Linear flows and Morse graphs: Topological consequences in low dimensions

Víctor Ayala¹, Ivan Jirón *¹,²

Departamento de Matemática, Universidad Católica del Norte, Antofagasta, Chile

A R T I C L E   I N F O

Article history:
Received 7 March 2012
Accepted 7 June 2013
Available online 26 July 2013
Submitted by R. Brualdi

MSC:
37B20
37B25
05C21

Keywords:
Morse graphs
Dynamic
Topological equivalence of linear flows
Grassmannians

A B S T R A C T

The main aim of this paper is to show some specific connections between linear dynamic and graphs. Precisely, the Morse decomposition of a linear flow on the Grassmannians induces a directed graph. We apply the results appearing in Ayala et al. (2006, 2005) [2,3] and Colonius et al. (2002) [4] and compute the associated graphs for linear flows in dimensions two and three.

© 2013 Elsevier Inc. All rights reserved.

1. Introduction

The main aim of this paper is to show some connections between linear dynamic and graphs. Precisely, a linear flow \( \Phi \) on \( \mathbb{R}^d \) induces a well defined non-linear flow on the projective space and on every level \( G_i = \text{Grass}(i, d) \): the Grassmannian manifold of the \( i \)-planes on \( \mathbb{R}^d \), for \( i \geq 1 \). In order to analyze topological equivalence of linear flows in [3] and in a more general set up: the topological equivalence of bilinear control systems in [4], the authors introduce a directed graph \( Gr(\Phi) \) associated with \( \Phi \). On the other hand, they also consider some spectral objects through the definition of the Lyapunov normal form, the short Lyapunov normal form and the zero short Lyapunov normal form of \( \Phi \). In particular, in the hyperbolic case, i.e., when \( \Phi_1 \) and \( \Phi_2 \) are hyperbolic, it is possible to

* Corresponding author.

E-mail address: ijiron@ucn.cl (I. Jirón).

¹ This research was partially supported by Proyecto FONDECYT No. 1100375.
² This research was partially supported by Proyecto D.G.I.P. No. 220202-10301284 VRIDT UCN.
characterize their dynamic. In fact, \( \Phi_1 \) and \( \Phi_2 \) are topological equivalent if and only if the dimensions of the stable (or unstable) subspaces are the same. This is something that can be seen from the short-zero Lyapunov form.

For representative linear flows in the plane and in the space we explicitly compute in this paper their dynamics on the Grassmannians and the corresponding associated graphs. In particular, it is possible in both dimensions to analyze the topological equivalence of a couple of linear hyperbolic dynamics just by looking at the graphs. Furthermore, in [3] it is proved that two arbitrary matrices in \( gl(d, \mathbb{R}) \) have isomorphic associated graphs if and only if their short Lyapunov forms are equal.

The article is organized as follows: Section 2 contains the basic notion of flows, Morse decomposition, the Lyapunov normal form, the Short Lyapunov normal form and the chain transitive set. Section 3 includes the Selgrade theorem [6], its generalization to the Grassmannian manifolds [3,4] and the construction of the direct graph associated to any linear differential equation. Finally, Section 4 contains the analysis on dimension two and three. We also include some examples.

2. Preliminaries

In this section the main definitions and results motivating this article are presented. We follow Refs. [2,3] and [4]. For a flow \( \Phi \) on a compact metric space \( Y \), a compact subset \( K \subset Y \) is said to be isolated invariant if it is invariant and there exists a neighborhood \( N \) of \( K \), i.e., a set \( N \) with \( K \subset \text{int}(N) \) such that \( \Phi(t, x) \subset N, \forall t \in \mathbb{R} \) implies \( x \in K \).

By using \( \Phi(t, x) \) and \( t \in \mathbb{R} \), a point \( x \in Y \) can be moved toward the future or past. The \( \omega \)-limit and \( \alpha \)-limit sets of \( x \) are respectively defined as

\[
\omega(x) = \left\{ y \in Y : \exists n \to \infty: \lim_{n \to \infty} \Phi(t_n, x) = y \right\},
\]

\[
\alpha(x) = \left\{ y \in Y : \exists n \to -\infty: \lim_{n \to -\infty} \Phi(t_n, x) = y \right\}.
\]

**Definition 1.** A Morse decomposition \( \mathcal{M} \) of a flow \( \Phi \) on a compact metric space \( Y \) is a finite collection \( \{ M_i : i = 1, \ldots, n \} \) of non-empty, pairwise disjoint and isolated compact invariant sets such that

(i) \( \forall x \in Y : \omega(x), \alpha(x) \subset \bigcup_{i=1}^{n} M_i \).

(ii) Suppose that there are \( M_{j_0}, \ldots, M_{j_l} \) and \( x_1, \ldots, x_l \in Y \setminus \bigcup_{i=1}^{n} M_i \) such that \( \alpha(x_k) \in M_{j_{k-1}} \) and \( \omega(x_k) \in M_{j_k} \) for \( k = 1, \ldots, l \), then \( M_{j_0} \neq M_{j_l} \).

The property (ii) implies that cycles are not allowed. The elements of a Morse decomposition are called Morse sets. Furthermore, an order is defined on a Morse decomposition in the following form:

For \( i, i' \in \{ 1, \ldots, n \} : M_i \preceq M_{i'} \) if there exists \( x \in Y \) with \( \alpha(x) \subset M_i \) and \( \omega(x) \subset M_{i'} \).

Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be Morse decompositions on a compact metric space \( Y \). Then, \( \mathcal{M}_2 \) is finer than \( \mathcal{M}_1 \), if every element of \( \mathcal{M}_2 \) is contained in some element of \( \mathcal{M}_1 \).

**Definition 2.** Consider a flow \( \Phi \) on a compact metric space \( (Y, \delta) \), where \( \delta \) is a metric. Let \( \epsilon, T > 0 \). A \( (\epsilon, T) \)-chain from \( x \in Y \) to \( y \in Y \) is given by a natural number \( n \in \mathbb{N} \), together with points \( x = x_0, \ldots, x_n = y \) in \( Y \), and times \( T_0, \ldots, T_{n-1} > T \), such that \( \delta(\Phi(T_i, x_i), x_{i+1}) < \epsilon \), for \( i = 0, 1, \ldots, n - 1 \).

**Definition 3.** A subset \( A \subset Y \) is chain transitive if for all \( x, y \in A \) and \( \epsilon, T > 0 \) there exists an \( (\epsilon, T) \)-chain from \( x \) to \( y \). A point \( x \in Y \) is chain recurrent if for all \( \epsilon, T > 0 \) there exists an \( (\epsilon, T) \)-chain from \( x \) to \( x \). The chain recurrent set \( R \subset Y \) is the set of all chain recurrent points.

**Theorem 4.** (See [2] ) The connected components of the chain recurrent set \( R \) coincide with the maximal (with respect to set inclusion) chain transitive subsets of \( R \). Furthermore, the flow restricted to a connected component of \( R \) is chain transitive.

It is well known that except for the order of the blocks, any matrix \( A \) on a real vector space can be represented just by using its real Jordan form (RJF). \( J(A) \) see [5]. Thus, in this article only the RJF is used to represent the corresponding system of differential equations \( \dot{x} = Ax \).
Let $d$ be a positive integer number. For any real matrix $A$ in $\text{gl}(d, \mathbb{R})$ the spectrum and the Lyapunov spectrum of $A$ are respectively given by

(a) $\text{Spec}(A) = \{\mu: \mu$ is an eigenvalue of $A\}$.
(b) $\text{Spec}_{Ly}(A) = \{\lambda = \text{Re}(\mu): \mu \in \text{Spec}(A)\}$.

A matrix $A \in \text{gl}(d, \mathbb{R})$ is called hyperbolic if $0 \notin \text{Spec}_{Ly}(A)$. The stability of a linear differential equation (system) depends on the Lyapunov exponents of $A$, i.e., the elements of $\text{Spec}_{Ly}(A)$. In order to explicitly show this dependence, by following [3] we define in the sequel certain normal forms of $A$. For that let $\lambda_1 < \cdots < \lambda_m$ the distinct real parts of the eigenvalues of $A$, with associated Lyapunov spaces $L_i = \bigoplus_j J_{j,i}$. Here, the sets $J_{j,i}$ are the subspaces of $\mathbb{R}^d$ corresponding to the Jordan blocks of $J(A)$ where the real part of the eigenvalues of $A$ is $\lambda_i$. Then $\mathbb{R}^d = \bigoplus_{i=1}^m L_i$.

**Definition 5.** The Lyapunov normal form of $A \in \text{gl}(d, \mathbb{R})$ is the diagonal matrix

$$L(A) = \text{diag}(A_1, \ldots, A_m).$$

Here $A_i = \text{diag}(\lambda_1, \ldots, \lambda_i)$ and the block size of $A_i$ is the dimension $d_i = \text{dim}(L_i)$ of the Lyapunov space $L_i$. The blocks are arranged according to the order $\lambda_1 < \cdots < \lambda_m$. Two matrices $A$ and $B$ are called Lyapunov equivalent if $L(A) = L(B)$.

**Definition 6.** For $A \in \text{gl}(d, \mathbb{R})$, the short Lyapunov form $SL(A)$ is given by the array

$$SL(A) = (m, d_1, \ldots, d_m)$$

where $m$ is the number of distinct Lyapunov exponents and the dimensions $d_i$ of the Lyapunov spaces, in the natural order of their Lyapunov exponents and $\sum_{j=1}^m d_j = d$.

Additionally, the short-zero Lyapunov form is

$$S^0L(A) = (m, m^\mathbb{R}, d_1, \ldots, d_m)$$

and can be used for the study of the dynamic induced by a linear flow. Here $m^\mathbb{R} = \sum_{j=1}^m d_j$ is the total multiplicity of the negative Lyapunov exponents.

**Example 7.** Consider the matrix $A = L(A) = \text{diag}(-4, -3, 2)$. The Lyapunov spaces are given by $L_1 = \text{span}[e_1]$, $i = 1, 2, 3$. Then, $SL(A) = (3, 1, 1, 1)$ and $S^0L(A) = (3, 2, 1, 1, 1)$.

Next we show how any linear differential system on $\mathbb{R}^d$ induces a non-linear dynamic at any level $\mathbb{G}_i$ of the Grassmannians. First, an equivalence relation is defined on the open set $\mathbb{X} = \mathbb{R}^d \setminus \{0\}$ of the Euclidean space by

$$\text{for } x, y \in \mathbb{X}: x \sim y \iff \exists t \in \mathbb{R} \setminus \{0\} \text{ with } y = tx.$$  

The $(d-1)$-dimensional projective space is defined by the quotient $\mathbb{P}^{d-1} = \mathbb{X} / \sim$. The map $\mathbb{P}: \mathbb{X} \rightarrow \mathbb{P}^{d-1}$ is the projection and $\mathbb{P}(x) = [x]$ is the equivalence class of $x$. Obviously, the space $\mathbb{P}^{d-1}$ can also be obtained by the antipodal equivalence relation on $\mathbb{S}^{d-1}$ as follows:

$$\text{for } x, y \in \mathbb{S}^{d-1}: x \sim y \iff y = \pm x.$$  

The induced linear flow $\varphi(t, .) = e^{tA}(.)$ by the differential system $\dot{x} = Ax$ projects down onto a flow $\mathbb{G}_1\varphi$ on $\mathbb{P}^{d-1}$. Explicitly, $\mathbb{G}_1\varphi(t, .)$ is induced by the non-linear differential equation

$$\dot{s} = (A - s^T As)\dot{s}: s \in \mathbb{P}^{d-1}.$$  

Here $I$ denotes the identity and $s^T$ the transpose of the matrix $s$. In a more general set up we have $e^{tA}$ is a one parameter group of invertible matrices. Thus, if $V \in \mathbb{G}_i$ is an $i$-dimensional subspace
Fig. 1. $\dot{x} = Ax$ induces a differential equation on the compact manifold $S^2$.

of $\mathbb{R}^d$, it follows that $e^{tA}V$ is a differentiable curve on the compact manifold $G_i$. Then by derivation, the matrix $A$ induces a non-linear differential equation on $G_i$ for any $i = 1, 2, \ldots, d$. Furthermore, by definition, the $k$-th flag on $\mathbb{R}^d$ is given by

$$F_k = \{F_k = (V_1, \ldots, V_k) : V_i \subset V_{i+1}, \dim(V_i) = i, \ i = 1, \ldots, k\}.$$  

For $k = d$ the complete flag $F = F_d$ is obtained. We denote by $G_i\varphi$ and $F_k\varphi$ the corresponding induced flows.

Then, for any $i = 1, \ldots, d$, $\dot{x} = Ax$ induces a differential equation on the compact manifold $G_i$. Fig. 1 shows a picture for the case $i = 1$ with $d = 3$ on the sphere $S^2$. By the antipodal identification we get the dynamics on $\mathbb{P}^2$.

3. Dynamics on Grassmannians and Morse graphs

In this section, we build an associated directed graph $\text{Gr}(A)$ to any linear map $A \in \text{gl}(d, \mathbb{R})$ where its nodes are the Morse sets. The first level of the graph contains the elements of the finest Morse decomposition of $G_1\varphi$ on the projective space $G_1$. The second level contains the elements of the finest Morse decomposition of $G_2\varphi$ on the Grassmannian manifold $G_2$ and so on. The unique element on the top of the graph is determined by the space $\mathbb{R}^d$, i.e., the only $d$-dimensional subspace of $\mathbb{R}^d$. We observe that the Lyapunov exponents determine a total order in $G_1$ but in general at any superior level $G_i$, $i > 1$, the graph just has a partial order. The existence of the finest Morse decomposition on the Grassmannians is given by the Selgrade theorem [6] and its generalization by the Colonius–Kliemann theorem [3].

**Theorem 8** (Selgrade). Let $A \in \text{gl}(d, \mathbb{R})$ a matrix with flow $\varphi$ on $\mathbb{R}^d$. For $k = 1, \ldots, m$, consider the Lyapunov space $L_k$ of $A$. Then,

$$\{\mathbb{P}L_k = \mathcal{M}_k : k = 1, \ldots, m\}$$

is the finest Morse decomposition of $\mathbb{P}\varphi$ on the projective space $G_1$.

In [4] the authors extend the Selgrade theorem from $G_1$ to any flag bundle. In particular, they prove that there exists the finest Morse decomposition of $G_i\varphi$ on the Grassmannian $G_i$ for $1 \leq i \leq d$. This fact is presented in the following theorem.

**Theorem 9** (Colonius–Kliemann). Let $A \in \text{gl}(d, \mathbb{R})$ a matrix. For $i = 1, \ldots, d$ and $k = 1, \ldots, m$ define the index set

$$I(i) = \{(i_1, \ldots, i_m) : i_1 + \cdots + i_m = i, \ 0 \leq i_k \leq d_k = \dim(L_k)\}.$$
Therefore, the finest Morse decomposition of $G_1 \varphi$ on the Grassmannian $G_1$ is given by the sets

$$\mathcal{N}^j_{i_1, \ldots, i_m} = G_{i_1} L_1 \oplus \cdots \oplus G_{i_m} L_m$$

where $(i_1, \ldots, i_m) \in I(i)$. Here, $G_{i_k} L_k$ represents an $i_k$-dimensional subspace spanned by vectors of $L_k$.

The previous results allow us to build upon the Morse graph, $Gr(A)$, of $A \in gl(d, \mathbb{R})$ through the following algorithm.

1. The finest Morse decomposition associated to the flow $G_1 \varphi$ on $G_1$ has $m$ Morse sets

$$\{ M_j / j = 1, \ldots, m \}.$$  

This set is linearly ordered according to the Lyapunov exponents.

2. The canonical base $\{ e_1, \ldots, e_d \}$ of $\mathbb{R}^d$ is associated with the Morse sets in such a way that each Morse set $M_i$ corresponds to $d_i$ basis vectors (in the respective orders):

$$M_1 \sim \{ e_1, \ldots, e_{d_1} \} \quad \text{and} \quad M_j \sim \{ e_{\alpha_j}, \ldots, e_{\beta_j} \} \quad \text{where} \quad \alpha_j = \sum_{k=1}^{j-1} d_k + 1, \quad \beta_j = \sum_{k=1}^{j} d_k.$$  

3. On $G_1$ the Morse decomposition is indexed as $M_i$.

4. On $G_2$ the Morse decomposition is indexed with two indices $(j_1, j_2)$. Furthermore, there exists a partial vertical order between $G_1$ and $G_2$ as follows:

$$M_i \sqsubseteq M_{j_1, j_2} \iff i \in \{ j_1, j_2 \}.$$  

This partial vertical order between $G_1$ and $G_2$ implies that the set $M_{j_1, j_2}$ in $G_2$ is obtained from the sets $M_{j_1}$ and $M_{j_2}$ in $G_1$. In other words, $M_{j_1, j_2}$ in $G_2$ is projected down to both $M_{j_1}$ and $M_{j_2}$ in $G_1$.

Several of the sets $M_{j_1, j_2}$ on $G_2$ may be identical. In this case the index pair with smallest number in each entry is used. Observe that the order relations are retained.

5. Continuing with the same structure for $G_3, \ldots, G_d$ we obtain unique indexes for all Morse sets on $G_i$, $i = 3, \ldots, d$ and hence on the maximal flag $F$. Then, for $M_{j_1, \ldots, j_i}$ in $G_1$ and $M_{j_1, \ldots, j_{i+1}}$ in $G_{i+1}$ we use $M_{j_1, \ldots, j_i} \sqsubseteq \{ j_{i-1} \} M_{j_1, \ldots, j_{i+1}}$ to indicate that $M_{j_1, \ldots, j_{i+1}}$ is projected onto $M_{j_1, \ldots, j_i}$.

6. The order on $G_i$ is defined by

$$M_{j_1, \ldots, j_l} \leq M_{j_1', \ldots, j_l'} \iff j_l \leq j_l' \quad \text{for all} \quad l = 1, \ldots, i.$$  

7. In this horizontal level, the order $\leq$ implies that for some point $x \in G_i$, $\alpha(x) \subset M_{j_1, \ldots, j_i}$ and $\omega(x) \subset M_{j_1', \ldots, j_l'}$, i.e. the $x$-orbit starts at $M_{j_1, \ldots, j_l}$ and ends at $M_{j_1', \ldots, j_l'}$.

Since the orders $\leq$ and $\sqsubseteq$ are transitive, we consider only arrows between nearest neighbor nodes in the construction of the Morse graph. For example, in Fig. 2 the arrow linking the nodes $A$ and $C$ can be removed, because it is the equivalent of the path formed by the arrows going from $A$ to $B$ and from $B$ to $C$.

**Example 10.** Let $A = \text{diag}(-2, \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix})$. The Lyapunov spaces are $L_1 = \text{span}(e_1)$, $L_2 = \text{span}(e_2, e_3)$. Furthermore, $\text{SL}(A) = (2, 1, 2)$ and $\text{S}^0 L(A) = (2, 1, 1, 2)$.  

![Fig. 2. Equivalent paths.](image-url)
4.1. Dimension two

For \( A \in \text{gl}(2, \mathbb{R}) \), we analyze this situation according to the following representative cases:

4.1.1. Case: The eigenvalues \( \lambda_1, \lambda_2 \in \mathbb{R}, 0 < \lambda_1 < \lambda_2, \) with \( m_a(\lambda_i) = 1 \) and \( m_g(\lambda_i) = 1 \) for \( i = 1, 2 \)

The Lyapunov normal form of \( A \) is \( L(A) = \text{diag}(\lambda_1, \lambda_2) \). The Lyapunov spaces are given by \( L_1 = \text{span}(e_1) \) and \( L_2 = \text{span}(e_2) \). Thus, the elements of the Morse decomposition of \( A \) on \( G_1 \) are the projections \( P_{L_1} = M_1 \) and \( P_{L_2} = M_2 \). For \( G_1 \) we have: \( N_{01}^1 = G_1 L_1 \oplus G_0 L_2 = G_1 L_1 = \text{span}(e_1) = M_1 \) and \( N_{01}^2 = G_0 L_1 \oplus G_1 L_2 = G_1 L_2 = \text{span}(e_2) = M_2 \). For \( G_2 \) we have: \( N_{11}^2 = G_1 L_1 \oplus G_1 L_2 = \text{span}(e_1, e_2) = M_{12} \).

The projected dynamic on \( S^1 \) is shown in Fig. 4. In this case, the points \((\pm 1, 0)\) are sources and the points \((0, \pm 1)\) are sinks. The Morse graph of \( A \) is shown in Fig. 5.

The cases \( \lambda_1 < 0 < \lambda_2 \) and \( \lambda_1 < \lambda_2 < 0 \) have the same projected dynamic on \( S^1 \) and of course the same Morse graph.

The cases \( \lambda_1 = 0 < \lambda_2 \) and \( \lambda_1 < 0 = \lambda_2 \) have the same projected dynamic on \( S^1 \), and these are shown in Figs. 6 and 7, respectively. And the Morse graph is shown in Fig. 5.
4.1.2. Case: An eigenvalue $\lambda_1 \in \mathbb{R}$, $0 < \lambda_1$, with $m_\sigma(\lambda_1) = m_\pi(\lambda_1) = 2$

In this case the orbits in $\mathbb{R}^2$ are lines that escape from the origin.

The Lyapunov normal form of $A$ is $L(A) = \text{diag}(\lambda_1, \lambda_1)$. The Lyapunov space is $L_1 = \mathbb{R}^2$. In this case for $G_1$, an infinite number of singular points are contained in the Morse set $\mathbb{P}L_1 = \mathcal{M}_1$. In other words, $\mathcal{M}_1$ is a chain transitive set, because for all $x, y \in \mathcal{M}_1$ and $\varepsilon, T > 0$ there exists an $(\varepsilon, T)$-chain from $x$ to $y$.

For $G_2$: The projection of $\mathbb{R}^2$ on $S^1$ produces only one element in $\mathcal{M}_{12}$, which is $S^1$. The projected dynamic on $S^1$ is shown in Fig. 8 and Morse graph for $A$ is shown in Fig. 9.

For the case $\lambda_1 = \lambda_2 < 0$ the orbits are lines that tend toward the origin and have the same dynamic on $S^1$ and the same Morse graph.

Furthermore, for the case $\lambda_1 = \lambda_2 = 0$ the orbits are points and the projected dynamic on $S^1$. Therefore, the circle is chain transitive then $\mathcal{M}_1$ is a single set and its Morse graph is given in Fig. 9.

4.1.3. Case: A non-diagonalizable matrix with eigenvalue $\lambda_1 \in \mathbb{R}$, $0 \leq \lambda_1$, and $m_\sigma(\lambda_1) = 2$, $m_\pi(\lambda_1) = 1$

For a non-diagonalizable matrix $A = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{bmatrix}$ the flow is given by $\varphi(t, x) = e^{t\lambda_1} (x_1 e_1 + x_2 e_2)$. If $\lambda_1 \neq 0$, then the orbits in $\mathbb{R}^2$ are curves escaping from the origin.

On $S^1$, the singularities are the points $(0, \pm 1)$. For any initial condition in the first or fourth quadrants, the corresponding orbit starts at the point $(0, -1)$ and ends at the point $(0, 1)$, and they remain in these quadrants. Whereas for an initial condition in the second or third quadrants its orbit starts at point $(0, 1)$ and ends at point $(0, -1)$, and also remain in these quadrants. Next, the projected dynamic on $S^1$ is shown in Fig. 10.

The Lyapunov normal form of $A$ is $L(A) = \text{diag}(\lambda_1, \lambda_1)$. The Lyapunov space is $L_1 = \mathbb{R}^2$. In this case for $G_1$, $\mathbb{P}L_1 = \mathcal{M}_1 \sim \text{span}[e_1, e_2]$. 

![Fig. 4. The projected dynamic on $S^1$.](image)

![Fig. 5. Morse graph for $A$.](image)
Fig. 6. Orbits and projected dynamic on $\mathbb{S}^1$ for case $\lambda_1 = 0 < \lambda_2$.

Fig. 7. Orbits and projected dynamic on $\mathbb{S}^1$ for case $\lambda_1 < 0 = \lambda_2$. 
Fig. 8. The projected dynamic on $S^1$.

For $G_2$: The projection of $\mathbb{R}^2$ on $S^1$ produces only one element in $\mathcal{M}_{12}$, which is $S^1$. And the Morse graph is given in Fig. 11.

For $\lambda_1 = 0$ the projected dynamic on $S^1$ is given in Fig. 10 and the Morse graph for $A$ is given in Fig. 11.

4.1.4. Case: The complex eigenvalues $\mu_1 = a + ib$, $a, b \in \mathbb{R}$, $\mu_2 = \bar{\mu}_1$ and $a > 0$

The Lyapunov normal form is $L(A) = \text{diag}(a, a)$. The Selgrade decomposition is given by $L_1 = \mathbb{R}^2$. The Morse set comes from the projection $P L_1 = \mathcal{M}_{1} \sim \text{span}\{e_1, e_2\}$. For $G_2$ it follows that $\mathcal{M}_{12} \sim \text{span}\{e_1, e_2\} = S^1$. Fig. 12 shows the projected dynamic on $S^1$ and the Morse graph is given in Fig. 13.

Furthermore, the cases $a = 0$ and $a < 0$ have the same projected dynamic on $S^1$ and the Morse graph.

Next, for $A \in gl(2, \mathbb{R})$ we show in Tables 1 and 2 the general situation in dimension two. With real eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ and complex eigenvalues $\mu = a \pm ib \in \mathbb{C}$. The last column on the right side identifies the systems whose Morse graphs are equals.

4.2. Dimension three

Let $A \in gl(3, \mathbb{R})$. Again, we analyze the different possibilities according to the spectrum of some representative matrix $A$. 
Fig. 10. The projected dynamic on $S^1$.

Fig. 11. Morse graph for $A$.

Table 1
The general situation in dimension two for real eigenvalues.

<table>
<thead>
<tr>
<th>$L(A)$</th>
<th>$\text{Spec}_{\mathbb{C}}(A)$</th>
<th>$SL(A)$</th>
<th>Morse graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>diag($\lambda_1, \lambda_2$)</td>
<td>$0 &lt; \lambda_1 &lt; \lambda_2$</td>
<td>(2, 1, 1)</td>
<td>Fig. 5</td>
</tr>
<tr>
<td></td>
<td>$\lambda_1 = 0 &lt; \lambda_2$</td>
<td>(2, 1, 1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda_1 &lt; 0 &lt; \lambda_2$</td>
<td>(2, 1, 1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda_1 &lt; 0 = \lambda_2$</td>
<td>(2, 1, 1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda_1 &lt; \lambda_2 &lt; 0$</td>
<td>(1, 2)</td>
<td>Fig. 9</td>
</tr>
<tr>
<td></td>
<td>$0 \leq \lambda_1 = \lambda_2 &lt; 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda_1 = \lambda_2 &lt; 0^*$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* A non-diagonalizable matrix is included in these cases.

Table 2
The general situation in dimension two for complex eigenvalues.

<table>
<thead>
<tr>
<th>$L(A)$</th>
<th>$\text{Spec}_{\mathbb{C}}(A)$</th>
<th>$SL(A)$</th>
<th>Morse graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>diag$(a, a)$</td>
<td>$a &gt; 0$</td>
<td>(1, 2)</td>
<td>Fig. 9</td>
</tr>
<tr>
<td></td>
<td>$a = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$a &lt; 0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4.2.1. Case: The eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $0 < \lambda_1 < \lambda_2 < \lambda_3$, with $m_{q}(\lambda_i) = 1 = m_{g}(\lambda_i)$ for $i = 1, 2, 3$.

We have $L(A) = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. The Lyapunov spaces are given by $L_i = \text{span}(e_i)$, $i = 1, 2, 3$. The elements of the finest Morse decomposition are $\mathcal{P}_i = \mathcal{M}_i$ and the Morse decomposition at any level is as follows:

For $G_1$: $N^1_{100} = G_1 L_1 = M_1 \sim \text{span}(e_1)$, $N^1_{010} = G_1 L_2 = M_2 \sim \text{span}(e_2)$, $N^1_{001} = G_1 L_3 = M_3 \sim \text{span}(e_3)$.

For $G_2$: $N^2_{110} = G_1 L_1 \oplus G_1 L_2 = M_{12} \sim \text{span}(e_1, e_2)$, $N^2_{101} = G_1 L_1 \oplus G_1 L_3 = M_{13} \sim \text{span}(e_1, e_3)$, $N^2_{011} = G_1 L_2 \oplus G_1 L_3 = M_{23} \sim \text{span}(e_2, e_3)$.

For $G_3$: $N^3_{111} = G_1 L_1 \oplus G_1 L_2 \oplus G_1 L_3 = M_{123} \sim \text{span}(e_1, e_2, e_3)$.

The projected dynamic on $S^2$ is shown in Fig. 14. In this case, the points $(\pm 1, 0, 0)$ are sources and the points $(0, 0, \pm 1)$ are sinks. While the points $(0, \pm 1, 0)$ are neither sources nor sinks. The Morse graph is given in Fig. 15.

For the case $\lambda_1 < \lambda_2 < \lambda_3 < 0$, the Morse graph is given in Fig. 15. Here the dynamic consider the same integral curves on $S^2$ but changes the directions. In fact, the points $(\pm 1, 0, 0)$ are sinks and the points $(0, 0, \pm 1)$ are sources.
4.2.2. Case: The eigenvalues $\lambda_1, \lambda_3 \in \mathbb{R}$ and $0 < \lambda_1 < \lambda_3$, with $m_u(\lambda_1) = 2 = m_p(\lambda_1), m_u(\lambda_3) = 1 = m_p(\lambda_3)$.

The Lyapunov normal form is $L(A) = \text{diag}(\lambda_1, \lambda_1, \lambda_3)$. The Lyapunov spaces are $L_1 = L_1(\lambda_1) = \text{Ker}[A - \lambda_1 I]^2 = \text{span}\{e_1, e_2\}, L_3 = L_3(\lambda_3) = \text{Ker}[A - \lambda_3 I] = \text{span}\{e_3\}$. And the projections of the Lyapunov spaces are $P_{L_1} = M_1$ and $P_{L_3} = M_3$.

Morse decomposition is given by: For $G_1$ it follows: $\mathcal{N}_{10}^2 = G_1L_1 = M_1 \sim \text{span}\{e_1, e_2\}, \mathcal{N}_{01}^2 = G_1L_2 = M_3 \sim \text{span}\{e_3\}$. For $G_2$ it follows: $\mathcal{N}_{11}^2 = G_1L_1 \oplus G_1L_2 = M_{13} \sim \text{span}\{x, e_3\} \wedge x \in \text{span}\{e_1, e_2\}, \mathcal{N}_{20}^2 = G_2L_1 = M_{12} \sim \text{span}\{e_1, e_2\}$. For $G_3$ it follows: $\mathcal{N}_{21}^3 = G_2L_1 \oplus G_1L_2 = M_{123} \sim \text{span}\{e_1, e_2, e_3\}$.

In this case, there are infinitely many singularities in $S^1 \subset XY$-plane. The points $(0, 0, \pm 1)$ are sinks. The projected dynamic on $S^2$ is shown in Fig. 16. The Morse graph is given in Fig. 17.
4.2.3. Case: An eigenvalue $\lambda_1 \in \mathbb{R}$ and $0 < \lambda_1$, with $m_a(\lambda_1) = 3 = m_g(\lambda_1)$

In this case the corresponding Lyapunov normal form is $L(A) = \text{diag}(\lambda_1, \lambda_1, \lambda_1)$. The Lyapunov space is $L_1 = \text{Ker}[A - \lambda_1 I] = \text{span}[e_1, e_2, e_3]$. For $G_1$, an infinite number of singular points are contained in the Morse set $\mathbb{P}L_1 = \mathcal{M}_1$. In other words, $\mathcal{M}_1$ is a chain transitive set, because for all $x, y \in \mathcal{M}_1$ and $\varepsilon, T > 0$ there exists an $(\varepsilon, T)$-chain from $x$ to $y$. The Morse decomposition at any level is as follows:

For $G_1$: $\mathcal{M}_1 \sim \text{span}[e_1, e_2, e_3]$. In this case, every single point of $\mathcal{M}_1 = \mathbb{S}^2$ is a singularity.

For $G_2$: $\mathcal{M}_{12} \sim \text{span}[e_1, e_2, e_3]$. This set is an infinite collection of two-dimensional subspaces.

For $G_3$: $\mathcal{M}_{123} \sim \text{span}[e_1, e_2, e_3]$. In this case, the only element of $\mathcal{M}_{123}$ is the own sphere $\mathbb{S}^2$.

The dynamics on $\mathbb{S}^2$ is shown in Fig. 18. The Morse graph is given in Fig. 19.

The real case $\lambda_1 < 0$ with multiplicity three is analogous.

4.2.4. Case: The eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\lambda_1 < \lambda_2 = 0 < \lambda_3$, with $m_a(\lambda_i) = 1 = m_g(\lambda_i)$ for $i = 1, 2, 3$

For a matrix $A$ in $\text{gl}(3, \mathbb{R})$ with Lyapunov exponents $\lambda_2 = 0, \lambda_1 < 0 < \lambda_3$ we have: $L(A) = \text{diag}(\lambda_1, 0, \lambda_3)$. The Lyapunov spaces are $L_i = \text{span}[e_i]$ and the projections of the Lyapunov spaces are $\mathbb{P}L_i = \mathcal{M}_i$. The Morse decomposition at any level is given by:

For $G_1$: $G_1L_1 = \mathcal{M}_1 \sim \text{span}[e_1]$, $G_1L_2 = \mathcal{M}_2 \sim \text{span}[e_2]$, $G_1L_3 = \mathcal{M}_3 \sim \text{span}[e_3]$.

For $G_2$: $G_1L_1 \oplus G_1L_2 = \mathcal{M}_{12} \sim \text{span}[e_1, e_2]$, $G_1L_1 \oplus G_1L_3 = \mathcal{M}_{13} \sim \text{span}[e_1, e_3]$ and $G_1L_2 \oplus G_1L_3 = \mathcal{M}_{123} \sim \text{span}[e_2, e_3]$.

For $G_3$: $G_1L_1 \oplus G_1L_2 \oplus G_1L_3 = \mathcal{M}_{123} \sim \text{span}[e_1, e_2, e_3]$.

The points $(\pm 1, 0, 0)$ are sources and the points $(0, 0, \pm 1)$ are sinks. While the points $(0, \pm 1, 0)$ are neither sources nor sinks. The projected dynamics on $\mathbb{S}^2$ is shown in Fig. 20, and its Morse graph is given in Fig. 21.

4.2.5. Case: The eigenvalues $\lambda_1 \in \mathbb{R}$, $\mu_2 = a + ib \in \mathbb{C}$, $\mu_3 = \bar{\mu}_2$ and $0 < \lambda_1 < a$, with $m_a(\lambda_1) = 1 = m_g(\lambda_1)$, $m_a(\mu_2) = 1$ and $m_g(\mu_2) = 2$

For a matrix $A$ in $\text{gl}(3, \mathbb{R})$ with eigenvalues $\lambda_1 \in \mathbb{R}$, $\mu_2 = a + ib \in \mathbb{C}$, and $\mu_3 = \bar{\mu}_2$ with $0 < \lambda_1 < a$. The Lyapunov normal form is $L(A) = \text{diag}(\lambda_1, a, a)$. The Lyapunov spaces are $L_1 = \text{span}[e_1]$, $L_2 = \text{span}[e_2, e_3]$. The Morse decomposition at any level is as follows:

For $G_1$: $\mathcal{M}_1 \sim \text{span}[e_1]$ and $\mathcal{M}_2 \sim \text{span}[e_2, e_3]$.

For $G_2$: $\mathcal{M}_{12} \sim \text{span}[e_1, x]$ and $x \in \text{span}[e_2, e_3]$.
For $G_3$: $M_{123} \sim \text{span}\{e_1, e_2, e_3\}$. In this case, the points $(\pm 1, 0, 0)$ are sources. The projected dynamic on $S^2$ is shown in Fig. 22, and its Morse graph is given in Fig. 23.

4.2.6. Case: The eigenvalues $\mu_1 = a + ib \in \mathbb{C}$, $\mu_2 = \bar{\mu}_1$, $\lambda_3 = a \in \mathbb{R}$ and $a < 0$, with $m_a(\mu_1) = 1$ and $m_a(\lambda_3) = 2$, $m_a(\lambda_3) = 1 = m_a(\lambda_3)$.

For a matrix $A$ in $\text{gl}(3, \mathbb{R})$ with eigenvalues $\mu_{1,2} \in \mathbb{C}$ and $\lambda_3 \in \mathbb{R}$ with $\lambda_3 = a < 0$. The Lyapunov normal form is $L(A) = \text{diag}(a, a, \lambda_3)$. The Lyapunov space is $L_3 = L_3(\lambda_3) = \text{span}\{e_1, e_2\} \oplus \text{span}\{e_3\}$. And the Morse decomposition is as follows:
4.2.7. Case: A non-diagonalizable matrix with eigenvalues \( \lambda_1, \lambda_3 \in \mathbb{R} \) and \( 0 \leq \lambda_1 < \lambda_3 \), with \( m_d(\lambda_1) = 2 \), \( m_s(\lambda_1) = 1 \), \( m_d(\lambda_3) = 1 = m_s(\lambda_3) \) for a non-diagonalizable matrix \( A = \text{diag}(\lambda_1, 0, \lambda_3) \) the flow is \( \varphi(t, x) = \left( \begin{array}{c} x_1 e^{t\lambda_1} \\ (x_1 + x_2) e^{t\lambda_1} \\ x_2 e^{t\lambda_3} \\ (x_2 + x_3) e^{t\lambda_3} \end{array} \right) \).

The Lyapunov spaces are \( L_1 = L_1(\lambda_1) = \text{span}\{e_1, e_2\} \) and \( L_2 = L_2(\lambda_3) = \text{span}\{e_3\} \). The elements of the finest Morse decomposition are \( \mathbb{P} L_i = M_i \) and the Morse decomposition at any level is as follows:

- For \( G_1' \): \( N_{10}^1 = G_1 L_1 = M_1 \sim \text{span}\{e_1, e_2\} \), \( N_{01}^1 = G_1 L_2 = \mathcal{M}_3 \sim \text{span}\{e_3\} \).
- For \( G_2' \): \( \mathcal{N}_{11}^2 = G_1 L_1 \oplus G_1 L_3 = M_{13} \sim \text{span}\{x, e_3\}, x \in \text{span}\{e_1, e_2\} \), \( N_{20}^2 = G_2 L_1 = \mathcal{M}_{12} \sim \text{span}\{e_1, e_2\} \).
- For \( G_3' \): \( \mathcal{N}_{21}^3 = G_2 L_1 \oplus G_1 L_2 = \mathcal{M}_{123} \sim \text{span}\{e_1, e_2, e_3\} \).

The projected dynamic on \( S^2 \) is shown in Fig. 26. In this case, the points \( (0, 0, \pm 1) \) are sinks. While the points \( (0, \pm 1, 0) \) are neither sources nor sinks. And its Morse graph is given in Fig. 27.

4.2.8. Case: A non-diagonalizable matrix with eigenvalues \( \lambda_1, \lambda_3 \in \mathbb{R} \) and \( 0 \leq \lambda_1 < \lambda_3 \), with \( m_d(\lambda_1) = 1 = m_s(\lambda_1), m_d(\lambda_3) = 2, m_s(\lambda_3) = 1 \) for a non-diagonalizable matrix \( A = \text{diag}(\lambda_1, 0, \lambda_3) \) the flow is \( \varphi(t, x) = \left( \begin{array}{c} x_1 e^{t\lambda_1} \\ x_2 e^{t\lambda_2} \\ x_3 e^{t\lambda_3} \end{array} \right) \).

The Lyapunov spaces are \( L_1 = L_1(\lambda_1) = \text{span}\{e_1\} \) and \( L_3 = L_3(\lambda_3) = \text{span}\{e_2, e_3\} \). The elements of the finest Morse decomposition are given by \( \mathbb{P} L_i = M_i \) and the Morse decomposition at any level is as follows:

- For \( G_1' \): \( N_{10}^1 = G_1 L_1 = M_1 \sim \text{span}\{e_1\}, N_{01}^1 = G_1 L_3 = M_2 \sim \text{span}\{e_2, e_3\} \).
- For \( G_2' \): \( \mathcal{N}_{11}^2 = G_1 L_1 \oplus G_1 L_3 = \mathcal{M}_{12} \sim \text{span}\{e_1, x\}, x \in \text{span}\{e_2, e_3\} \), \( N_{02}^2 = G_2 L_3 = \mathcal{M}_{23} \sim \text{span}\{e_2, e_3\} \).
For \( G_3 \): \( \mathcal{N}_{12}^3 = G_1 L_1 \oplus G_2 L_3 = M_{123} \sim \text{span}\{e_1, e_2, e_3\} \).

The projected dynamic on \( S^2 \) is shown in Fig. 28. In this case, the points \((\pm 1, 0, 0)\) are sources and the points \((0, 0, \pm 1)\) are sinks. And its Morse graph is given in Fig. 29.

4.2.9. Case: A non-diagonalizable matrix with an eigenvalue \( \lambda_1 \in \mathbb{R} \) and \( 0 \leq \lambda_1 \), with \( m_d(\lambda_1) = 3 \) and \( m_g(\lambda_1) = 1 \)

For a non-diagonalizable matrix \( A = \begin{pmatrix} \lambda_1 & 1 & 1 \\ 1 & \lambda_1 & 1 \\ 1 & 1 & \lambda_1 \end{pmatrix} \) the flow is given by \( \phi(t, x) = \begin{pmatrix} x_1 e^{\lambda_1 t} \\ (x_1 + x_2) e^{\lambda_1 t} \\ (x_1 + x_2) e^{\lambda_1 t} \end{pmatrix} \).

The Lyapunov space is given by \( L_1 = L_1(\lambda_1) = \text{span}\{e_1, e_2, e_3\} \). And the Morse decomposition at any level is as follows:
Table 3
The general situation in dimension three for real eigenvalues.

<table>
<thead>
<tr>
<th>( L(A) )</th>
<th>Spec( L(A) )</th>
<th>SL(( A ))</th>
<th>Morse graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>diag(( \lambda_1, \lambda_2, \lambda_3 ))</td>
<td>( 0 &lt; \lambda_1 &lt; \lambda_2 &lt; \lambda_3 )</td>
<td>(3, 1, 1)</td>
<td>Fig. 15</td>
</tr>
<tr>
<td></td>
<td>( 0 = \lambda_1 &lt; \lambda_2 &lt; \lambda_3 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \lambda_1 &lt; 0 &lt; \lambda_2 &lt; \lambda_3 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \lambda_1 &lt; \lambda_2 &lt; 0 &lt; \lambda_3 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \lambda_1 &lt; \lambda_2 &lt; \lambda_3 &lt; 0 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>diag(( \lambda_1, \lambda_2, \lambda_3 ))</td>
<td>( \lambda_1 &lt; \lambda_2 = \lambda_3 )</td>
<td>(2, 1, 2)</td>
<td>Fig. 23</td>
</tr>
<tr>
<td>diag(( \lambda_1, \lambda_2, \lambda_3 ))</td>
<td>( 0 &lt; \lambda_1 = \lambda_2 &lt; \lambda_3 )</td>
<td>(2, 2, 1)</td>
<td>Fig. 17</td>
</tr>
<tr>
<td>diag(( \lambda_1, \lambda_2, \lambda_3 ))</td>
<td>( \lambda_1 = \lambda_2 &lt; \lambda_3 )</td>
<td>(1, 3)</td>
<td>Fig. 19</td>
</tr>
<tr>
<td>diag(( \lambda_1, \lambda_2, \lambda_3 ))</td>
<td>( \lambda_1 = \lambda_2 = \lambda_3 \leq 0 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>diag(( \lambda_1, \lambda_2, \lambda_3 ))</td>
<td>( \lambda_1 = \lambda_2 &lt; 0 &lt; \lambda_3 )</td>
<td>(2, 2, 1)</td>
<td>Fig. 17</td>
</tr>
<tr>
<td>diag(( \lambda_1, \lambda_2, \lambda_3 ))</td>
<td>( \lambda_1 = \lambda_2 &lt; \lambda_3 &lt; 0 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>diag(( \lambda_1, \lambda_2, \lambda_3 ))</td>
<td>( 0 &lt; \lambda_1 &lt; \lambda_2 = \lambda_3 )</td>
<td>(2, 1, 2)</td>
<td>Fig. 23</td>
</tr>
<tr>
<td>diag(( \lambda_1, \lambda_2, \lambda_3 ))</td>
<td>( \lambda_1 &lt; \lambda_2 = \lambda_3 &lt; 0 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>diag(( \lambda_1, \lambda_2, \lambda_3 ))</td>
<td>* A non-diagonalizable matrix is included in these cases.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For \( G_1 \): \( N_1^1 = G_1 L_1 = M_1 \sim \text{span}\{e_1, e_2, e_3\} \). This set is an infinite collection of one-dimensional subspaces.

For \( G_2 \): \( N_2^2 = G_2 L_1 = M_{12} \sim \text{span}\{e_1, e_2, e_3\} \). This set is an infinite collection of two-dimensional subspaces.

For \( G_3 \): \( N_3^3 = G_3 L_1 = M_{123} \sim \text{span}\{e_1, e_2, e_3\} \). This set has a point, which is the whole space. In this case, the only element of \( M_{123} \) is the own sphere \( S^2 \).

The projected dynamic on \( S^2 \) is shown in Fig. 30. The orbits are closed curves which begin and end at the points \((0, 0, \pm 1)\) and they remain in their respective hemispheres, with respect to \(YZ\)-plane. Furthermore, in the \(YZ\)-plane there are two orbits:
Table 4
The general situation in dimension three for real and complex eigenvalues.

<table>
<thead>
<tr>
<th>L(A)</th>
<th>Spec(_{\text{Ly}})(A)</th>
<th>SL(A)</th>
<th>Morse graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>diag((\lambda_1, a, a))</td>
<td>(0 &lt; \lambda_1 &lt; a)</td>
<td>(2, 1, 2)</td>
<td>Fig. 23</td>
</tr>
<tr>
<td>diag((\lambda_1, a, a))</td>
<td>(\lambda_1 &lt; 0 &lt; a)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>diag(a, a, (\lambda_1))</td>
<td>(0 &lt; a &lt; \lambda_3)</td>
<td>(2, 2, 1)</td>
<td>Fig. 17</td>
</tr>
<tr>
<td>diag(a, a, (\lambda_1))</td>
<td>(a = 0 &lt; \lambda_3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>diag(a, a, (\lambda_1))</td>
<td>(a = \lambda_3 &lt; 0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>diag(a, a, (\lambda_1))</td>
<td>(\lambda_1 = a &lt; 0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>diag(a, a, (\lambda_1))</td>
<td>(a &lt; 0 &lt; \lambda_3)</td>
<td>(1, 3)</td>
<td>Fig. 19</td>
</tr>
<tr>
<td>diag(a, a, (\lambda_1))</td>
<td>(a = 0 &lt; \lambda_3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>diag(a, a, (\lambda_1))</td>
<td>(a = \lambda_3 &lt; 0)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* A non-diagonalizable matrix is included in these cases.

1. The orbit starting at \((0, 0, -1)\) and ending at \((0, 0, 1)\).
2. The orbit starting at \((0, 0, 1)\) and ending at \((0, 0, -1)\).

The Morse graph is given in Fig. 31.

Next, for \(A \in \text{gl}(3, \mathbb{R})\) we show in Tables 3 and 4 the general situation in dimension three. The last column on the right side identifies the systems which have same Morse graphs. The eigenvalues are given by \(\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\) and \(\mu_{2,3} = a \pm ib \in \mathbb{C}\).

References