}^2$ represents a rotation and translation in the plane, respectively. Again, any derivation is inner, and each linear vector field is associated with $\mathcal{D} = -ad(U)$, $U \in \mathfrak{e}$.

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