On the level-continuity of fuzzy integrals

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Abstract

In this paper we define the level-convergence of measurable functions on a fuzzy measure space, by using the closure operator in the Moore sense. We study some of the properties of this convergence and give conditions for the continuity of the fuzzy integral in relation to the level-convergence.

Keywords: Generalized measure theory; Fuzzy measure; Fuzzy integration; Levelwise-convergence

1. Introduction

The theory of fuzzy integration with respect to a fuzzy measure was introduced by Sugeno in [8] as a model for the treatment of non-deterministic problems. In particular, the continuity of the fuzzy integral with respect to different kinds of convergence has been exhaustively studied in the last years.


Finally, Greco and Bassanezi [2] and Román-Flores et al. [6] by using the concept of F-continuity and autocontinuity, respectively, of a fuzzy measure \( \mu \) with respect to another fuzzy measure \( \nu \), removed the continuity condition and improved the Wang results. Recently [7], we studied different kinds of multivalued convergences for fuzzy sets on \( \mathbb{R}^n \) and its relationships.

The aim of this paper is to analyze the continuity of the fuzzy integral with respect to multivalued convergences, more precisely we introduce the concept of level-convergence (L-convergence) on a fuzzy measure space \( X \), with respect to a closure operator on \( X \), study some of its properties and give conditions for the continuity of the fuzzy integral in relation to the L-convergence.

The structure of this paper is as follows. In Section 2 we give the previous results that will be used in the article. In Section 3 we introduce the concept of level-convergence (L-convergence) on a fuzzy measure space \( X, \Sigma, \mu \) and we compare these concepts with pointwise convergence. In Section 4 we introduce the concept of endographic-convergence (\( \Gamma \)-convergence) on a fuzzy measure space and, under adequate conditions, we prove the equivalence between \( \Gamma \)-convergence and L-convergence. Finally, in Section 5, we investigate the continuity of the
fuzzy integral with respect to the L-convergence and, under suitable conditions, we prove that this continuity is equivalent to the continuity of the fuzzy measure \( \mu \), i.e.: 

\[
(f_n \xrightarrow{L} f \Rightarrow \int f_n \, d\mu \to \int f \, d\mu) \iff \mu \text{ continuous.}
\]

2. Preliminaries

**Definition 2.1.** Let \( X \) be a set and \( \Sigma \) be a \( \sigma \)-algebra of subsets of \( X \). By fuzzy measure we mean a positive, extended real-valued set function \( \mu : \Sigma \to [0, \infty] \) with properties:

- (FM1) \( \mu(\emptyset) = 0 \).
- (FM2) \( A, B \in \Sigma \) and \( A \subseteq B \Rightarrow \mu(A) \leq \mu(B) \).

Furthermore, if:

- (FM3) \( A_1 \supseteq A_2 \supseteq \cdots, A_n \in \Sigma \Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n) \),

then \( \mu \) is upper continuous. Analogously, we say that \( \mu \) is lower continuous if:

- (FM4) \( A_1 \supseteq A_2 \supseteq \cdots, A_n \in \Sigma \) and there exists \( n_0 \) such that \( \mu(A_{n_0}) < \infty \), then

\[
\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n).
\]

If \( \mu \) satisfies (FM3) and (FM4) we say that \( \mu \) is continuous.

Throughout this paper \((X, \Sigma, \mu)\) will be a fuzzy measure space and \( M(X) \) the family of all measurable functions \( f : X \to [0, \infty] \).

If \( f \in M(X) \), then the fuzzy integral of \( f \) is defined in [8] as

\[
\int_{\Delta} f \, d\mu = \bigvee_{x \geq 0} \mu(A \cup \{ f \geq x \}), \quad A \subseteq \Sigma,
\]

where \( \bigvee, \bigwedge \) denote the operations sup and inf in \([0, \infty]\). The following properties of the fuzzy integral are well known:

**Theorem 2.2** (Ralescu and Adams [3] and Wang [10]).

- (P1) \( \int_{A} I_A \, d\mu = \mu(A) \) (\( I_A \) is the indicator function of \( A \), i.e. \( I_A(x) = \infty \) if \( x \in A \) and \( I_A(x) = 0 \) if \( x \notin A \)).
- (P2) \( \int_{A} k \, d\mu = k \wedge (\mu(A)) \), \( k \) constant.
- (P3) (i) If \( A \subseteq B \) then \( \int_{A} f \, d\mu \leq \int_{B} f \, d\mu \).
- (ii) If \( f \leq g \) in \( A \) then \( \int_{A} f \, d\mu \leq \int_{A} g \, d\mu \).

**Remark 2.3.** From (P6) we conclude that the fuzzy integral is continuous with respect to uniform convergence.

**Theorem 2.4** (Ralescu and Adams [3]). If \( f : X \to [0, \infty] \) is a measurable function, then

\[
\int_{X} f \, d\mu = \int_{0}^{\infty} \mu\{ f \geq x \} \, dx,
\]

where the integral on the right-hand side of the last equation is the fuzzy integral of \( F(x) = \mu\{ \{ f \geq x \} \) with respect to the Lebesgue measure in \([0, \infty]\).

**Theorem 2.5** (Ralescu and Adams [3]). If \( \mu \) is subadditive (i.e. \( \mu(A \cup B) \leq \mu(A) + \mu(B) \)) and \( f_n \to f \) in measure, then \( \int f_n \, d\mu \to \int f \, d\mu \).

**Theorem 2.6** (Ralescu and Adams [3]). If \( \mu \) is subadditive, \( \mu(X) < \infty \) and \( f_n \to f \) pointwise then \( \int f_n \, d\mu \to \int f \, d\mu \).

Other interesting properties and applications of this integral are discussed in [4, 5, 9].

3. Level-convergence and closure operators

**Definition 3.1.** If \((X, \Sigma, \mu)\) is a fuzzy measure space and \( - : \Sigma \to \Sigma, A \mapsto \overline{A} \) verifies the properties

- (i) \( A \subseteq \overline{A}, \forall A \in \Sigma, \)
- (ii) \( \overline{\overline{A}} = \overline{A}, \forall A \in \Sigma, \)
- (iii) \( A, B \in \Sigma, A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}, \)

then \( - \) is called a closure operator on \( X \), in the Moore sense [1].

**Definition 3.2.** Let \((X, \Sigma, \mu)\) be a fuzzy measure space and \( - \) a closure operator on \( X \). We say that:

- (a) \( A \in \Sigma \) is a strongly measurable set (s-measurable) if \( \mu(A) = \mu(\overline{A}) \).
- (b) \( A \in \Sigma \) is a closed set (with respect to \( - \)) if \( A = \overline{A} \).
Note: If “−” is any closure operator, then

$$A_i \text{ closed } \forall i \in I \Rightarrow \bigcap_{i \in I} A_i \text{ closed.}$$

**Definition 3.3.** We say that a sequence of sets 

$$(A_n), A_n \in \Sigma,$$ converges to $A \in \Sigma$, denoted by $A = \lim A_n$ (in short, $A_n \rightarrow A$), if

$$A = \lim \inf A_n = \lim \sup A_n,$$

where

$$\lim \sup A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k,$$

and

$$\lim \inf A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

**Definition 3.4.** Let $f \in M(X)$ and $\alpha \in [0, \infty]$. Then, the $\alpha$-level of $f$ is defined by

$$L_\alpha f = \{x \in X \mid f(x) \geq \alpha\}.$$

The support of $f$ is defined by

$$\text{supp}(f) = L_0 f = \{x \in X \mid f(x) > 0\} = \bigcup_{\alpha > 0} L_\alpha f.$$

**Definition 3.5.** We say that a sequence of functions 

$$(f_n), f_n \in M(X),$$ L-converges to $f \in M(X)$ (in short, $f_n \xrightarrow{L} f$) if for every $\alpha > 0$, $L_\alpha f_n \rightarrow L_\alpha f$.

The next proposition shows that, under suitable conditions, L-convergence is stronger than pointwise convergence.

**Proposition 3.6.** If $f_n \xrightarrow{L} f$ and $L_\alpha f_n$ is closed $\forall \alpha$, then $f_n \rightarrow f$ pointwise.

**Proof.** Suppose that $f_n \xrightarrow{L} f$ and let $x_0 \in X$ with $f(x_0) = \alpha_0$. Then

$$x_0 \in L_{\alpha_0} f = \lim \inf L_{\alpha_0} f_n = \bigcup_{n=1}^{\infty} \left( \bigcap_{k=n}^{\infty} L_{\alpha_0} f_k \right).$$

Consequently, $\exists n_0 \in N$ such that $x_0 \in \bigcap_{k=n_0}^{\infty} L_{\alpha_0} f_k$. That is, $f_n(x_0) \geq \alpha_0 \forall n \geq n_0$. Hence, $f_n(x_0) \geq \alpha_0 \forall n \geq n_0$. Thus, $\lim \inf f_n(x_0) \geq \alpha_0$.

Now suppose that $\beta_0 = \lim \sup f_n(x_0) > \alpha_0$ and let $\varepsilon > 0$ such that $\beta_0 - \varepsilon > \alpha_0$. Then, $f_n(x_0) \geq \beta_0 - \varepsilon$ for infinite values of $n$. Hence, $x_0 \in L_{\beta_0 - \varepsilon} f_n$ for infinite values of $n$. Consequently,

$$x_0 \in \bigcap_{n=1}^{\infty} \left( \bigcup_{k=n}^{\infty} L_{\beta_0 - \varepsilon} f_k \right) \subseteq \lim \sup L_{\beta_0 - \varepsilon} f_n = L_{\beta_0 - \varepsilon} f.$$

Thus $f(x_0) \geq \beta_0 - \varepsilon > \alpha_0$. But this is impossible since $f(x_0) = \alpha_0$. This implies that $\alpha_0 \leq \lim \inf \ f_n(x_0) \leq \lim \sup f_n(x_0) \leq \alpha_0$. Consequently, $\lim \inf \ f_n(x_0) = f(x_0)$, i.e., $f_n \rightarrow f$ pointwise.

**Corollary 3.7.** With the same conditions of Proposition 3.6, then: $f_n \xrightarrow{\mu} f$ and $\mu$ finite implies $f_n \xrightarrow{L} f$.

In the next example we show that pointwise convergence does not imply L-convergence.

**Example 3.8.** Let $X = R, \Sigma = \sigma$-algebra of Lebesgue measurable sets on $X$, and “−” the usual topological closure on $R$. Define $f_n, f$ by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1, \\ 0 & \text{elsewhere} \end{cases}$$

and

$$f_n(x) = \begin{cases} x + 1 - \frac{1}{n} & \text{if } 0 \leq x < 1, \\ 0 & \text{elsewhere}. \end{cases}$$

Clearly $f_n \rightarrow f$ uniformly, but $L_1 f = [0, 1]$ whereas $L_1 f_n = \{1\}, \forall n$. Thus, $\lim \inf L_1 f_n = \lim \sup L_1 f_n = \{1\} \neq [0, 1] = L_1 f$. Consequently, $(f_n)$ does not converge levelwise to $f$.

**4. Endographic convergence (F-convergence)**

It is possible, under adequate conditions, to give an endographic characterization of L-convergence. If $f \in M(X)$ then the endograph of $f$, denoted by $\text{End}(f)$, is the subset of $X \times [0, \infty]$ defined by

$$\text{End}(f) = \{(x, \alpha) \mid f(x) \geq \alpha\}.$$

**Definition 4.1.** If $f_n, f \in M(X)$ then we say that $(f_n)$ F-converges to $f$ if $\text{End}(f_n) \rightarrow \text{End}(f)$.
a sequence in \( \Sigma \). Then, in this case,

\[
\lim \sup A_n = \bigcap_{n=1}^{\infty} \left( \bigcup_{k=n}^{\infty} A_k \right)
\]

and

\[
\lim \inf A_n = \bigcup_{n=1}^{\infty} \left( \bigcap_{k=n}^{\infty} A_k \right).
\]

**Remark 4.2.** In this case, \( \lim \sup A_n \) consists of all \( x \) which are in infinitely many of the \( A_n \) and \( \lim \inf A_n \) consists of all \( x \) which are in all but finitely many of the \( A_n \).

**Proposition 4.3.** Let \( f_n, f \in M(X) \) and suppose that "\( \rightarrow \)" \( \equiv \) identity. Then the following properties are equivalent:

(i) \( f_n \xrightarrow{L} f \).

(ii) \( f_n \xrightarrow{L} f \).

**Proof.** (i) \( \Rightarrow \) (ii): Suppose that \( f_n \xrightarrow{L} f \) and let \( x \in \bigcap_{n=1}^{\infty} \End(f_n) \) such that \( f(x) \geq \alpha \). Then

\[
(x, \alpha) \in \End(f) = \lim \inf \End(f_n)
\]

\[
= \bigcup_{n=1}^{\infty} \left( \bigcap_{k=n}^{\infty} \End(f_k) \right).
\]

Hence, \( \exists n_0 \in \mathbb{N} \) such that \( (x, \alpha) \in \bigcap_{k=n_0}^{\infty} \End(f_k), \) i.e. \( f_k(x) \geq \alpha, \forall k \geq n_0 \).

This implies that \( x \in \{f_k \geq \alpha\}, \forall k \geq n_0 \), therefore \( x \in \lim \inf \{f_n \geq \alpha\}, \) i.e. \( \{f \geq \alpha\} \subseteq \lim \inf \{f_n \geq \alpha\} \). On the other hand, if \( x \in \lim \sup \{f_n \geq \alpha\} \) then \( x \in \{f_n \geq \alpha\} \), for infinite values of \( n \) (see Remark 4.2), i.e. \( (x, \alpha) \in \End(f_n) \) for infinite values of \( n \), consequently \( (x, \alpha) \in \lim \sup \End(f_n) = \End(f) \). Hence \( \lim \sup \{f_n \geq \alpha\} \subseteq \{f \geq \alpha\} \). Consequently, \( f_n \xrightarrow{L} f \).

(ii) \( \Rightarrow \) (i): Let \( (x, \alpha) \in \lim \sup \End(f_n) \). Then \( (x, \alpha) \in \End(f_n) \) for infinite values of \( n \), therefore

\[
(x, \alpha) \in \lim \sup \{f_n \geq \alpha\} \subseteq \{f \geq \alpha\}.
\]

Thus, \( (x, \alpha) \in \End(f) \). This implies that \( \lim \sup \End(f_n) \subseteq \End(f) \).

Now, let \( (x, \alpha) \in \End(f) \). Then \( f(x) \geq \alpha \) and, by hypothesis,

\[
x \in \lim \inf \{f_n \geq \alpha\}.
\]

Hence \( (x, \alpha) \in \lim \inf \End(f_n) \). Consequently, \( \End(f_n) \rightarrow \End(f) \) and \( f_n \xrightarrow{L} f \).

**5. L-convergence and fuzzy integral**

Here we investigate the continuity of the fuzzy integral with respect to L-convergence.

**Lemma 5.1.** If \( (A_n) \) is a sequence of s-measurable sets with

\[
A_n \rightarrow A, \mu \text{ continuous, and there exists } n_0 \text{ such that}
\]

\[
\mu \left[ \bigcup_{k=n_0}^{\infty} A_k \right] < \infty,
\]

then \( \mu(A_n) \rightarrow \mu(A) \).

**Proof.** \( A = \lim A_n \) implies \( A = \lim \inf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \). Since \( \bigcap_{k=n}^{\infty} A_k \) is an increasing sequence then

\[
\bigcup_{k=n}^{\infty} A_k \subseteq A = \lim \inf A_n.
\]

Now, looking to the fact that \( \bigcap_{k=n}^{\infty} A_k \subseteq \overline{A} \) and the continuity of \( \mu \) on monotone sequences, we obtain

\[
\mu(A) = \mu(\lim \inf(A_n)) = \lim_{n \rightarrow \infty} \mu \left[ \bigcap_{k=n}^{\infty} A_k \right] = \lim \inf \mu \left[ \bigcap_{k=n}^{\infty} A_k \right] \leq \mu(\overline{A}).
\]

Thus, since \( A_n \) is s-measurable, we obtain \( \mu(A) \leq \lim \inf \mu(A_n) \).

Analogously, since \( \bigcup_{k=n}^{\infty} A_k \) is a decreasing sequence, then

\[
\bigcup_{k=n}^{\infty} A_k \supseteq A = \lim \sup A_n.
\]

So, since \( \bigcup_{k=n}^{\infty} A_k \supseteq \overline{A} \) then, by (FM4), we obtain

\[
\mu(A) = \mu(\lim \sup A_n) = \lim_{n \rightarrow \infty} \mu \left[ \bigcup_{k=n}^{\infty} A_k \right] = \lim \sup \mu \left[ \bigcup_{k=n}^{\infty} A_k \right] \geq \mu(\overline{A}).
\]
Consequently, \( \mu(A) \geq \lim \sup \mu(A_n) \). Thus, \( 0 \leq \mu(A) \leq \lim \inf \mu(A_n) \leq \mu(A) \), i.e. \( \mu(A_n) \to \mu(A) \). □

**Theorem 5.2.** Let \( f_n, f \in M(X) \) with \( L_\infty f_n \) s-measurable, \( L_\infty f \) closed \( \forall \alpha > 0 \),

\[
\mu \left( \bigcup_{n=1}^{\infty} \text{supp}(f_n) \right) < \infty \text{ and } \mu \text{ continuous.}
\]

Then \( f_n \xrightarrow{L_\infty} f \Rightarrow \int f_n \, d\mu \to \int f \, d\mu \).

**Proof.** If \( f_n \xrightarrow{L_\infty} f \) then, by definition of \( L_\infty \)-convergence, it follows that

\[
L_\infty f_n \to L_\infty f, \forall \alpha > 0.
\]

Hence, by Lemma 5.1, \( \mu(L_\infty f_n) \to \mu(L_\infty f), \forall \alpha > 0 \).

So, making use of Theorem 2.6 we obtain that

\[
\int \mu(L_\infty f_n) \, dx \to \int \mu(L_\infty f) \, dx.
\]

Thus, by Theorem 2.4, we conclude that \( \int f_n \, d\mu \to \int f \, d\mu \). □

**Corollary 5.3.** Let \( f_n, f \in M(X) \) with \( L_\infty f_n \) closed, \( L_\infty f \) closed \( \forall \alpha > 0 \),

\[
\mu \left( \bigcup_{n=1}^{\infty} \text{supp}(f_n) \right) < \infty \text{ and } \mu \text{ continuous.}
\]

Then, \( f_n \xrightarrow{L_\infty} f \Rightarrow \int f_n \, d\mu \to \int f \, d\mu \).

**Proof.** It is sufficient to see that: \( A \) closed \( \Rightarrow A \) s-measurable. □

The hypothesis \( \mu \left[ \bigcup_{n=1}^{\infty} \text{supp}(f_n) \right] < \infty \) in Theorem 5.2 cannot be avoided as shown by the following example:

**Example 5.4.** Let \((X, \Sigma, \mu, \"-\")\) be as in Example 3.8 and \( \mu \) the usual Lebesgue measure on \( X \). Define \( f_n, f \) by

\[
f_n(x) = \begin{cases} 
\frac{|x|}{n} & \text{if } -n \leq x \leq n, \\
1 & \text{elsewhere}
\end{cases}
\]

and

\[
f(x) = 0, \forall x.
\]

Then \( \text{supp}(f_n) = \{ 0 \}, \forall n \). So, \( \mu \left[ \bigcup_{n=1}^{\infty} \text{supp}(f_n) \right] = \infty \). On the other hand, \( f_n \xrightarrow{L_\infty} f \) and \( \int f_n \, d\mu = 1 \forall n \), whereas \( \int f \, d\mu = 0 \).

**Lemma 5.5.** Let \( A_n, A \in \Sigma \), then \( A_n \to A \) if and only if \( I_{A_n} \xrightarrow{L_\infty} I_A \).

**Proof.** A direct consequence of the fact that \( L_\infty I_{A_n} = I_A \). □

In the case \( \\sim \) \( \equiv \) identity, we obtain the following equivalence:

**Theorem 5.6.** Let \((X, \Sigma, \mu)\) be a finite fuzzy measure space, and \( \sim : \Sigma \to \Sigma \) the identity closure operator (i.e. \( A = A, \forall A \in \Sigma \)). Then the following properties are equivalent:

(i) \( \mu \) is continuous.

(ii) \( f_n \xrightarrow{L_\infty} f \Rightarrow \int f_n \, d\mu \to \int f \, d\mu \).

**Proof.** (ii) \( \Rightarrow \) (i): Let \((A_n)\) be a monotone sequence in \( \Sigma \) and \( A = \lim A_n \). Then, by Lemma 5.5, \( A_n \to A \) implies \( I_{A_n} \xrightarrow{L_\infty} I_A \). Thus, by hypothesis, \( \int I_{A_n} \, d\mu \to \int I_A \, d\mu \).

That is \( \mu(A_n) \to \mu(A) \) (see Remark 2.3). Therefore, \( \mu \) is continuous. □

**References**


