A generalization of the Minkowski embedding theorem and applications

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Abstract

Puri and Ralescu (1985) gave, recently, an extension of the Minkowski Embedding Theorem for the class $E_L$ of fuzzy sets $u$ on $\mathbb{R}^n$ with the level application $\alpha \mapsto L_u$ Lipschitzian on the $C([0,1] \times S^{n-1})$ space. In this work we extend the above result to the class $E_C$ of level-continuous applications. Moreover, we prove that $E_C$ is a complete metric space with $E_L \subset E_C$ and $E_C = E_L$. To prove the last result, we use the multivalued Bernstein polynomials and the Vitali's approximation theorem for multifunction. Also, we deduce some properties in the setting of fuzzy random variable (multivalued).

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1. Introduction

Recently, Puri and Ralescu [15] showed that there is an embedding $j: E_L \rightarrow C([0,1] \times S^{n-1})$, where $E_L$ is the subspace of $(E^n, D)$ with Lipschitzian levels and $E^n$ denote the class of normal convex fuzzy sets with compact support. This fact is very important, since $(E^n, D)$ is not separable and this is an empeachment to develop clearly an integration theory for the fuzzy random variables. Unfortunately $(E_L, D)$ is not a complete subspace of $(E^n, D)$ as will be shown in Section 4. We observe that the application $j$ can be defined by the same expression as in [14, 15] for all $E^n$ (obviously with a different image space). So the following question is raised: is there some subspace of $(E^n, D)$ that is separable and complete for the metric $D$ and in such a manner that it is still embedded in $C([0,1] \times S^{n-1})$ for all $n$?

In this paper, we prove that the space that answers the above question is $E_C$, which consists of the fuzzy sets with levels continuous; also we prove that $E_L \not\subset E_C$ and that $E_C$ is the maximal subspace of $E^n$ with this property and $E_L = E_C$ (see Section 3). Also, we give some applications for the theory of fuzzy random variables (see Section 4). Actually, it was only for simplicity that we derived our results

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in $\mathbb{R}^n$; they extend easily to the case of real separable Banach Spaces.

To stress the importance of the embedding $j$, we recall that Kaleva [10] used the embedding $j$ together with one characterization of the fuzzy compact subsets of $\mathbb{E}^n$, see [7], in the subclass of $\mathbb{E}_L^n$, which he called equi-Lipschitzian, to demonstrate the existence of the solutions of the Cauchy Problem for fuzzy differential equations with values in the equi-Lipschitzian subsets. Also, Puri and Ralescu used the embedding to study fuzzy random variables and the convergences of fuzzy martingales.

In a previous work [18], we proved the equivalence of the various notions of convergence in the class of fuzzy sets with continuous levels, but not necessarily with convex levels. Obviously, these results are true for $\mathbb{E}^2$.

In a forthcoming paper we will describe applications to the problem of the convergence of fuzzy martingales (multivalued).

2. Preliminaries

In the sequel $\mathcal{K}(\mathbb{R}^n)$ will denote the set of the nonempty compact-convex subsets of $\mathbb{R}^n$. The Hausdorff metric $H$ over this class is defined by

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\},$$

where $d$ is usual distance, and $d(a, B) = \sup_{b \in B} d(a, b)$.

It is well known that $(\mathcal{K}(\mathbb{R}^n), H)$ is a separable complete metric space (see [6, 14]).

We can understand a fuzzy set in $\mathbb{R}^n$ as a function $u: \mathbb{R}^n \to [0, 1]$. As an extension of $\mathcal{K}(\mathbb{R}^n)$, we define the space $\mathbb{E}^n$ of fuzzy sets $u: \mathbb{R}^n \to [0, 1]$, with the following properties:

(i) $u$ is normal, i.e., $\{x \in \mathbb{R}^n | u(x) = 1\} \neq \emptyset$;

(ii) $u$ is fuzzy-convex, i.e., for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ we have,

$$u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\};$$

(iii) $u$ is upper semicontinuous;

(iv) The closure of the set $\{x \in \mathbb{R}^n | u(x) > 0\}$ is a nonempty compact subset in $\mathbb{R}^n$. This set is called the support of $u$ and it is denoted by $L_0 u$.

The linear structure in $\mathbb{E}^n$ is defined by the operations

$$(u + v)(x) = \sup_{y + z = x} \min\{u(y), v(z)\},$$

$$\lambda u(x) = \begin{cases} u(x/\lambda) & \text{if } \lambda \neq 0, \\ \chi_{\{0\}}(x) & \text{if } \lambda = 0, \end{cases}$$

where $u, v \in \mathbb{E}^n$, $\lambda \in \mathbb{R}$ and $\chi_A$ denote the characteristic function of $A$.

Recall that every fuzzy set is characterized by its family of $\alpha$-level ($\alpha \in (0, 1]$), where the $\alpha$-level of $u$ is defined by

$$L_\alpha u = \{x \in \mathbb{R}^n | u(x) \geq \alpha\}.$$ 

We observe that $L_0 u \supseteq L_\alpha u \supseteq L_\beta u$ for all $0 \leq \alpha \leq \beta$. So if $u \in \mathbb{E}^n$, then $L_{\alpha} u \in \mathcal{K}(\mathbb{R}^n)$ for all $\alpha \in [0, 1]$.

Moreover, the linear structure in terms of the family $(L_\alpha u)$ is given by

$$L_\alpha (u + v) = L_\alpha u + L_\alpha v \quad (1)$$

and

$$L_\alpha (\lambda u) = \lambda L_\alpha u \quad (2)$$

for all $\alpha \in [0, 1]$.

We extend the Hausdorff metric by defining the $D$-metric:

$$D(u, v) = \sup_{0 < \alpha < 1} H(L_\alpha u, L_\alpha v).$$

Concerning the properties of this space, Puri and Ralescu [15] proved that $(\mathbb{E}^n, D)$ is a complete metric space; Kaleva proved that $(\mathbb{E}^n, D)$ is not separable ([10], see Example 2.1).

The $D$-metric is homogeneous and invariant by translations under the operations (1) and (2), and consequently, applying the theorem of Radström [17], Diamond and Kloeden [7], Kaleva [10] and, Puri and Ralescu [15] showed that $\mathbb{E}^n$ can be embedded as a convex cone in certain Banach spaces.

We denote by $\mathbb{E}_L^n$ the subspace of $\mathbb{E}^n$ for which the elements $u$ are such that the mapping $\alpha \to L_\alpha u$ is $H$-continuous on $[0, 1]$, i.e., given $\epsilon > 0$, there is a $\delta > 0$ such that

$$|\alpha - \beta| < \delta \Rightarrow H(L_\alpha u, L_\beta u) < \epsilon.$$ 

Since $[0, 1]$ is a compact metric space, the application $\alpha \to L_\alpha u$ is, in fact, uniformly continuous.
Also, we denote by \( E_\beta \) the subspace of \( E_\alpha \) for which the elements \( u \) are such that the application \( x \rightarrow L_\alpha u \) is Lipschitz continuous, i.e., there is a \( v > 0 \) such that, for all \( x, \beta \in [0, 1] \)
\[
H(L_\alpha u, L_\beta u) \leq v |x - \beta|.
\]

The following example shows that \( E_\alpha \not\subset E_\beta \).

**Example 2.1.** Let \( u: \mathbb{R} \rightarrow [0, 1] \) defined by
\[
u(x) = \begin{cases} x^2 & \text{if } x \in [0, 1] \\ 0 & \text{if } x \in (0, 1) \cup (1, \infty) \end{cases}
\]
Then, \( L_\alpha u = \sqrt{x}, 1 \) for all \( x \in [0, 1] \). Consequently,
\[
H(L_\alpha u, L_\beta u) = |\sqrt{x} - \sqrt{\beta}| = \frac{1}{\sqrt{x} + \sqrt{\beta}} |x - \beta|
\]
for all \( x \neq \beta \). So \( u \in E_\alpha \setminus E_\beta \).

By using the following properties of the \( H \)-metric,
\[
H(A + B, C + D) \leq H(A, C) + H(B, D),
\]
\[
H(\lambda A, \lambda B) = \lambda H(A, B)
\]
for all \( A, B \in \mathcal{K}(\mathbb{R}^n) \) and \( \lambda > 0 \), we deduced that \( E_\alpha \) and \( E_\beta \) are closed under the operations (1) and (2).

Moreover, recall that the support function of a nonempty subset \( A \) of \( \mathbb{R}^n \) is the function \( s_A: S^{n-1} \rightarrow \mathbb{R} \) defined by
\[
s_A(x) = \sup \{ \langle x, \alpha \rangle / \alpha \in A \},
\]
where \( S^{n-1} = \{ x \in \mathbb{R}^n / \| x \| = 1 \} \) and \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( \mathbb{R}^n \). If we take \( A \in \mathcal{K}(\mathbb{R}^n) \), then
\[
s_A(x) = \max \{ \langle x, \alpha \rangle / \alpha \in A \}.
\]

Some properties of the function \( s_A(\cdot) \) are
\[
s_{A+B} = s_A + s_B, \quad (3)
\]
\[
s_{\lambda A} = \lambda s_A, \quad (4)
\]
\( s_A \) is Lipschitz continuous with constant:
\[
\| A \| = H(\{0\}, A). \quad (5)
\]

Moreover, the \( H \)-metric can be written as
\[
H(A, B) = \max \{ |s_A(x) - s_B(x)| ; x \in S^{n-1} \}. \quad (6)
\]

The above result can be seen in [2] or [4]. We consider \( C(S^{n-1}) = \{ f: S^{n-1} \rightarrow \mathbb{R} ; f \text{ is continuous} \} \) with usual norm \( \| . \|_\infty \) of uniform convergence.

The following Embedding Theorem is due to Minkowski:

**Theorem 2.2.** The application \( j: \mathcal{K}(\mathbb{R}^n) \rightarrow C(S^{n-1}) \) defined by \( j(A) = s_A \) is positively homogeneous, additive, and it is also an isometry.

Puri and Ralescu [15] extended the definition of support functions to the fuzzy context setting
\[
s_u(\alpha, x) = s_{L_\alpha u}(x)
\]
for all \( (\alpha, x) \in [0, 1] \times S^{n-1} \) (see also [5]).

It is easily seen that \( s_{u+v} = s_u + s_v \) and \( s_{\lambda u} = \lambda s_u \) for all \( u, v \in \mathbb{E}_\alpha^\beta \) and \( \lambda > 0 \). We denote by \( C([0, 1] \times S^{n-1}) = \{ f: [0, 1] \times S^{n-1} \rightarrow \mathbb{R} ; f \text{ continuous} \} \) with the usual norm.

One of the principal results of [15] is

**Theorem 2.3.** The application \( j: E_\alpha^\beta \rightarrow C([0, 1] \times S^{n-1}) \) defined by \( j(u) = s_u \) is positively homogeneous, additive, and it is also an isometry. Moreover, \( j(u) \) is Lipschitz continuous.

### 3. The isometry \( j \) defined on \( E_\alpha^\beta \)

Our purpose in this section is to show that \( E_\alpha^\beta \) is a complete metric space and that \( j \) is an isometry when defined on \( E_\alpha^\beta \) with values on \( C([0, 1] \times S^{n-1}) \). Also, we will show that \( \mathcal{E}_\beta \) is dense in \( E_\alpha^\beta \) and that \( E_\alpha^\beta \) is the maximal subspace of \( E_\alpha^\beta \) with this property.

**Theorem 3.1.** \((E_\alpha^\beta, D)\) is a complete metric space.

**Proof.** Let \((u_p)\) a \( D \)-Cauchy sequence in \( E_\alpha^\beta \). Then, by using the completeness of \( E_\alpha^\beta \), we deduce that there exist \( u \in E_\alpha^\beta \) such that \( u_p \xrightarrow{D} u \).

In continuation, we prove that \( u \in E_\alpha^\beta \). In fact, given \( \varepsilon > 0 \) there is \( n \in \mathbb{N} \) such that \( D(u_p, u) < \frac{1}{n} \varepsilon \) for all \( p \geq n \). For a fixed \( p_0 > n \), we have that there exist \( \delta = \delta(\varepsilon, p_0) > 0 \) such that
\[
|\alpha - \beta| < \delta \Rightarrow H(L_\alpha u_p, L_\beta u_p) < \frac{1}{2} \varepsilon,
\]
for all \( |\alpha - \beta| < \delta \). So, \( u \in E_\alpha^\beta \) and this completes the proof. \( \Box \)
To show the density of \( \mathbb{E}_L^n \) in \( \mathbb{E}_F^n \), we will use the Kuratowski’s limits related with the \( H \)-convergence.

If \( A_q \) is a sequence of subsets of \( \mathbb{R}^n \), we define the lower and upper limits in the Kuratowski sense as

\[
\lim_{q \to \infty} \inf A_q = \left\{ x \in \mathbb{R}^n \mid x = \lim_{q \to \infty} x_q, x_q \in A_q \right\}
\]

and

\[
\lim_{q \to \infty} \sup A_q = \left\{ x \in \mathbb{R}^n \mid x = \lim_{j \to \infty} x_q, x_q \in A_q \right\}
\]

respectively.

We say that a sequence of sets \( A_q \) converges to a set \( A \), \( A \subseteq \mathbb{R}^n \), in the Kuratowski sense, if \( \lim_{q \to \infty} \inf A_q = \lim_{q \to \infty} \sup A_q = A; \) in this case, we write \( A = \lim_{q \to \infty} A_q \) or \( A_q \xrightarrow{K} A \), and we say that \( A_q \) \( K \)-converges to \( A \).

The Kuratowski limits are closed sets. Moreover, the following relations are true:

\[
\lim_{q \to \infty} \inf A_q \subseteq \lim_{q \to \infty} \sup A_q,
\]

\[
\lim_{q \to \infty} \inf A_q = \lim_{q \to \infty} \inf A_q,
\]

\[
\lim_{q \to \infty} \sup A_q = \lim_{q \to \infty} \sup A_q.
\]

The following result is well known (see [14]).

**Lemma 3.2.** A sequence \( (A_q) \subseteq \mathcal{K}(\mathbb{R}^n) \) converges to a compact set \( A \) with respect to the Hausdorff metric if and only if there is a \( K \in \mathcal{K}(\mathbb{R}^n) \) such that \( A_q \subseteq K \) for all \( q \) and

\[
\lim_{q \to \infty} \inf A_q = \lim_{q \to \infty} \sup A_q = A.
\]

**Theorem 3.3.** \( \mathbb{E}_L^n = \mathbb{E}_F^n \).

**Proof.** Let \( u \in \mathbb{E}_F^n \), then the multifunction \( F: [0, 1] \to \mathcal{K}(\mathbb{R}^n) \) given by \( F(\alpha) = L_{x, u} \) is \( H \)-continuous on \([0, 1]\). We consider the \( q \)th Bernstein polynomial \( B_q(F; \alpha) \) associated with \( F \):

\[
B_q(F; \alpha) = \sum_{j=0}^{q} \binom{q}{j} F\left(\frac{j}{q}\right) \alpha^j(1 - \alpha)^{q-j}, \quad 0 \leq \alpha \leq 1
\]

Vitale [20] has proved that

\[
D(F, B_q(F, \cdot)) \to 0 \quad \text{as} \quad q \to +\infty.
\]

We observe that \( B_q(F; \alpha) \in \mathcal{K}(\mathbb{R}^n) \) for each \( q \in \mathbb{N} \) and \( \alpha \in [0, 1] \). Now we verify the hypothesis of the Representation Theorem given by Negoita and Ralescu [13] to show that the family \( N_q = B_q(F; \alpha) \), for each \( q \in \mathbb{N} \), define an unique fuzzy set. If \( \alpha \leq \beta \), then \( F(\alpha) \supseteq F(\beta) \) and, consequently, \( B_q(F; \alpha) \supseteq B_q(F; \beta) \) (see [20, p. 312]). So, we only have to prove that, if \( x_1 \leq x_2 \leq \cdots \leq x_l \to \alpha \neq 0 \) as \( l \to \infty \), then

\[
B_q(F; \alpha) = \bigcap_{l=1}^{\infty} B_q(F; x_l).
\]

We observe that \( \alpha \to B_q(F; \alpha) \) is a \( H \)-continuous multifunction, consequently, for each fixed \( q \), we have

\[
B_q(F; \alpha) \xrightarrow{H} B_q(F; \alpha) \quad \text{as} \quad l \to \infty,
\]

as \( B_q(F; \alpha) \in \mathcal{K}(\mathbb{R}^n) \), and we deduce from the Lemma 3.2 that

\[
B_q(F; \alpha) \xrightarrow{K} B_q(F; \alpha) \quad \text{as} \quad l \to \infty. \tag{7}
\]

Being \( \{B_q(F; \alpha)\}_{l \in \mathbb{N}} \), one decreasing sequence, we have

\[
B_q(F; \alpha) = \bigcap_{l=1}^{\infty} B_q(F; \alpha) \quad \text{as} \quad l \to \infty.
\]

So it follows from (7) that it holds the required equality.

This completes the hypothesis of the Negoita–Ralescu theorem.

Finally, we prove that \( \alpha \to B_q(F; \alpha) \) for each \( q \in \mathbb{N} \) is a Lipschitzian application. By virtue of (6) it is sufficient to show that

\[
\max \{ |s_{B_q(F, \alpha)}(x) - s_{B_q(F, \beta)}(x)| ; \ x \in S^{n-1} \} \leq C|\alpha - \beta|
\]

with \( C > 0 \) independent of \( \alpha \) and \( \beta \).

Note that the support function of Bernstein approximant of \( F \) is given by

\[
s_{B_q(F, \alpha)}(x) = \sum_{j=0}^{q} \binom{q}{j} x^j(1 - x)^{q-j} s_{F(\cdot)}(x)
\]
with \( x \in S^{n-1} \), so that
\[
|s_{B_{\epsilon}}(x) - s_{B_{\beta}}(x)|
\leq \sum_{j=0}^{q} \binom{q}{j} |s_{F_{j}}(x)||x^{j}(1-x)^{q-j} - \beta^{j}(1-\beta)^{q-j}|
\leq |s_{F_{0}}|_{\infty} \sum_{j=0}^{q} \binom{q}{j} |x^{j}(1-x)^{q-j} - \beta^{j}(1-\beta)^{q-j}|
\leq C|\alpha - \beta|.
\]

Since

\[
F(0) \supseteq F\left(\frac{1}{q}\right) \supseteq F(1) \quad \forall 0 < j < q
\]

implies that
\[
s_{F_{1}}(x) \leq s_{F_{q}}(x) \leq s_{F_{0}}(x)
\]

for all \( x \in S^{n-1} \). This completes the proof.

Now, we give an extension of the Theorem 2.3.

**Theorem 3.4.** The application \( j: \mathbb{E}_{C}^{n} \to C([-1, 1] \times S^{n-1}) \) defined by \( j(u) = s_{u} \) is positively homogeneous, additive and it is also an isometry.

**Proof.** Since \( \mathbb{E}_{L} \) is dense in \( \mathbb{E}_{C}^{n} \), \( j: \mathbb{E}_{C}^{n} \to C([-1, 1] \times S^{n-1}) \) has a unique uniformly continuous extension to \( \mathbb{E}_{C}^{n} \) and it is easy to show that the extension is also an isometry, see, for instance, [8] or [1].

Since \([0, 1] \times S^{n-1}\) is compact, we can deduce immediately the following:

**Corollary 3.5.** \((\mathbb{E}_{C}^{n}, D)\) is a separable metric space.

**Corollary 3.6.** If \( u_{p}, u \in \mathbb{E}_{C}^{n} \), then \( u_{p} \to u \) iff \( s_{u_{p}} \to s_{u} \) uniformly on \([0, 1] \times S^{n-1}\).

In what follows, we show that \( \mathbb{E}_{C}^{n} \) is the maximal subspace of \( E^{n} \) that can be embedded in \( C([-1, 1] \times S^{n-1}) \) through the isometry \( j \).

**Theorem 3.7.** Let \( u \in \mathbb{E}_{C}^{n} \backslash \mathbb{E}_{C}^{n} \) be, then \( j(u) \notin C([-1, 1] \times S^{n-1}) \).

**Proof.** Let \( j(u) \in C([-1, 1] \times S^{n-1}) \) then, for all \( \varepsilon > 0 \), there exists \( \delta > 0 \), such that,
\[
|| (\alpha, x), (\beta, y) || < \delta \Rightarrow |j(u)(\alpha, x) - j(u)(\beta, y)| < \varepsilon. \tag{\ast}
\]

If we suppose that \( u \notin \mathbb{E}_{C}^{n} \) with \( j(u) \in C([-1, 1] \times S^{n-1}) \), we have that the map \( \alpha \to L_{u} \) is not continuous: consequently there exists \( \varepsilon > 0 \) such that for all \( \delta > 0 \), we can take \( \alpha \) and \( \beta \) with \( |\alpha - \beta| < \delta \) and
\[
H(L_{u}, L_{\beta}) = \sup_{x \in S^{n-1}} \{|j(u)(\alpha, x) - j(u)(\beta, x)|\} > \varepsilon.
\]

Now through the compactness of \( S^{n-1} \) and the continuity of \( j(u) \), there exists \( x_{0} \in S^{n-1} \) such that
\[
|j(u)(\alpha, x_{0}) - j(u)(\beta, x_{0})| > \varepsilon
\]

i.e. a contradiction with (\ast). \( \Box \)

**Remark 3.8.** Theorem 3.4, together with Theorem 3.7 provides a complete characterization for \( j \) to be an isometry with values in \( C([-1, 1] \times S^{n-1}) \).

We proved in [18, Lemma 3.13] the following result: \( u \in \mathbb{E}_{C}^{n} \) if and only if
\[
L_{u} = \{ x | u(x) > \alpha \} \quad \forall \alpha \in (0, 1).
\]

This property does not suppose any convexity hypothesis on \( L_{u} \). The above characterization is equivalent to say that \( u \) is without proper local maximum points (see [18, p. 223]). Another characterization was given by Ming [12, p. 316]: \( u \in \mathbb{E}_{C}^{n} \) if and only if for any \( \alpha \in (0, 1) \)
\[
u^{\alpha} = \{ x | u(x) = \alpha \}
\]

has no interior point in the subspace \( L_{u} \).

We were informed by the anonymous referee that Theorem 3.4 is coincident with Theorem 3.3 in [12]. However, the techniques and arguments used are totally different.

**Remark 3.9.** Theorem 3.3 shows that \( \mathbb{E}_{C}^{n} \) is an incomplete metric space.

4. Applications

In this section we give some applications of our previous results to the convergence of the fuzzy random variables in \( \mathbb{E}_{C}^{n} \).

We will briefly go over some basic material on the measurability and integration of multifunctions that we will need in the sequel. For more details we
refer to Aumann [3], Castaing and Valadier [6], Hukuhara [9] and, Klein and Thompson [14].

Let \( P(\mathbb{R}^n) \) be the set of nonempty subsets of \( \mathbb{R}^n \) and \((\Omega, \Sigma, \mu)\) a complete finite measure space. Let \( F: \Omega \to P(\mathbb{R}^n) \) be a multifunction from \( \Omega \) onto \( \mathbb{R}^n \). Let \( \text{Gr}(F) = \{(w, x) \in \Omega \times \mathbb{R}^n / x \in F(w)\} \) be the graph of \( F \). We say that \( F \) is measurable if \( \text{Gr}(F) \subseteq \Sigma \times \mathcal{B}(\mathbb{R}^n) \), where \( \mathcal{B}(\mathbb{R}^n) \) is the Borel \( \sigma \)-field of \( \mathbb{R}^n \).

For any multifunction \( F: \Omega \to P(\mathbb{R}^n) \) we can define the set \( S(F) = \{ f \in L^1(\Omega, \mathcal{B}(\mathbb{R}^n)) / f(w) \in F(w), \mu - \text{a.e.} \} \), i.e., \( S(F) \) contains all integrable selectors of \( F \). The integral introduced by Aumann [2] as a generalization of the single-valued integral and of the Minkowski sum of sets is defined by

\[
\int_{\Omega} F(w) \, d\mu(w) = \left\{ \int_{\Omega} f(w) \, d\mu(w) \mid f \in S(F) \right\}.
\]

and denoted simply by \( \{F\} \).

It is natural to ask under what conditions \( \{F\} \) (or equivalently, \( S(F) \)) is nonempty. The multifunction \( F \) will be called integrably bounded if there exist \( \varphi \in L^1(\Omega, \mathbb{R}) \) such that \( \|x\| \leq \varphi(w) \mu - \text{a.e.} \), almost all \( x \) and \( w \) such that \( x \in F(w) \).

Theorem 4.1. If the measure \( \mu \) on the \( \sigma \)-algebra \( \Sigma \) of \( \Omega \) is atomless, then the integral \( \{F\} \) is a convex set.

Theorem 4.2. If \( F \) is integrably bounded and \( F(w) \) is closed for almost all \( w \in \Omega \), then \( \{F\} \subseteq \mathcal{K}(\mathbb{R}^n) \).

Also, we mention the following generalization of Lebesgue’s dominated convergence theorem.

Theorem 4.3. If \( F_\alpha: \Omega \to P(\mathbb{R}^n) \) are measurable and there is an \( f \in L^1(\Omega, \mathbb{R}) \) such that \( \sup_{p \geq 1} \|g_p(w)\| \leq f(w) \) for all \( g_p \in S(F_\alpha) \), then if \( F_\alpha(w) \subseteq F(w) \) we have

\[
\int_{\Omega} F_\alpha \to \int_{\Omega} F \quad \text{as} \quad p \to \infty.
\]

Theorem 4.4. Let \( F: \Omega \to P(\mathbb{R}^n) \) be measurable and integrably bounded, if \( \pi \) is a linear form over \( \mathbb{R}^n \), then

\[
\sup \pi \left( \int_{\Omega} F \, d\mu \right) = \int \sup \pi(F(w)) \, d\mu(w).
\]

A fuzzy random variable is a function \( \Gamma: \Omega \to \mathbb{E}^n \) such that for every \( \alpha \in [0, 1] \) the multifunction \( \Gamma_\alpha: \Omega \to P(\mathbb{R}^n) \) defined by \( \Gamma_\alpha(w) = L_\alpha \Gamma(w) \) is measurable [10]. Moreover, we say that \( \Gamma \) is integrably bounded if \( \Gamma_\alpha \) is integrably bounded for all \( \alpha \in [0, 1] \). We observe that for \( \Gamma \) to be integrably bounded it is necessary and sufficient that \( \Gamma_0 \) be integrably bounded; this is a consequence of the following fact: \( 0 \leq \alpha \leq \beta \) implies \( \Gamma_\beta(w) \subseteq \Gamma_\alpha(w) \subseteq \Gamma_0(w) \), for all \( w \in \Omega \).

The following theorem due to Puri and Ralescu [15] allows us to define the integral of a fuzzy random variable \( \Gamma: \Omega \to \mathbb{E}^n \).

Theorem 4.5. If \( \Gamma: \Omega \to \mathbb{E}^n \) is an integrably bounded fuzzy variable, there exists a unique fuzzy set \( u \in \mathbb{E}^n \) such that \( L_\alpha u = \int \Gamma_\alpha \, d\mu \), for every \( \alpha \in [0, 1] \).

The element \( u \in \mathbb{E}^n \) obtained in Theorem 4.5 defines the integral of the fuzzy random variable \( \Gamma \), i.e.,

\[
\int \Gamma \, d\mu = u \iff L_\alpha u = \int \Gamma_\alpha \, d\mu, \quad \text{for every} \quad \alpha \in [0, 1].
\]

Theorem 4.6. Let \( \Gamma: \Omega \to \mathbb{E}^n \) be a fuzzy random variable integrably bounded, then \( \{\Gamma\} \subseteq \mathbb{E}^n \).

Proof. We consider a sequence \( (\alpha_p) \subseteq [0, 1] \) such that \( \alpha_p \to \alpha \), \( \alpha \in [0, 1] \). Since \( F(w) \in \mathbb{E}^n \), it follows that \( L_{\alpha_p} \Gamma(w) \to L_\alpha \Gamma(w) \) for all \( w \in \Omega \) as \( p \to \infty \). Thus, we deduce that for all \( w \in \Omega \), \( \Gamma_{\alpha_p}(w) \to \Gamma_\alpha(w) \) as \( p \to \infty \). Moreover, being \( \Gamma \) integrably bounded, we conclude that each \( \Gamma_{\alpha_p} \) is also integrably bounded, and if \( f \in L^1(\Omega, \mathbb{E}^n) \) is such that for all \( \alpha \in \Gamma_0(w): \|x\| \leq f(w) \), we also conclude that \( \sup_{p \geq 1} \{\|x\| \mid x \in \Gamma_{\alpha_p}(w)\} \leq f(w) \). Consequently, using Theorem 4.3, we have \( \{\Gamma_{\alpha_p}\} \to \{\Gamma_\alpha\} \) as \( p \to \infty \). In other words, \( L_{\alpha_p} \Gamma = L_\alpha \Gamma \) as \( p \to \infty \), and therefore \( \{\Gamma\} \subseteq \mathbb{E}^n \).

Corollary 4.7. Let \( \Gamma_\alpha: \Omega \to \mathbb{E}^n \) be an integrably bounded fuzzy random variable. Then \( \int \Gamma_\alpha \to \int \Gamma \) on \( (\mathbb{E}^n, D) \leftrightarrow s_{|f_{\alpha_p}} \to s_{|f} \) on \( C([0, 1] \times S^{n-1}), \|\cdot\|_\infty \).

Also, we have
Theorem 4.8. Let $\Gamma: \Omega \to \mathbb{E}^n_C$ be an integrably bounded fuzzy random variable. Then,

$$s_{\Gamma}(x, x) = \int s_{\Gamma(x)}(x, x) \, d\mu(w).$$

Proof. It follows immediately from Theorem 4.4. In fact,

$$s_{\Gamma}(x, x) = s_{L_{\Gamma}}(x) = s_{L_{\Gamma}}(x) = \int s_{\Gamma(x)}(x) \, d\mu(w) = \int s_{\mu}(x, x) \, d\mu(w).$$

Remark 4.9. It is well known that if $A \in \mathcal{K}(\mathbb{R}^n)$ then

$$A = \bigcap_{y \in S^{n-1}} \{ x \in \mathbb{R}^n | \langle x, y \rangle \leq s_A(y) \},$$

see [2] or [4].

If we apply this in the fuzzy context, we have that if $u \in E^p_C$, then for each $x \in [0, 1]$

$$L_x u = \bigcap_{y \in S^{n-1}} \{ x \in \mathbb{R}^n | \langle x, y \rangle \leq s_u(x, x) \}.$$

Thus, given some relations involving fuzzy sets in $E^p_C$, we obtain the corresponding relations for the fuzzy support function $s_u$. On the other hand, from relations involving fuzzy support functions $s_u$ we can obtain analogous relations for the $z$-level of the fuzzy set $u \in E^p_C$ and, consequently, for $u$. Thus, we can apply the duality theory between support functions and $\mathcal{K}(\mathbb{R}^n)$ in the fuzzy context.

Remark 4.10. For related results see [11].

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References


