About the continuity of reachable sets of restricted affine control systems

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\textbf{A R T I C L E    I N F O}

Article history:
Received 30 August 2016
Revised 11 November 2016
Accepted 14 November 2016
Available online 26 November 2016

MSC:
93B03
93B99
93C15

Keywords:
Affine system
Accessible sets
Lower semi-continuity
Hausdorff metric

\textbf{A B S T R A C T}

In this paper we prove that for a restricted affine control system on a connected manifold \(M\), the associated reachable sets up to the time \(t\) varies continuously in each independent variable: time, state and the range of the admissible control functions. However, as a global map it is just lower semi-continuous. We show a bilinear control system on the plane where the global map has a discontinuity point. According to the Pontryagin Maximum Principal, in order to synthesizes the optimal control the Hausdorff metric continuity is crucial. We mention some references with concrete applications. Finally, we apply the result to the class of Linear control systems on Lie groups.

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1. Introduction

A control system \(\Sigma = (M, \mathcal{D})\) is determined by a manifold \(M\) and a family of differential equations \(\mathcal{D}\) induced by a class of admissible control functions. For \(x \in M\) the accessible set of \(\Sigma\) from \(x\), i.e., the set of points that can be reached from \(x\) through all possible \(\mathcal{D}\)-trajectories in positive time, have been investigated in several works from different points of view. For instance, a control system has the accessibility property from \(x\) if the reachable set from \(x\), has non-empty interior in the \(M\) topology.\cite{25,42}.

The description of this class of sets have been analyzed by, Darken\cite{21}, Gronski\cite{23}, Lobry\cite{34} and Sussmann and Jurdjevic\cite{41}.

Also, in\cite{30,31} the author makes an effort to describe the structure of these sets for special systems on low dimension. Actually, the accessible sets are difficult to describe because they are boundary points that can only be reached by chattering controls, i.e., infinite number of switched of controls in finite time.

From a particular state \(x \in M\), the controllability property of \(\Sigma\) means that starting from \(x\) it is possible to reach any point of the space state by using the available controls in positive time. In other words, the reachable set from \(x\) must be the whole \(M\). The study of controllability has been a subject of huge interest and has generated an enormous activity in research for different classes of control system. Specially, on Linear and Bilinear systems on Euclidean spaces,\cite{20,24,43} and Linear and Invariant systems on Lie groups.

For linear systems we mention \cite{1–9,12–14,16} and \cite{27}. For invariant we refer to the father of this class of systems\cite{19}, and\cite{38} and a complete list of references therein.

Furthermore, for a restricted admissible class of control \(\mathcal{L}\), in\cite{20} the authors introduce the notion of control set, a subset \(C\) of \(M\) where controllability holds at the interior \(\text{int}(C)\) of \(C\) and approximately controllable at the boundary \(\partial C\) of \(C\). Then, they prove that the map

\[\mathcal{L}(\rho) \rightarrow \rho\text{-}control\text{ set}\]

is lower semi-continuous. Here, \(\rho > 0\) is a parameter which allows to increase (respect to \(C\)) the admissible class of control function \(\mathcal{L}\) by increment the range of the controls. See also,\cite{17,36}.

On the other hand, in his book\cite{35}, Pontryagin shows that for a restricted classical linear control system on Euclidean spaces, the accessible set up to the positive time \(t\) is compact, convex and having the form changed continuously on time with the Hausdorff metric. The Pontryagin Maximum Principal is a very powerful theorem for concrete applications in a broad spectrum of disciplines.

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\textsuperscript{1} Supported by Proyecto Fondecyt no. 1150292, Conicyt, Chile
\textsuperscript{2} Supported by Proyecto Fondecyt no. 1151159, Conicyt, Chile
\textsuperscript{3} Supported by Fapesp grant no. 2016/11135-2.

http://dx.doi.org/10.1016/j.chaos.2016.11.006
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For instance, for application in mechanics see [28], in control of rail vehicles [32], in aerospace systems [33,40], in economy, [39], etc.

In our particular case, given an initial condition $x$ and an arbitrary by fix compact and convex subset $\mathbb{O}$ of $\mathbb{R}^m$, the continuity of the application

$$R_{x,\mathbb{O}} : t \mapsto R_{st,\mathbb{O}}(x) \subset M$$

is crucial in the proof of the celebrated Lenin Price Pontryagin Theorem. Actually, in the classical optimal time for a Linear Control System on $\mathbb{R}^n$, the continuity of $R_{x,\mathbb{O}}$ allows to build the optimal control. In fact, in this particular case, $R_{st,\mathbb{O}}(x)$ is also convex and if $t^*$ is the optimal time associated to the optimal control $u^*$, then the ending point of the optimal curve $\varphi(t, x, u^*)$, i.e., the point $\varphi(t^*, x, u^*)$, must belong to the boundary of $R_{st,\mathbb{O}}(x)$, otherwise is interior! By applying the Banach Theorem, there exists a hyperplane $H_t$, which leave the whole reachable set in one side of $H_t$. It turns out that there exists a covector $\eta_t$, orthogonal to $H_t$, such that

$$\langle \eta_t, z \rangle \leq 0 \text{ for any } z \in R_{st,\mathbb{O}}(x)$$

and the maximum equals to zero is attainable exactly on the boundary point $\varphi(t^*, x, u^*)$. By the Bellman Maximum Principle, any point of the curve must be optimal. Hence, the existence of a 1-parameter curve of covectors follows, which is the main ingredient of the PMP to synthesize the optimal control and solve the problem.

Our work is the first attempt to prove a similar result for the class of Linear Control Systems on Lie Groups introduced in [12]. In this article we just take care of the Hausdorff continuity part. But, for a more general class of systems. In the near future we expect to analyze convexity through some notion of geodesic of the system and to try to get the same Pontryagin result for linear system on Lie groups.

Precisely, consider a restricted affine control system on a connected Riemannian $C^\infty$-manifold $M$, determined by the family of differential equations

$$\Sigma_\mathbb{O} : \dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t)f_i(x(t)), \text{ with } u \in U_{\mathbb{O}}.$$ 

Where

$$U_{\mathbb{O}} = \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m); \ u(t) \in \mathbb{O}\}$$

is the class of restricted admissible control functions with $\mathbb{O}$ being a compact and convex subset of $\mathbb{R}^m$ with $0 \in \mathbb{int}(\mathbb{O})$.

If $x \in M$ and $u \in U_{\mathbb{O}}$, we denote by $\varphi(t, x, u)$ the $\Sigma_{\mathbb{O}}$-solution satisfying $\varphi(0, x, u) = x$. The reachable set $R_{st,\mathbb{O}}(x)$ of $\Sigma_{\mathbb{O}}$ is built with the points of $M$ which are possible to reach starting from the initial condition $x$, through all $\Sigma_{\mathbb{O}}$-solutions in nonnegative time less or equal than $t$.

It is well known that the map

$$(t, x, u) \in \mathbb{R} \times M \times U_{\mathbb{O}} \rightarrow \varphi(t, x, u) \in M$$

is continuous. Furthermore, the set $U_{\mathbb{O}}$ is a compact metrizable space in the weak* topology of $L^\infty(\mathbb{R}, \mathbb{R}^m) = L^1(\mathbb{R}, \mathbb{R}^m)^*$. (see for example [29]). As usual $V^*$ denotes the dual of the vector space $V$.

In this paper we give a direct proof that for a restricted affine control system $\Sigma_{\mathbb{O}}$ on a connected manifold $M$, the associated reachable sets up to time $t$ varies continuously on each variable separately by fixing the others. Precisely, the maps

$$t \mapsto R_{st,\mathbb{O}}(x), \ x \mapsto R_{st,\mathbb{O}}(x) \text{ and } \mathbb{O} \mapsto R_{st,\mathbb{O}}(x)$$

are continuous.

The variable $\mathbb{O}$ belongs to the metric space $\text{Co}(\mathbb{R}^m, d_H)$ where

$$\text{Co}(\mathbb{R}^m) = \{\mathbb{O} \subset \mathbb{R}^m; \mathbb{O} \text{ is a non-empty compact convex subset}\}$$

and $d_H$ is the Hausdorff metric. Moreover, $(\mathcal{C}(M), d_H)$ is the metric space of all non-empty compact subsets of $M$ with the Hausdorff metric.

As a consequence, every continuous functional $J$ defined on the accessible set $R_{st,\mathbb{O}}(x)$ has a minimum and maximum at any continuity point $(t, x, \mathbb{O})$. In fact, $J(R_{st,\mathbb{O}}(x)) \subset \mathbb{R}$ is compact.

The main theorem of the paper establish that the map

$$(t, x, \mathbb{O}) \in \mathbb{R} \times M \times \text{Co}(\mathbb{R}^m) \rightarrow R_{st,\mathbb{O}}(x)$$

is lower semi-continuous.

Finally, we notice that no preliminary knowledge of control system is required to read the paper.

2. Control affine systems

Let $M$ be a connected Riemannian $C^\infty$-manifold and $f_0, f_1, \ldots, f_m \in \mathcal{C}^\infty(M), m + 1$ vector fields.

Definition 1. An affine control system is determined by the family of ordinary differential equations

$$\Sigma_{\mathbb{O}} : \dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t)f_i(x(t)), \text{ where } u \in U_{\mathbb{O}}.$$ 

The set of the control functions $U_{\mathbb{O}}$ is defined as

$$U_{\mathbb{O}} = \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m); \ u(t) \in \mathbb{O}\}$$

with $\mathbb{O}$ being a compact and convex subset of $\mathbb{R}^m$.

It is well known that the set of the control functions is a compact metrizable space in the weak* topology of $L^\infty(\mathbb{R}, \mathbb{R}^m) = L^1(\mathbb{R}, \mathbb{R}^m)^*$. (see for instance Proposition 1.14 of [29]). As usual $V^*$ means the dual of the vector space $V$.

For a given initial state $x \in M$ and $u \in U_{\mathbb{O}}$ we denote the solution of $\Sigma_{\mathbb{O}}$ by $\varphi(t, x, u)$. The curve $t \mapsto \varphi(t, x, u)$ is the only solution of $\Sigma_{\mathbb{O}}$ satisfying $\varphi(0, x, u) = x$ in the sense of Carathéodory. That is, it is an absolutely continuous curve satisfying the corresponding integral equation.

Throughout the paper we assume that all the solutions are defined in the whole real line. Even though this assumption is in general restrictive, there are several cases where the assumption of completeness goes without loss of generality, such as the class of linear systems on Lie groups, [15], and control affine systems on compact manifolds, [26]. Moreover, the map

$$(t, x, u) \in \mathbb{R} \times M \times U_{\mathbb{O}} \rightarrow \varphi(t, x, u) \in M$$

is a continuous map (see for instance Theorem 1.1 of [29]).

For a given state $x \in M$ and a positive time $t$ let us introduce the sets

$$R_{st,\mathbb{O}}(x) = \{y \in M; \exists u \in U_{\mathbb{O}}, \ s \in [0, t] \text{ with } y = \varphi(s, x, u)\},$$

and

$$R_{\mathbb{O}}(x) = \bigcup_{t=0}^{\infty} R_{st,\mathbb{O}}(x).$$

$R_{\mathbb{O}}(x)$ is called the set of reachable point from $x$ up to time $t$ and $R_{\mathbb{O}}(x)$ the set of reachable points from $x$.

Our goals include first to prove the partial continuity of the map

$$(t, x, \mathbb{O}) \in \mathbb{R} \times M \times U_{\mathbb{O}} \rightarrow R_{st,\mathbb{O}}(x).$$

Means, continuity in each variable: time $t$, state $x \in M$ and the range $\mathbb{O}$ of the admissible class of control $U_{\mathbb{O}}$. Secondly, we prove that the global map $R$ is lower semi-continuous.

First, we notice that it is possible to reduce the proof by considering a special class of control. In fact, let us consider the set of the piecewise control functions $U_{\mathbb{O}}^{PC} \subset U_{\mathbb{O}}$ and define the corresponding reachable sets as

$$R_{st,\mathbb{O}}^{PC}(x) = \{y \in M; \exists u \in U_{\mathbb{O}}^{PC}, s \in [0, t] \text{ with } y = \varphi(s, x, u)\}$$

where
and
\[ R^{PC}_{Ω_2}(x) = \bigcup_{t \geq 0} R^{PC}_{t,Ω_2}(x). \]

It turns out that
\[ cl(R^{PC}_{Ω_2}(x)) = R_{t,Ω_2}(x), \quad \text{for any } t > 0, \ x \in M \]
where \( cl(R^{PC}_{Ω_2}(x)) \) stands for the closure of \( R^{PC}_{Ω_2}(x) \) (Proposition 1.16 of [29]). The closure of a set \( A \) will be denoted by \( cl(A) \).

**Remark 2.** Let \( Ω_1 \) and \( Ω_2 \) two compact and convex subsets of \( \mathbb{R}^m \) with the property that \( Ω_1 ⊆ Ω_2 \). We consider the control affine systems \( Σ_1 \) and \( Σ_2 \) associated with the corresponding set of control functions \( U_{Ω_1} \) and \( U_{Ω_2} \), respectively. By the very definition it follows that \( U_{Ω_2} \subseteq U_{Ω_1} \). The uniqueness of the solutions means that any \( Σ_1 \)-solution is also a \( Σ_2 \)-solution. Furthermore, in the weak-*topology \( U_{Ω_1} \) is a compact subset of \( U_{Ω_2} \).

Finally, we recall that for arbitrary \( u ∈ U_{Ω_2} \) and \( γ > 0 \) the sets
\[ W_{u,γ}(x_1, . . . , x_k) = \left\{ u' ∈ U_{Ω_2} : \left| \int_{Ω} (u(s) - u'(s), x_i(s))ds \right| < γ \text{ for } i = 1, . . . , k \right\}. \]
where \( k ∈ \mathbb{N} \) and \( x_i ∈ L^1(\mathbb{R}, \mathbb{R}^m) \) for \( 1 ≤ i ≤ k \), form a subsessee for the weak-*topology (see [22]).

3. Partial continuity

This section is devoted to show that the global map
\[ (t, x, Ω) ∈ \mathbb{R} \times M \times Co(\mathbb{R}^m) ↦ R_{t,Ω}(x) \subset \mathbb{R}^m \]
is continuous in each variable. Precisely, continuous with respect to the usual topology for the time \( t ∈ \mathbb{R} \), with respect to the topology of the manifold \( M \) for the state \( x \) and for the topology determined by the Hausdorff metric relative to the compact subsets of \( \mathbb{R}^m \). It is important to mention here that in the last case the fixed time \( t \) is finite.

We begin by proving the continuity of \( R \) on the third variable

**Proposition 3.** Respect to the Hausdorff metric the map \( Ω ↦ R_{t,Ω}(x) \) is continuous.

**Proof.** Fix \( Ω \) and consider \( \tilde{Ω} \) such that \( Ω ⊆ \text{int} \tilde{Ω} \). We already know that
\[ (t, x, u) ∈ \mathbb{R} \times M \times U_{Ω} \Rightarrow ϕ(t, x, u) \in M \]
is a continuous map. Fix \( x ∈ M \). From the compactness of \( [0, t] \) and \( U_{Ω} \) it follows that the map
\[ (s, u) ∈ [0, t] × U_{Ω} ↦ ϕ_s(t, u) = ϕ(t, x, u) \in M \]
is uniformly continuous.

By Proposition 1.6 of [29], for any \( u ∈ U_{Ω} \) there exists a piecewise-constant function \( u' ∈ U_{Ω}' \) with the property that
\[ ϕ(u, s, u') < ε/2, \quad \text{for any } s ∈ [0, t]. \]
By the compactness of \( [0, t] × U_{Ω} \) and the continuity of the map \((t, u) ↦ ϕ_s(t, u))\), there are \( u_1, . . . , u_m ∈ U_{Ω} \), \( u_1, . . . , u_m ∈ U_{Ω}' \) and \( γ_1, . . . , γ_m > 0 \) such that
\[ U_{Ω} \subseteq \bigcup_{i=1}^m W_{u_i,γ_i}(x_{i,1}, . . . , x_{i,k_i}) \]
and for any \( s ∈ [0, t] \) and \( u ∈ W_{u_i,γ_i}(x_{i,1}, . . . , x_{i,k_i}) \)
\[ ϕ(u, s, u') < ε/2, \quad 1 ≤ i ≤ m. \]

**Claim 1:** There exists \( ε > 0 \) with the property that for each \( u ∈ U_{Ω} \), \( 3ε' ∈ [1, . . . , m] \) with
\[ W_{u,ε}(Δ) ⊆ W_{u_i,ε'}(x_{i,1}, . . . , x_{i,k_i}) \]
where
\[ Δ = \{ x_{i,j} : 1 ≤ j ≤ k_i, 1 ≤ i ≤ m \}. \]
In fact, for \( v ∈ U_{Ω} \) let \( ε_v \) be a positive number such that
\[ W_{u,ε_v}(Δ) ⊆ W_{u_i,γ_i}(x_{i,1}, . . . , x_{i,k_i}) \]
for some \( 1 ≤ i ≤ m \). Since \( U_{Ω} \) is compact, there exist \( v_1, . . . , v_l ∈ U_{Ω} \) and \( ε_1, . . . , ε_l > 0 \) with
\[ U_{Ω} ⊆ \bigcup_{k=1}^l W_{v_k,ε_k/2}(Δ) \]
with \( W_{v_k,ε_k}(x_{k,1}, . . . , x_{k,k}) \)
for \( 1 ≤ i ≤ m \). Let us select \( ε \) such that \( 0 < ε < ε_k/2 \) for all \( k ∈ [1, . . . , l] \). For any \( u ∈ U_{Ω} \) let \( u_0 = W_{u,ε}(Δ) \). Consider \( k ∈ [1, . . . , l] \) such that \( u ∈ W_{v_k,ε_k/2}(Δ) \). Then, we get
\[ \left| \int_{Ω} (\bar{u}(s) - v_k(s), x_{i,j}(s))ds \right| < ε \]
\[ 0 ≤ \int_{Ω} (u(s) - v_k(s), x_{i,j}(s))ds < ε + ε_k/2 < ε_k, \]
showing that for some \( i' ∈ [1, . . . , m] \)
\[ W_{u,ε}(Δ) ⊆ W_{v_{i'},ε}(x_{i',1}, . . . , x_{i',k_i}) \]
as stated.

**Claim 2:** For any \( ε > 0 \) there exists \( δ_1 > 0 \) with the property that
\[ \text{for any } Ω \text{ and } γ ∈ (0, δ_1) \text{ with } N_γ(Ω') ⊆ \text{int} \tilde{Ω} \text{ it follows that } \]
\[ Ω ⊆ N_γ(Ω') ↦ R_{t,Ω}(x) ⊆ N_δ(R_{t,Ω}(x)) \]
\[ \text{In fact, by the continuity of } (s, u) ↦ ϕ_s(s, u) \text{ in } [0, t] × U_{Ω} \text{ there exists } δ_1 > 0 \text{ such that for any } s ∈ [0, t] \text{ and } i = 1, . . . , m, \]
\[ ϕ_{s, worthless}(s, u) ∈ ε/2 \text{ if } |u - u_i|_β < δ_1, \]
Let \( Ω' \) and \( γ ∈ (0, δ_1) \) satisfying \( Ω ⊆ N_γ(Ω') ⊆ \text{int} \tilde{Ω} \). The control \( u_i \) is piecewise constant thus we can construct a piecewise constant function \( u' ∈ U_{Ω}' \) such that \( |u' - u_i|_∞ < γ \) for any \( i = 1, . . . , m \).

Let us consider \( y ∈ R_{t,Ω}(x), u ∈ U_{Ω}, s ∈ [0, t] \) and \( i ∈ [1, . . . , m] \)
with the property that
\[ y = ϕ_{s, worthless}(s, u) \text{ with } u ∈ W_{u_i,γ_i}(x_{i,1}, . . . , x_{i,k_i}). \]
The triangular inequality shows that
\[ ϕ_{s, worthless}(s, u') ≤ ϕ_{s, worthless}(s, u)+\left| ϕ_{s, worthless}(s, u') - ϕ_{s, worthless}(s, u) \right| < ε. \]
Because \( ϕ_{s, worthless}(s, u') ∈ R_{t,Ω}(x) \) we obtain \( y ∈ N_ε(R_{t,Ω}(x)) \) and consequently
\[ R_{t,Ω}(x) ⊆ N_ε(R_{t,Ω}(x)) \]
as stated.

**Claim 3:** For any \( ε > 0 \) there exists \( δ_2 > 0 \) with \( N_{δ_2}(Ω) ⊆ \text{int} \tilde{Ω} \) and
\[ Ω' ⊆ N_{δ_2}(Ω) ↦ R_{t,Ω}(x) ⊆ N_{δ_2}(R_{t,Ω}(x)). \]
We know that \( R_{t,Ω}(x) = cl(R_{t,Ω}(x)) \). Hence, in order to prove Claim 3 it is enough to show that for any \( ε > 0 \) there is \( δ_2 > 0 \) such \( N_{δ_2}(Ω) ⊆ \text{int} \tilde{Ω} \) and
\[ Ω' ⊆ N_{δ_2}(Ω) ↦ R_{t,Ω}(x) ⊆ N_{δ_2}(R_{t,Ω}(x)). \]
Take \( δ_2 > 0 \) with \( δ_2 < ε/2M \) and \( N_{δ_2}(Ω) ⊆ \text{int} \tilde{Ω} \). For any \( Ω' ⊆ N_{δ_2}(Ω) \) let \( z ∈ R_{t,Ω}(x), s ∈ [0, t] \) and \( u' ∈ U_{Ω}' \) such that \( z =
\(\psi(s, u')\). If \(c'_1, \ldots, c'_m\) are the values assumed by \(u'\) in \([-T, T]\), there are elements \(c_1, \ldots, c_m \in \Omega\) such that \(|c_i - c'_i| < \delta_2\). Define the piecewise constant function \(u \in \mathcal{U}_\Omega\) by

\[ u(s) = \begin{cases} c_i, & \text{if } u'(s) = c'_i \text{ and } s \in [0, t] \\ p, & \text{if } s \in \mathbb{R} \setminus [-T, T]. \end{cases} \]

where \(p \in \Omega\) is an arbitrary point. Therefore, for \(1 \leq j \leq k_i, 1 \leq i \leq m\), we get

\[ \left| \int_{t_j}^{t} (u'(s) - u(s), x_{i,j}(s)) ds \right| \leq \delta_2 \]

which shows that \(u' \in \mathcal{W}_0, \epsilon\) and \(u' \in \mathcal{W}_1, (x_{1,1}, \ldots, x_{1,k})\) for some \(1 \leq i \leq m\). Thus, \(\mathcal{G}(\psi_x(s, u'), \psi_x(s, u)) < \epsilon\) and \(z \in N_\epsilon(R_{\Omega, \mathcal{U}})\).

Since \(z \in R_{\Omega, \mathcal{U}}\) was arbitrary we can conclude that

\[ R_{\Omega, \mathcal{U}} \subset N_\epsilon(R_{\Omega, \mathcal{U}}) \]

as claimed.

**Claim 4:** The map \(\Omega \mapsto R_{\Omega, \mathcal{U}}\) is continuous in the Hausdorff metric.

For a given \(\Omega\) and \(\epsilon > 0\) let \(\delta = \min\{\delta_1, \delta_2/2\}\) where \(\delta_1, \delta_2\) are defined as in the claims 2 and 3 respectively. It turns out that

\[ d_H(\Omega, \Omega') < \delta \Leftrightarrow \Omega \subset N_\delta(\Omega') \quad \text{and} \quad \Omega' \subset N_\delta(\Omega). \]

By Claim 3, we get

\[ \Omega' \subset N_\delta(\Omega) \Rightarrow R_{\Omega, \mathcal{U}} \subset N_\epsilon(R_{\Omega, \mathcal{U}}). \]

In addition, guaranteed by the previous analysis we obtain

\[ \Omega' \subset N_\delta(\Omega) \Rightarrow N_\delta(\Omega') \subset N_\epsilon(\Omega) \subset N_\delta(\Omega) \subset \text{int}\Omega. \]

On the other hand, Claim 2 implies that

\[ \Omega \subset N_\delta(\Omega') \Rightarrow R_{\Omega, \mathcal{U}} \subset N_\epsilon(R_{\Omega, \mathcal{U}}). \]

Hence,

\[ d_H(R_{\Omega, \mathcal{U}}, R_{\Omega', \mathcal{U}}) < \delta \Rightarrow d_H(R_{\Omega, \mathcal{U}}, R_{\Omega, \mathcal{U}}) \subset N_\epsilon(R_{\Omega, \mathcal{U}}) \]

finishing the proof.

Next, for a given state \(x \in M\) we show the continuity of the global map on time.

**Proposition 4:** The map \(t \mapsto R_{t, \mathcal{U}}\) is continuous.

By continuity, for any \(u \in \mathcal{U}\) there exists \(\delta_u > 0\) and a neighborhood \(V_u\) of \(u\) in \(\mathcal{U}\) with the property

\[ |s - t| < \gamma_u \quad \text{and} \quad u \in V_u \Rightarrow \mathcal{G}(\phi_x(t, u), \phi_x(s, u)) < \epsilon/2. \]

The set \(\mathcal{U}\) is compact, so there exist \(V_1, \ldots, V_n\) such that

\[ \mathcal{U} = \bigcup_{i=1}^{n} V_i. \]

By taking \(\gamma = \min_{1 \leq i \leq n} \gamma_u\) we can conclude that for any \(s, s' \in (t - \gamma/2, t + \gamma/2)\) and \(u, u' \in V_i\)

\[ \mathcal{G}(\psi_x(s, u), \psi_x(s', u')) < \epsilon, \quad \text{for some} \quad 1 \leq i \leq n. \quad (2) \]

Let us take \(\delta = \gamma/2\) and \(s \in (t - \delta, t + \delta)\). We have two cases depending on the relative position of \(s\) and \(t\)

1. If \(s \geq t\) it follows that

\[ R_{t, \mathcal{U}}(x) \subset R_{s, \mathcal{U}}(x) \subset N_\epsilon(R_{s, \mathcal{U}}(x)). \]

Also, if \(z \in R_{s, \mathcal{U}}(x)\) we get \(z = \psi(s', x, u)\) for some \(s' \in [0, s]\) and \(u \in \mathcal{U}\). These are the possibilities

a) \(s' \leq t \Rightarrow z \in R_{t, \mathcal{U}}(x) \subset N_\epsilon(R_{s, \mathcal{U}}(x)) \)

b) \(s' > t \Rightarrow s' \in [t, s] \subset (t - \delta, t + \delta) \)

Thus, by taking \(V_t\) with \(u \in V_t\), the Eq. (2) gives us

\[ \mathcal{G}(\psi_x(s', u), \psi_x(t, u')) < \epsilon, \quad \text{for any} \quad u' \in V_t \]

implying that \(z \in N_\epsilon(R_{s, \mathcal{U}}(x))\).

Since \(z \in R_{s, \mathcal{U}}(x)\) was arbitrary we conclude

\[ R_{s, \mathcal{U}}(x) \subset N_\epsilon(R_{s, \mathcal{U}}(x)) \]

2. If \(s < t\) we obtain

\[ R_{s, \mathcal{U}}(x) \subset N_\epsilon(R_{s, \mathcal{U}}(x)) \subset N_\epsilon(R_{s, \mathcal{U}}(x)). \]

Also, for any \(z \in R_{s, \mathcal{U}}(x)\) we get \(z = \psi(t', x, u)\) for some \(t' \in [0, t]\) and \(u \in \mathcal{U}\). We get two possibilities

a) \(t' > t \Rightarrow z \in R_{s, \mathcal{U}}(x) \subset N_\epsilon(R_{s, \mathcal{U}}(x)) \)

b) \(t' < t \Rightarrow t' \in [s, t] \subset (t - \delta, t + \delta) \)

So, by taking \(u \in V_t\), the Eq. (2) gives us

\[ \mathcal{G}(\psi_x(t', u), \psi_x(t, u')) < \epsilon, \quad \text{for any} \quad u' \in V_t \]

implying that \(z \in N_\epsilon(R_{s, \mathcal{U}}(x))\).

Since \(z \in R_{s, \mathcal{U}}(x)\) was arbitrary we conclude that

\[ R_{s, \mathcal{U}}(x) \subset N_\epsilon(R_{s, \mathcal{U}}(x)) \]

Finally, by the preceding analysis if \(s \in (t - \delta, t + \delta)\) we obtain

\[ R_{s, \mathcal{U}}(x) \subset N_\epsilon(R_{s, \mathcal{U}}(x)) \quad \text{and} \quad R_{s, \mathcal{U}}(x) \subset N_\epsilon(R_{s, \mathcal{U}}(x)) \]

which is equivalent to

\[ \mathcal{G}(\psi_x(t, x, u), \psi_x(t, x, u')) < \epsilon \]

showing the desired result.

To end this section we prove the continuity of the global map on the state space \(M\).

**Proposition 5.** The map \(x \mapsto R_{t, \mathcal{U}}(x)\) is continuous.

**Proof.** Let \(x \in M\), fix \(t > 0\) and consider \(\epsilon > 0\). By continuity of the solutions and the compactness of the set \([0, t] \times \mathcal{U}\) we can find \(\delta > 0\) with the property that

\[ y \in B(x, \delta) \Rightarrow \mathcal{G}(\phi_x(s, u), \phi_x(s, u)) < \epsilon, \quad \text{for all} \quad (s, u) \in [0, t] \times \mathcal{U} \]

where \(\phi_x(s) = \phi(t, x, u)\). Then, for \(z \in R_{s, \mathcal{U}}(x)\) let \(s \in [0, t]\) and \(u \in \mathcal{U}\) such that \(z = \phi_x(s, u)\). If \(y \in B(x, \delta)\) we know that \(\mathcal{G}(\phi_x(s, u), \phi_x(s, u)) < \epsilon\). Therefore,

\[ z \in R_{s, \mathcal{U}}(y) \quad \text{and} \quad R_{s, \mathcal{U}}(y) \subset N_\epsilon(R_{s, \mathcal{U}}(y)). \]

In an analogous way we can show that

\[ R_{s, \mathcal{U}}(y) \subset N_\epsilon(R_{s, \mathcal{U}}(y)). \]

Hence,

\[ \mathcal{G}(\psi(x, y) < \delta \Rightarrow \mathcal{G}(\psi_x(s, u), \psi_x(s, u)) \]

concluding the proof. \(\square\)

**Remark 6.** We know that the following map is continuous

\[ (t, x, u) \in \mathbb{R} \times M \times \mathcal{U} \mapsto \phi(t, x, u) \in M. \]

So, by a fixed state \(x \in M\) the reachable set \(R_{t, \mathcal{U}}(x)\) under the continuous image \([0, t] \times \mathcal{U}\) by \(\phi\) is compact. Hence, any continuous functional \(J : R_{t, \mathcal{U}}(x) \to \mathbb{R}\) has a minimum and maximum point.
4. Lower semi-continuity

In this section we will be concerned with the global map
\((t, x, \Omega) \in \mathbb{R} \times M \times \text{Co}(\mathbb{R}^m) \rightarrow \mathcal{R}_{\text{set}, \Omega}(x) \subset \mathbb{R}^m.\)
It is shown that \(\mathcal{R}\) is lower semi-continuous on the product.
In order to do that, we use an equivalent concept called inner semi-continuity appears in [see [10]], as follows. We denote by \(\mathcal{P}(Z)\) the family of subsets of the metric space \(Z.\)

**Definition 7.** Let \(X, Y\) be two metric spaces, and consider a set-valued map \(F : X \rightarrow \mathcal{P}(Y).\) We say that \(F\) is inner semi-continuous at \(x_0 \in X\) if
\[
F(x_0) \subset \liminf_{s \rightarrow s_0} F(y_s) = \{ y \in Y, \forall x_k \rightarrow x_0, \exists y_n \in F(x_n) \text{ with } y_n \rightarrow y \}.
\]
Now, we are in a position to prove the main result of the section.

**Theorem 8.** The map \((t, x, \Omega) \mapsto \mathcal{R}_{\text{set}, \Omega}(x)\) is lower semi-continuous.

**Proof.** Let \(y_0 \in \mathcal{R}_{\text{set}, \Omega}(x_0)\) and consider the convergent sequence \((t_n, x_n, \Omega_n) \rightarrow (t_0, x_0, \Omega_0).\)
Since \(y_0 \in \mathcal{R}_{\text{set}, \Omega}(x_0)\) it holds that \(y_0 = \varphi(s_0, x_0, u_0)\) for \(s_0 \in [0, t_0]\) and \(u_0 \in U_{\Omega_0}.\) Actually, by the equality
\[
\mathcal{R}_{\text{set}, \Omega}(x_0) = \mathcal{R}_{\text{set}, \Omega}(x)\]
there is no loss of generality in assuming \(u_0 \in \mathcal{U}_{\Omega_0}^{PC}.\)
Let us denote by \(e_n = t_n - t_0\) and consider \(s_n = s_0 - e_n.\) If \(s_0 = 0,\) the result follows trivially, since \(x_n \rightarrow x_0.\) If \(s_0 > 0,\) the fact \(t_0 \rightarrow t\) implies \(s_n \rightarrow s_0.\) And, if \(n \in \mathbb{N}\) is large enough, \(s_n \in [0, t_0].\)
Let \(c_1, 0, \ldots, c_n, 0 \in \Omega_0\) the values assumed by \(u_0\) in \([0, t_0].\) By hypothesis \(c_n \rightarrow c_0,\) then, as we did before, it is possible to build piecewise constant control functions \(u_n \in U_{\Omega_n}\) with \(u_n \rightarrow u_0.\) By considering
\[
y_n = \varphi(s_n, x_n, u_n) \in \mathcal{R}_{\text{set}, \Omega}(x_n) \rightarrow y_0.
\]
we get by continuity that the sequence \((y_n)\) converges to \(y_0.\) It turns out that \(\mathcal{R}_{\text{set}, \Omega}(x_0) \subset \liminf_{(t, x, \Omega) \rightarrow (t_0, x_0, \Omega_0)} \mathcal{R}_{\text{set}, \Omega}(x)\)
concluding the proof. \(\square\)

It is worth pointing out that about the continuity of reachable sets of affine control systems, **Theorem 8** is the best that you can expect. Actually, in the next section we show that the global map \(\mathcal{R}\) could have discontinuity points. In fact, we give an explicitly example of a two dimensional bilinear control system on \(\mathbb{R}^2\) where the function
\[
(\mathcal{x}, \Omega) \in M \times \text{Co}(\mathbb{R}^m) \rightarrow \mathcal{R}_{\text{set}, \Omega}(\mathcal{x})
\]
is not continuous (see [17]). This also show that we should not expect the continuity of the global map \(\mathcal{R}_{\text{set}, \Omega}(x)\) as well.

5. An example and a counterexample

In this section we show the potential of **Theorem 8** through the class of linear control systems on Lie groups, [12]. On the other hand, based on the class of bilinear control systems we show an example where the global map has a discontinuity point.

Ayala and Tirao introduced the notion of linear control system \(\Sigma\) on a connected Lie group \(G\) as the family of ordinary differential equations
\[
\dot{x}(t) = X(g(t)) + \sum_{i=1}^{m} u_j(t)X_j(g(t)), \quad g(t) \in G.
\]
Here, \(X\) is a linear vector field, means that its flow \(\{X_t : t \in \mathbb{R}\}\) is a subgroup of the group \(\text{Aut}(G)\) of \(G\)-automorphism. The vector fields \(X\) are right-invariant on \(G,\) for \(j = 1, \ldots, m\) and \(u = (u_1, \ldots, u_m) \in U \subset L^\infty(\mathbb{R}, \mathbb{R}^m)\)
with \(\mathcal{O}\) compact, convex and \(0 \in \text{int}(\mathcal{O}).\)

Linear control systems on Lie groups are important for at least two reasons. First, they are a natural generalization of the classical linear control system on the Euclidean space \(\mathbb{R}^d,\) which is defined by
\[
\dot{x}(t) = Ax(t) + Bu, \quad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m} \quad \text{and} \quad u = (u_1, \ldots, u_m) \in \mathcal{U}.
\]
In fact, just observe that \(e^A \in \text{Aut}(\mathbb{R}^d)\) for any \(t \in \mathbb{R}.\) And, any column vector \(b_j\) of \(B\) is an invariant vector field on \(\mathbb{R}^d.\) Besides that, in [26] Jouan shows that \(\Sigma_G\) is relevant from theoretical and practical point of view. Actually, he shows that any affine control system on a connected Riemannian \(c^n\)-manifold \(M,\) as in **Definition 1**, whose dynamic generates a finite dimensional Lie algebra, i.e.
\[
\text{dim Span}_{\mathbb{C}} \{ X, Y^1, \ldots, Y^m \} < \infty
\]
is equivalent to a linear control system on a Lie group \(G\) or on a homogeneous space of \(G.\)

Since the \(\Sigma_G\)-solutions are defined for any time, [15], all the results of the previous sections apply to \(\Sigma_G.\)

**Example 9.** By definition a bilinear control system \(\Sigma\) in \(\mathbb{R}^d\) is determined by a family of differential equations
\[
\dot{x}(t) = \left( \begin{array}{c}
A + \sum_{i=1}^{m} u_i(t)B_i
\end{array} \right)x(t), \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{R}^d
\]
where \(A, B_1, \ldots, B_m \in \mathfrak{g}(d, \mathbb{R})\) are real \(d \times d\) matrices and \(u \in U = \{ u : \mathbb{R} \rightarrow \mathbb{R} \subset \mathbb{R}^m, u \text{ is locally integrable} \}\)
When \(\Omega\) is a compact and convex subset of \(\mathbb{R}^m\) with \(0 \in \text{int}(\Omega),\) is the set of the admissible class of controls.

We notice that the origin in \(\mathbb{R}^d\) is a singularity for any admissible control function \(u \in \Omega.\)
First, we consider the unrestricted two dimensional bilinear system
\[
\Sigma_{\Omega_{\mathbb{R}^2}} \dot{x}(t) = \left( \begin{array}{c}
A + u(t)B
\end{array} \right)x(t), \quad A, B \in \mathfrak{g} \quad \text{and} \quad u \in U \quad \text{with} \quad \Omega = \mathbb{R}^2.
\]
Here, \(\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})\) stands for the Lie algebra of the Lie group \(G = \mathbb{SL}(2, \mathbb{R})\) of the determinant 1 real matrices of order 2. For this class of systems in [18] the authors prove:

**Theorem 10.** \(\Sigma\) is controllable on \(\mathbb{R}^2 - \{0\} \Rightarrow \det[A, B] < 0.\)

As usual \([A, B] = AB - BA\) stands by the standard Lie bracket between matrices. In order to extend **Theorem 10** and also to show a geometric picture of this result, the authors in [17] identify \(\mathfrak{g} \equiv \mathbb{R}^3\) through the basis
\[
H = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & 0
\end{pmatrix}, \quad S = \begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
the hyperbolic, symmetric and skew-symmetric elements, respectively.

Any derivation of a semisimple Lie algebra is inner, in particular, the Cartan Killing form on \(\mathfrak{g}\)
\[
k(X, Y) = tr(ad(\mathbf{X}) \cdot ad(Y))
\]
is a multiple of the trace form \(k(X, Y) = 4tr(XY)\) which gives rise to a quadratic form \(Q(X) = tr(\mathbf{X}^2).\) The double circular cone \(C\) generated by the nilpotent matrix \(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\) separates \(\mathbb{R}^3\) in two connected components: the interior \(\text{int}(C)\) of \(C\) formed by matrices
with purely imaginary eigenvalues, and its exterior $\text{ext}(C)$ which is formed by the diagonalizable matrices of $g$ with real eigenvalues. Furthermore, any matrix in $C$ makes a 45° angle with the axis generated by $A$. They consider a positive real number $r$, the restricted bilinear system

$$\Sigma_r : \dot{x}(t) = (A + u(t)B)x(t),$$

$$A, B \in \mathfrak{sl}(2, \mathbb{R}) \text{ and } u \in u_t \text{ with } \Omega^r = [-r, r].$$

and prove the following theorem.

**Theorem 11.** Assume $\det[A, B] \neq 0$. Then, the controllability of $\Sigma_r$ depend on the relative position of the segment $s_r : A + uB : u \in \Omega^r$ as follows:

1. If $\det(A) \geq 0$ the system is controllable.
2. If $\det(A) < 0$ there are two possibilities:
   a) If $\det[A, B] < 0$ the line $A + uB : u \in \Omega^r$ crosses $\text{int}(C)$ and the system is controllable. The only bifurcation point is given by $r^* = \inf \{r : s_r \cap \text{int}(C) = \emptyset\}$.
   b) If $\det[A, B] > 0$ the system is not controllable for any $r > 0$.

The main geometric ingredient here is: the controllability property is equivalent to the fact that the segment $s_r$ must cross the bicone. In fact, the fundamental argument in the proof of the theorem above is the existence of $r > 0$ and a control $u \in \Omega^r$ with the property that

$$\text{Spec}(A + uB) \in \mathbb{R}^2 \Leftrightarrow s_r \cap \text{int}(C) \neq \emptyset.$$

Actually, for any $r < r^*$ controllability of $\Sigma_r$ follows from the fact that the only semigroup with non empty interior in the semisimple Lie group $G$ is the whole group $\text{SL}(2, \mathbb{R})$. [37]

Fix $x \in \mathbb{R}^2$. Respect to the continuity on the third variable of $R^r_{\mathbb{R}^2} : \mathbb{R}^2 \rightarrow R_{\mathbb{R}^2}(x)$ we can conclude the following: under the condition (2.a) the set valued map $R^r_{\mathbb{R}^2}$ is not continuous. Actually, $R^r_{\mathbb{R}^2}$ has a discontinuity at the point $\Omega^r$.

In fact, for $r < r^*$, Proposition 3 implies that for any $x \neq 0$, the reachable set $R^r_{\mathbb{R}^2}(\Omega^r)$ is compact. However, $R^r_{\mathbb{R}^2}(\Omega^r) = \mathbb{R}^2 - [0]$ for any $r > r^*$.

6. Conclusion

In this paper we analyze the continuity of the global map

$$(t, x, \Omega) \in \mathbb{R} \times M \times \text{Co}(\mathbb{R}^m) \rightarrow R_{\mathbb{R}^2, \Omega}(x) \subset \mathbb{R}^m$$

associated to any affine control system as in Definition 1. Precisely, we prove

1. $\mathcal{R}$ is continuous at any independent variable
2. $\mathcal{R}$ is lower semi-continuous on its domain
3. Theorem 8 is the best results you can expect. In fact, we show a restricted bilinear control system on the plane where $\mathcal{R}$ has a discontinuity point

A very powerful results called the Pontryagin Maximum Principle shows the compactness, convexity and Hausdorff metric continuity deformation of the reachable sets for the class of Restricted Linear Control Systems on Euclidean spaces. In order to compute the time optimal control the Hausdorff $t$-continuity property of $R_{\mathbb{R}^2, \Omega}(x)$ is crucial. Our paper is the first attempt to prove a similar result for the class of Restricted Linear Control Systems on Lie Groups (LCS) introduced in [12]. We apply Theorem 8 to the LCS class because its relevance due to the Equivalence Theorem of Jouan, [26].

The next step is to define an appropriate notion of convexity on Lie groups for LCS. We believe that this is possible through some notion of $\mathcal{C}_2$-geodesics. Actually, we know how to compute explicitly the hamiltonian vector fields and the Hamiltonian equations for this class of control systems in our recently paper published by a SIAM Journal, [11]. In the near future we hope to get the proposed aims.

**References**


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