A review on some classes of algebraic systems

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Abstract

In this paper, we review some algebraic control system. Precisely, linear and bilinear systems on Euclidean spaces and invariant and linear systems on Lie groups. The fourth classes of systems have a common issue: to any class, there exists an associated subgroup. From this object, we survey the controllability property. Especially, from those coming from our contribution to the theory.

Key words
Classical linear, bilinear, invariant and linear systems on Lie groups

1 Introduction

This review was intended as an attempt to motivate researchers to take attention of some special classes of *algebraic systems on Lie groups*. In this paper, we analyze the following categories of systems

1. $\Sigma_{Lin}(\mathbb{R}^d)$: Linear systems on the Euclidean space $\mathbb{R}^d$
2. $\Sigma_{Bil}(\mathbb{R}^d)$: Bilinear systems on the Euclidean space $\mathbb{R}^d$
3. $\Sigma_{Inv}(G)$: Invariant systems on a Lie group G
4. $\Sigma_{Lin}(G)$: Linear systems on a Lie group G

The fourth mentioned classes of control have a common issue: they have an associated semigroup which is a perfect algebraic object to study controllability, one of the most relevant problems in control theory. The controllability property gives the possibility to connect any two arbitrary elements through a solution of the system in nonnegative time. It has been a subject of enormous interest and

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has generated a vast research activity. However, still, there is no general criterion to characterize this property even for systems with strong differentiable and algebraic structures. After introducing the associated semigroup of a specific class, we mention some related controllability properties. We mainly refer those coming from our contribution to the theory.

In the sequel, we summarize the different chapters of this survey.

In chapter two, we study some general facts about control systems on differential manifolds. We introduce the notion of accessible, controllable and control sets. We mention the orbit theorem related to the idea of transitivity and the definition of Lie bracket. We also introduce the definition of normalizer when the manifold is a Lie group. This algebraic object contains all the systems involved in this article when we consider the piecewise constant function as the admissible class of control.

In chapter three we show that to the class of unbounded linear systems $\Sigma_{Lin}(\mathbb{R}^d)$, the accessible set from the origin is a subspace, which determines controllability as soon as this subspace has a nonempty interior, see (Kalman & Narendra 1962). When the controls are restricted, we mention a result from (Colonius & Kliemann, 2000-a), where the authors show the existence and uniqueness of a control set with a nonempty interior. Mostly, a control set $\mathcal{C}$ is a subset of the state space where the system is approximately controllable on the boundary $\partial\mathcal{C}$, and it is controllable on its interior $int(\mathcal{C})$.

Chapter four takes care of the controllability property of bilinear systems. Despite the fact that this class of system has been a source of research for more than 40 years, still, there is not a complete characterization. In this case, the origin is a global singularity, so the accessible set from the identity is trivial. However, there exists a semigroup associated with the system. We refer an algebraic approach, (Elliot, 2009), especially when the controls are unrestricted. For restricted control, we follow the reference (Colonius & Kliemann, 2000-a) where the authors give a dynamic analytic approach that can be applied to the bilinear case. The main ingredients are the projection of the system on the sphere (actually on the projective space), the notion of chain control set and the Morse spectrum. We describe some results on dimension two.

Chapter five contains a short review of Lie theory. In fact, linear and bilinear control systems on the Euclidean space $\mathbb{R}^d$ are relevant from the practical and theoretical point of views. But, many applications are coming from mechanical or physical problems where the state space is not the vector space $\mathbb{R}^d$ but a Lie group. For instance, see (Brockett, 1972), (Dubins, 1957), (Isidori, 1998), (Jurdjevic, 1997-a) and (Jurdjevic, 1997-b). Furthermore, a bilinear system can also be approached by Lie theory. In fact, the Lie algebra generated by the matrices of the system plays an essential role.

In chapter six, we study the class of invariant systems on Lie groups. There are many controllability results which are obtained on specific state spaces. We mention few of them, first (Brockett, 1972) which is the starting point of this class of systems. In (Jurdjevic & Sussmann 1972-a), the authors work on general Lie groups. In (Hilgert, Hofmann & Lawson, 1985) and (Ayala, 1995) they characterize the nilpotent case. In (Sachkov, 1999) an extension from nilpotent to completely solvable Lie groups is done. For a complete survey on the topic see references (Sachkov, 1999, 2000 and 2006).

For the class of unrestricted linear systems on Lie groups, in chapter seven we mention results coming from (Markus,1980), (Ayala & Tirao, 1999), (Ayala & San Martin, 2001), (Jouan, 2010), (Jouan, 2011). For the restricted case, we include recent results relative to the boundness and uniqueness of control sets, (Ayala & Da Silva, 2016-b). Despite the fact that the accessible set $\mathcal{A}$ from the identity is not a semigroup we associate to the linear system a semigroup $\mathcal{S}$ which depends on $\mathcal{A}$, (Ayala & Da Silva, 2016-a).

According to every particular class of systems, we include controllability results. Especially those coming from our contribution to the theory. Except for some results of the first class, in general, we do not add the proofs. However, when it is possible, we explain the meaning of theorems and notions through examples. We start the review in a more general set up.

2 Some general facts of systems on manifolds

Let us consider an ordinary differential equation X on a connected differentiable manifold M of dimension n which model a real dynamic system. A control system on M allows modifying the behavior of X according to different strategies denominated controls. From the mathematical side, in our case from the differential geometry point of view, a control system $\Sigma = (M, \mathcal{D})$ can be stated as a family of vector fields $\mathcal{D}$ coming from a family of differential equations

$$\dot{x}(t) = X(x(t)) + \sum_{j=1}^{m} u_j(t) Y^j(x(t)),$$

where X is the drift, i.e., the dynamic to be controlled. The vector fields Y^j, $j = 1, ..., m$, are defined on M and are weighted by the class of admissible locally integrable functions $u \in \mathcal{U} = \{u : \mathbb{R} \to \Omega \subset \mathbb{R}^m,\ u \text{ is locally integrable}\}$ is the set of the admissible controls with Ω closed, convex and $0 \in int(\Omega)$.

The set $\mathcal{U}$ of admissible controls is closed under concatenation. If u, $v \in \mathcal{U}$, then $w = u * v \in \mathcal{U}$, where

$$w(t) = \begin{cases} u(t), & t \in [0, T), \\ v(t - T), & t \in [T, \infty). \end{cases}$$

The system Σ should guarantee the existence and uniqueness of a solution $\phi(x, u, t)$ (also denoted by $\phi_t^u(x)$), of any differential equation associated to a specific control u an arbitrary initial state x of M and $t \in Dom(\phi(x, u, \cdot))$. In our particular classes of control systems any vector field is complete, means that the solution $\phi(x, u, t)$ is well defined at any $t \in \mathbb{R}$.

Given a control system Σ, some natural mathematical problems which are hard to answer. For instance, the controllability property. Let x be an initial fix state of M and $y \in M$ arbitrary, does y can be reached from x in nonnegative time through an admissible solution of Σ? If the answer is affirmative, we say that y is *accessible* from x through Σ. A control system Σ is said to be *controllable from* x if the accessible set $\mathcal{A}(x)$ of Σ from x defined by

$$\mathcal{A}(x) = \{y \in M : \exists u \in \mathcal{U},\ t \geq 0\ :\ \phi(x, u, t) = y\}$$

is the whole space. And Σ is said to be *controllable if it is controllable from any point of the manifold.* The *set of controllable points to* x is defined by

$$\mathcal{A}^*(x) = \{y \in M : \exists u \in \mathcal{U},\ t \geq 0\ :\ \phi(y, u, t) = x\}.$$

Furthermore, a state $x \in M$ is said to be accessible from x in $T \geq 0$ units of time (in exactly $T \geq 0$ units of time) if there exists $u \in \mathcal{U}$, and $t \in [0, T]$ such that $\phi(y, u, t) = x$, $(\exists u \in \mathcal{U} : \phi(y, u, T) = x)$. We denote by $\mathcal{A}(x, T)$ and $\mathcal{E}(x, T)$ the accessible set from x in T units of time and in exactly T units of time, respectively. Of course, $\mathcal{A}(x) = \cup_{t\,\geq\,0}\ \mathcal{A}(x, t) = \cup_{T\,\geq\,0}\ \mathcal{E}(x, T)$.

Not any arbitrary system is controllable, and it is of interest to know whether a system Σ is controllable or not. It is worth to point out that it is a challenge to characterize this property for general systems, especially when Ω is a proper subset of $\mathbb{R}^m$. In this more realistic way, sometimes it is possible to distinguish a particular subset of M with a nonempty interior where the system is approximately controllable, in the following sense.

A nonempty set $\mathcal{C} \subset M$ is called a *control set of* Σ if for every $x \in \mathcal{C}$

i) there exists $u \in \mathcal{U}$ such that $\phi(t, x, u) \in \mathcal{C}$, for any $t \geq 0$

ii) $\mathcal{C} \subset \mathrm{cl}(\mathcal{A}(x))$, where cl denotes the closure, and

iii) $\mathcal{C}$ is maximal concerning the conditions (*i*) and (*ii*).

See (Colonius & Kliemann, 2000-a) and (San Martin, 1993).

In this review, we consider a semigroup associated with some classes of algebraic control systems. Means, systems with an algebraic structure on the manifold and the dynamics. It turns out that on a connected Lie group, a semigroup with a nonempty interior containing a neighborhood of the identity element e generates all the group. Therefore, when the accessible set from the identity is a semigroup, local controllability from e implies global controllability from the identity. Unfortunately, in (Ayala & San Martin, 2001) it is shown that for a linear control system on the Lie group $SL(2, \mathbb{R})$ the accessibility set from e is not a semigroup. Furthermore, for a transitive linear system on a Lie group G the author in (Jouan, 2011) shows that $\mathcal{A}(e)$ is a semigroup if and only if it coincides with G.

For a general system $\Sigma = (M, \mathcal{D})$, a fundamental result is the Orbit Theorem (Sussmann, 1973), which allows reducing the state space M of any initial condition x to its orbit, i.e., the differentiable manifold determined by

$$G_\Sigma(x) = \left\{Z^1_{t_1}\ \circ ... \circ\ Z^k_{t_k}(x) : Z^j \in \mathcal{D},\ k \in \mathbb{N} \text{ and } t_j \in \mathbb{R}\right\} \subset M.$$

Here, $(Z_t)_{t\,\in\,\mathbb{R}}$ stands for the flow of the vector field Z, a 1-parameter group of M-diffeomorphisms. Observe that G_Σ is a group of global diffeomorphism and $S_\Sigma = \left\{Z^1_{t_1} \circ ... \circ\ Z^k_{t_k} : k \in \mathbb{N},\ Z^j \in \mathcal{D} \text{ and } t_j \geq 0\right\}$ is a semigroup.

In this review we consider *linear and bilinear systems on the Euclidean space* $\mathbb{R}^d$ *and invariant and linear systems on a connected Lie group* G *with Lie algebra* $\mathfrak{g}$. For all of these systems, the Lie algebra generated by the corresponding vector fields $\mathcal{D}$ is finite dimensional. In fact, as we will see, the normalizer

$$\mathfrak{n} = \{Z \in \mathcal{X}^\infty(G) : [Z, \mathfrak{g}] \subset \mathfrak{g}\}$$

of $\mathfrak{g}$ in the set $\mathcal{X}^\infty(G)$ of smooth vector fields on G, contains for any piecewise constant control every dynamic of our four classes of systems. Actually, in (Ayala & Tirao, 1999) it is shown that $\mathfrak{n}$ is diffeomorphic to the semidirect product between $\mathfrak{g}$ and the Lie algebra $\mathfrak{aut}(G)$ of the Lie group $Aut(G)$ of all G-automorphism, i.e., $\mathfrak{n} \cong \mathfrak{g} \times_s \mathfrak{aut}(G)$. The normalizer is finite dimensional with dimension $\dim(\mathfrak{g}) + ((\dim(\mathfrak{g}))^2$ at most. And $Span_{\mathcal{LA}}(\mathcal{D}) \subset \mathfrak{n}$.

Except for $\Sigma_{Lin}(\mathbb{R}^d)$, without loss of generality we assume that any system is transitive, which means that any two points can be connected considering positive and negative times. In other words, $G_\Sigma(e) = G \Leftrightarrow Span_{\mathcal{LA}}(\mathcal{D}) = \mathfrak{g}$.

The Lie bracket $[X, Y]$ is defined by

$$[X, Y](x) = (\frac{d}{dt})_{t\,=\,0+}\ Y_{-\sqrt{t}} \circ X_{-\sqrt{t}} \circ Y_{\sqrt{t}} \circ X_{\sqrt{t}}(x).$$

Since the bracket depends on negative times, $Span_{\mathcal{LA}}(\mathcal{D})$ is related to the controllability property but not too much. According to the orbit theorem, it is possible to recover the orbit just by a derivation. In fact, the tangent space of any orbit is given by $T_y G(x) = Span_{\mathcal{LA}}(\mathcal{D})(y)$.

Let us fix some ideas. In our context the Lie group G could be the real vector space $\mathbb{R}^d$, a sphere S^n, *i.e.*, when $n = 1, 3, 7$, a torus $T^n = S^1 \times S^1 \times ... \times S^1$, the Heisenberg group. We also consider any connected matrix group, such as $GL^+(n, \mathbb{R})$ the invertible real matrices of order n, or their subgroups $SL(n, \mathbb{R})$ and $SO(n, \mathbb{R})$, the matrix groups of determinant 1 and orthogonal, respectively.

3 Linear control systems on Euclidean spaces

We start with an example which contains important ingredients of the theory.

Example 1 *In the Pontryagin book, (Pontryagin, Boltianski, Gamkrelidze & Mishchenko, 1961) the authors establish the following optimal problem: how to stop a train at the station at a minimum time? Consider the ideal case of a straight railway line. For any* $t \geq 0$, *denotes by* $x(t)$ *the distance from the train to the station, that we consider as the origin. The force is giving* $F = ma$ *and* $a(t) = \dot{y}(t)$ *where* $y(t)$ *is the velocity and* $m = 1$. *We get on* $\mathbb{R}^2$

$$\dot{x}(t) \ = y(t),\ \dot{y}(t) \ = u(t),\ u \in \mathcal{U} = \{u : \mathbb{R} \to [-1.1]\,,\, u \text{ is locally integrable}\}.$$

The train is controlled by $u = a$ *and we get a restricted linear system on* $\mathbb{R}^2$:

$$\Sigma_{Lin}(\mathbb{R}^2) : \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t),\ u \in \mathcal{U}.$$

Geometrically, for a given point $(x_0, y_0) \in \mathbb{R}^2$ we need to find a solution of the system transferring the initial condition to the origin $(0,0)$ at minimum time. By some elementary computation $rank(BAB) = 2$ and $Spec(A) = \{0\}$ .According to Theorem 8, the system is controllable. Hence, there exists one control connecting (x_0, y_0) to $(0,0)$. By the Pontryagin Maximum Principal (Pontryagin et al.,1961), it turns out that the optimal control u^* exists and it takes it optimal values in the boundary $\partial\Omega = \{-1, 1\}$. The solutions of $u = -1$ and $u = 1$, are parabolas. We denote by u_{-1} the intersection of the solution through the origin for $u = -1$ with the half-plane $y > 0$ and by u_1 the intersection of the integral curve through the origin determined by $u = 1$ with the half-plane $y < 0$. So, the maximal breaking (u_{-1}) and acceleration (u_1) are optimal curves arriving at the origin, which allows to solve the problem starting at any point (x_0, y_0) and reaching the origin with at most one change: from -1 to 1 or conversely. It is worth to mention that the Pontryagin Maximum Principle got the Lenin Prize in Russia recognizing its great contribution to the Society.

The classical linear system $\Sigma_{Lin}(\mathbb{R}^d) = (\mathbb{R}^d, \mathcal{D})$ on the Euclidean space $\mathbb{R}^d$ is determined by the dynamics of $\mathcal{D}$ coming from

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{m} u_j(t)\ b^j, \qquad b^j \in \mathbb{R}^d \text{ and } u \in \mathcal{U}$$

with Ω is a closed and convex subset of $\mathbb{R}^m$ with $0 \in int(\Omega)$. Here $A \in \mathrm{gl}(d, \mathbb{R})$, the Lie algebra of the real matrices of order d. The classical cost matrix B is built with the columns vectors b^j, called the control vectors, $j = 1, ..., m$.

Essentially, this system depends on two kinds of dynamics. The linear and the invariant ones. In fact, the linear differential equation $\dot{x}(t) = Ax(t)$ on $\mathbb{R}^d$ is controlled by m invariant constant vector fields b^j, $j = 1, ..., m$. On the other hand, any vector b in the Abelian Lie algebra $\mathbb{R}^d$ induces by translation an invariant vector field $Z^b(x) = b$ on the commutative Lie group $\mathbb{R}^d$.

Given an initial condition $x_0 \in \mathbb{R}^d$ and $u \in \mathcal{U}$, it is possible to describe the solution of the system completely as follows

$$\phi_t^u(x_0) = e^{tA}\left(x_0 + \int_0^t e^{-\tau A}\ B\ u(\tau)\, d\tau\right)$$

which satisfy the Cauchy problem with initial value $\dot{x} = Ax + Bu,\ x(0) = x_0$. Thus, $\phi_t^u(x_0)$ with $t \in \mathbb{R}$ describes a curve in $\mathbb{R}^d$ such that starting from x_0 the elements on the curve are reached from x_0 forward and backward through the specific dynamic of the linear system determined by the controls.

Next, we show that in the unrestricted case, i.e., when $\Omega = \mathbb{R}^m$, the accessibility set of a linear control system on $\mathbb{R}^d$ is a semigroup. Means, a vector subspace of $(\mathbb{R}^d, +)$. Since the proof is direct, we include it here. We follow the reference (Ayala & Zegarra, 2001).

Theorem 2 *Let $\Sigma_{Lin}(\mathbb{R}^d)$ be an unrestricted linear control system. Then,*

1. For any $x_0 \in \mathbb{R}^d$ and $T > 0$

$$\mathcal{E}(x_0, T) \subset \mathcal{A}(x_0, T) \subset \mathcal{A}(x_0) \text{ and } e^{TA}x_0 + \mathcal{A}(0, T) \subset \mathcal{A}(x_0, T)$$

2. $0 \leq T_1 \leq T_2 \Rightarrow \mathcal{E}(0, T_1) \subset \mathcal{A}(0, T_2)$

3. For any $T > 0$ the sets $\mathcal{E}(0, T)$ and $\mathcal{A}(0)$ are vector subspaces

Proof. Property (1) comes directly from the solution shape. To prove (2) just observe that it is possible to rest at the origin with $u = 0$. Finally, by definition

$$\mathcal{E}(0, T) = \left\{ \phi_T^u(0) = e^{TA} \int_0^T e^{-\tau A} \, Bu(\tau) \, d\tau \; : \; u \in \mathcal{U} \right\}.$$

Under the hypothesis $\Omega = \mathbb{R}^m$ the set $\mathcal{U}$ is a vector space. Actually,

$$\phi_T^{u_1}(0) + \phi_T^{u_2}(0) = \phi_T^{u_1+u_2}(0) \text{ and } \phi_T^{\lambda u}(0) = \lambda \phi_T^u(0).$$

As the union of an increasing chain of subspaces, $\mathcal{E}(0, T)$ is a vector space. ∎

Furthermore, it is possible to characterize $\mathcal{A}(0)$ in an algebraic way. Denote by $\langle A, B \rangle$ the small A-invariant subspace of $\mathbb{R}^d$ containing $\text{Im}(B)$. The proof of the next two propositions are rather standard and we omit it.

Proposition 3 *Let $\Sigma_{Lin}(\mathbb{R}^d)$ be an unrestricted linear control system. Then, $\mathcal{E}(0, T) = \langle A, B \rangle$ for any $T > 0$.*

Proposition 4 *Let $\Sigma_{Lin}(\mathbb{R}^d)$ be an unrestricted linear control system. The following conditions are equivalent*

$\Sigma_{Lin}(\mathbb{R}^d)$ is controllable $\Leftrightarrow$ There exists $x_0 \in \mathbb{R}^d$ such that $\Sigma_{Lin}(\mathbb{R}^d)$ is controllable from $x_0 \Leftrightarrow \Sigma_{Lin}(\mathbb{R}^d)$ is controllable from the origin $\Leftrightarrow \mathcal{A}(0, T) = \mathbb{R}^d$, for any $T > 0 \Leftrightarrow \mathcal{E}(0, T) = \mathbb{R}^d$, for any $T > 0 \Leftrightarrow \mathbb{R}^d = \langle A, B \rangle$.

The next result due to Kalman, (Kalman, Ho & Narendra, 1962) provides a criterion for testing controllability. Let us denote by

$$K = \left(B \; AB \; A^2 \; B ... A^{n-1} B \right)$$

the $d \times dm$ matrix associated to the dynamic A and B of $\Sigma_{Lin}(\mathbb{R}^d)$.

Theorem 5 *(Kalman rank condition) The unrestricted linear control system $\Sigma_{Lin}(\mathbb{R}^d)$ is controllable $\Leftrightarrow$ rank$(K) = d$.*

Proof. At present we will merely show that

$$\langle A, B \rangle = \left\{ A^k \; b^j \; : \; j = 1, 2, ..., m \; , \quad k = 0, 1, 2, ..., n-1 \right\}.$$

But, the Cayley-Hamilton theorem said that it is not necessary to compute $A^k \; b^j$, for $j = 1, 2, ..., m$, when $k \geq n$. ∎

Remark 6 *When $\Omega = \mathbb{R}^m$ the accessible set from the origin is a subspace. But its interior could be empty. Equivalently,* $\dim \mathcal{A}(0) < n$.

What happens if $\Omega \subsetneq \mathbb{R}^m$? It turns out that in few cases $\mathcal{A}(0)$ is still a subspace. Precisely, in (Colonius & Kliemann, 2000-a) the authors show the following theorem. Assume that Ω is a compact set with $0 \in int(\Omega)$.

Theorem 7 *Let $\Sigma_{Lin}(\mathbb{R}^d)$ be a restricted linear control system which satisfies the Kalman condition. Hence, there exists one and only one control set with an nonempty interior containing the origin and it is given by $\mathcal{C} = \mathrm{cl}(\mathcal{A}(0)) \cap \mathcal{A}^*(0)$. Furthermore, the system is controllable at the open set $int(\mathcal{C})$.*

We suggest to the readers see a concrete example in (Colonius & Kliemann, 2000-a). Finally, as a direct consequence, we get

Theorem 8 *As before let $\Sigma_{Lin}(\mathbb{R}^d)$ be a restricted linear control system which satisfies the Kalman condition. Therefore,*

$$\Sigma_{Lin}(\mathbb{R}^d) \text{ is controllable on } \mathbb{R}^d \Leftrightarrow Spec(A)_{Ly} = \{0\}\,.$$

Here, $Spec(A)_{Ly}$ means the Lyapunov spectrum of the matrix A, i.e., the set of the real parts of the eigenvalues in $Spec(A)$. In this particular case, the accessibility set $\mathcal{A}(0)$ is also a subspace and equals to $\mathbb{R}^d$.

4 Bilinear control systems on Euclidean spaces

Bilinear control systems are relevant from both, the theoretical (Colonius & Kliemann, 2000-a), (Elliot, 2009) and (Isidori, 1998); and from the practical point of view, (Ledzewick & Shattler, 2006) and (Mohler, 1973). A bilinear control system $\Sigma_{Bil}(\mathbb{R}^d)$ is determined by the family of differential equations

$$\dot{x}(t) = \left(A + \sum_{j=1}^{m} u_j(t) B_j \right) x(t),\ t \in \mathbb{R}, \qquad x(t) \in \mathbb{R}^d. \tag{bilinear}$$

Here, $A, B_1, \ldots, B_m \in gl(d, \mathbb{R})$ and in this case we take $u \in \mathcal{U}$ where

$$\mathcal{U} = \{u : \mathbb{R} \to \Omega \subset \mathbb{R}^m,\ u \text{ is piecewise constant control}\}$$

is the set of the admissible controls with Ω compact, convex and $0 \in int(\Omega)$.

At once you can see that the Lie algebra generated by the matrices $\nabla = Span_{\mathcal{LA}}\{A, B_1, \ldots, B_m\}$ is a Lie subalgebra of $gl(d, \mathbb{R})$. In fact, the Lie bracket $[A, B] = AB - BA \in gl(d, \mathbb{R})$, for any $A, B \in gl(d, \mathbb{R})$. In particular, there exists a connected Lie subgroup $G \subset GL(d, \mathbb{R})$ with Lie algebra ∇, (Warner, 1971).

If $\Sigma = (M, \mathcal{D})$ has a global singularity let say at x_0, then it is possible to linearize the vector fields of the system obtaining a bilinear system $\Sigma_{Lin}(\mathbb{R}^d, x_0)$. Global information of $\Sigma_{Lin}(\mathbb{R}^d, x_0)$ gives local information of Σ, precisely like in the classical approach of Hartman-Grobman (Hartman, 1960). Contrary to the linear system on Euclidean spaces, the control vectors of $\Sigma_{Bil}(\mathbb{R}^d)$ have an influence on the state. This situation allows using the Lie theory systematically. For any constant control u, the bilinear system determines a linear differential equation and $\mathcal{D}$ is a family of matrices. Just observe that the differential equation

$$\dot{x}(t) = Ax(t),\ x(0) = x_0,\ x(t) \in \mathbb{R}^d$$

has the solution $x(t) = e^{tA}x_0$. At the same time the solution of

$$\dot{X}(t) = A(t),\ X(0) = Id,\ X(t) \in GL^+(d, \mathbb{R})$$

is given by $X(t) = e^{tA}$. As we will see in the next section, it is possible to know the behavior of the control system $\Sigma_{Bil}(\mathbb{R}^d)$ at the point x_0 through the action of $\Sigma_{Inv}(G)$ defined on the invertible matrix group $GL(d, \mathbb{R})$ of order d. Actually, G is built as the connected subgroup with Lie algebra $Span_{\mathcal{LA}}(\mathcal{D})$.

Controllability of bilinear systems has been a source of research since more than 40 years, but still, there is not a complete characterization. There exists an algebraic approach, (Elliot, 2009), especially when the controls are unrestricted. For the restricted case, there exists an analytic approach on fiber bundles that can be applied to $\Sigma_{Bil}(\mathbb{R}^d)$. For the following definitions and results, we refer to (Colonius & Kliemann, 2000-a), Chapters 7 and 12.

Assume the system satisfies the LARC property at any $x \in \mathbb{R}^d$-$\{0\}$, *i.e.*,

$$\dim(Span_{\mathcal{LA}}\{A + \sum_{i=1}^{m} u_i B_i :\ u \in \mathcal{U}\}(x)) = d. \tag{1}$$

The control system $\Sigma_{Bil}(\mathbb{R}^d)$ has the following associated systems:

a) The *angle system* $\mathbb{P}\Sigma$ defined by the projection of $\Sigma_{Bil}(\mathbb{R}^d)$ onto $\mathbb{P}^{d-1}$,

$$\mathbb{P}\Sigma : \qquad \dot{s}(t) = h(A, s(t)) + \sum_{i=1}^{m} u_i(t) h(B_i, s(t)), \qquad s \in \mathbb{P}^{d-1}, \tag{2}$$

here $h(A, s) = (A - s^T AsI)s$, where I is the identity matrix and $u \in \mathcal{U}$

b) The *radial system* defined on $\mathbb{R}^+$ by $r(t) = ||\varphi(t, x, u)||$.

The solutions of the projected system (2) are denoted by $\mathbb{P}\varphi(t, s, u)$ for the initial value $\mathbb{P}\varphi(0, s, u) = s \in \mathbb{P}^{d-1}$. The real projective space $\mathbb{P}^{d-1}$ is determined by a quotient manifold $\mathbb{P}^{d-1} \cong S^{d-1}/\sim$, where the antipodal differentiable equivalence relation is defined by $x \sim y \Leftrightarrow y = \pm x$.

Since for any control $u \in \mathcal{U}$ the origin is a singularity of any bilinear system, we need to define what we understand for controllability. Let us introduce the associated semigroup to $\Sigma_{Bil}(\mathbb{R}^d)$ as follows

$$S_\Sigma = \left\{ e^{t_1(A+u_1B)} \cdots\ e^{t_k(A+u_kB)} : t_j \geq 0,\ j = 1, ..., k \text{ and } u_j \in \mathbb{R} \right\}.$$

Definition 9 *A bilinear control system $\Sigma_{Bil}(\mathbb{R}^d)$ is said to be controllable on $\mathbb{R}^d$-$\{0\}$ if given any two points $x,\ y \in \mathbb{R}^d$-$\{0\}$ there exists $u \in \mathcal{U}$ and $t \geq 0$ transferring x to y at t units of time. In other words, $\exists \psi \in S_{\Sigma_{Bil}(\mathbb{R}^d)} : y = \psi(x)$.*

There are many controllability results to this class of control system. In this paper, we concentrate on the specific case of dimension two. In particular, we show some contribution that we have done, (Ayala & San Martin, 1994) and (Ayala, Cruz, Lauro & Kliemann, 2016).

Next, we establish a fundamental result of this theory due to Colonius and Kliemann, (Colonius & Kliemann, 2000-a). For that, we need to introduce some notions. For a solution $\varphi(t, x, u)$ with $x \neq 0$ and $u \in \mathcal{U}$ the Lyapunov exponent of the pair (u, x) is defined as $\lambda(u, x) = \limsup\limits_{t\to\infty} \frac{1}{t} \log \|\varphi(t, x, u(\cdot))\|$. The Lyapunov spectrum consists of all Lyapunov exponents, i.e.$\Sigma_{Ly} = \{\lambda(u, x) : (u, x) \in \mathcal{U} \times \mathbb{R}^d\text{-}\{0\}\}$. Extremal Lyapunov exponents are defined globally by $\kappa^* = \inf_{u \in \mathcal{U}} \inf_{x \neq 0} \lambda(u, x)$, $\kappa = \sup_{u \in \mathcal{U}} \sup_{x \neq 0} \lambda(u, x)$. In particular, when Ω is compact $-\infty < \kappa^* \leq \kappa < \infty$. It turns out that

Theorem 10 *(Colonius & Kliemann, 2000-a) Consider the bilinear control system $\Sigma_{Bil}(\mathbb{R}^d)$ and its projected system (2) satisfying* (1)*. Are equivalents*

1. *$\Sigma_{Bil}(\mathbb{R}^d)$ is controllable in $\mathbb{R}^d$-$\{0\}$*
2. *The system $\mathbb{P}\Sigma$ (2) is controllable on $\mathbb{P}^{d-1}$ and $0 \in (\kappa^*, \kappa)$.*

Theorem 11 *(Colonius & Kliemann, 2000-a) Consider the bilinear control system $\Sigma_{Bil}(\mathbb{R}^d)$ and its projected system* (2) *satisfying* (1)*. If* (2) *is controllable on $\mathbb{P}^{d-1}$, it turns out that $[\kappa^*, \kappa] = \Sigma_{Ly}$.*

In the sequel, we concentrate on bilinear systems on dimension 2.

Theorem 12 *(Ayala, Cruz, Lauro & Kliemann, 2016). Consider the unrestricted bilinear control system with $d = 2$, $\Omega = \mathbb{R}$ and assume that the Lie algebra rank condition (1) holds. Then, the projected angle system is controllable on $\mathbb{P}^1$ $\Leftrightarrow$ there exists a constant control $u \in \mathbb{R}$ such that $A + uB$ has a complex eigenvalue.*

Example 13 *First we consider the bilinear single control system*

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + u(t) \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The projected angular system on $\mathbb{S}^1$ satisfies the differential equation

$$\dot{\theta} = [(1 + u(t)\cos(\theta) - 2u(t)\sin(\theta)] \ \ \sin(\theta).$$

Since $\theta = 0$ is a common fixed point it does not satisfy LARC. In particular, the projected system is not controllable on $\mathbb{P}^1$. The same situation happens for

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + v(t) \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

But, if you combine both systems as

$$\begin{pmatrix} \dot{x_1} \\ x_2 \end{pmatrix} = \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + u(t) \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} + v(t) \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The projected angle system on $\mathbb{S}^1$ reads as

$$f((u(t), v(t)), \alpha) = -(1 + u(t)\sin(\alpha) + 2(u(t) + v(t))\cos(\alpha) + 2(u(t) - v(t))$$

and one can check easily that (1) is satisfied. Computing the eigenvalues for a constant control (u, v) we get

$$\lambda(u, v) = \frac{3}{2} + \frac{3}{2}u + v \pm \frac{1}{2}\sqrt{(1+u)^2 + 16uv}.$$

By taking $u = 1$ and $v = -1$ we obtain a pair of complex eigenvalues. Hence the system is controllable on $\mathbb{P}^1$.

Controllability conditions for an unrestricted system $\Sigma = \Sigma_{Bil}(\mathbb{R}^d)$

$$\Sigma : \dot{x}(t) = (A + uB)\, x(t), \qquad t \in \mathbb{R}, \qquad x(t) \in \mathbb{R}^2 \tag{3}$$

where given in (Bragas et al. 1996), (Joó & Tuan, 1992). In this case $u \in \mathcal{U}$ with $\Omega = \mathbb{R}^m$, and $A, B \in \mathfrak{gl}(2, \mathbb{R})$. Furthermore,

Theorem 14 *(Ayala & San Martin, 1994) Assume $A, B \in \mathfrak{sl}(2, \mathbb{R})$. Then, the unrestricted system*

$$\Sigma : \dot{x}(t) = Ax(t) + uBx(t), \ u \in \mathbb{R}$$

is controllable in $\mathbb{R}^2$-$\{0\}$ if and only if $\det[A, B] < 0$.

We would like to show that the algebraic condition in the previous theorem is very interesting geometric condition. In the sequel, we follow. For the Lie theory, see next section.

Given a Lie algebra $\mathfrak{g}$ and $X \in \mathfrak{g}$, the adjoint map $ad(X) : \mathfrak{g} \to \mathfrak{g}$ is defined by $ad(X)(Y) = [X, Y]$. The Cartan-Killing form, $k(X, Y) = tr(ad(X) \circ ad(Y))$ induced the quadratic non degenerate form $Q(Z) = tr(Z^2)$. Through this form, it is possible to identify the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ with $\mathbb{R}^3$ with axis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

corresponding to the hyperbolic, symmetric and skew-symmetric basis elements. The set $\mathcal{C} = Q(0)$ is a double cone with axis the line generated by A. Denote by $\mathcal{C}_{int} = \{Z : Q(Z) < 0\}$ and $\mathcal{C}_{ext} = \{Z : Q(Z) > 0\}$. The elements of $\mathcal{C}$ are nilpotent matrices. While $\mathcal{C}_{int}$ contain all the matrices with imaginary eigenvalues the elements of $\mathcal{C}_{ext}$ are the diagonal real matrices.

Theorem 15 *(Ayala & San Martin, 1994) Assume* $\det[A,B] < 0$*. Then*

$$\Sigma : \dot{x}(t) = Ax(t) + uBx(t),\ u \in \Omega = [-1,1]$$

is controllable if and only if the straight line $l = \{A + uB : u \in \Omega = [-1,1]\}$ *meet the interior of* $\mathcal{C}$*, i.e.,* $l \cap int(\mathcal{C}) \neq \varnothing$.

In other words, if $Span_{\mathcal{LA}}\{A,B\} = \mathfrak{sl}(2,\mathbb{R})$, the restricted system is controllable if and only $\exists u_0 : A + u_0 B$ has an imaginary eigenvalue.

More general, for a fix non zero element $x_0 \in \mathbb{R}^2$ we consider the map

$$\mathcal{X} : [0,\infty) \rightarrow \mathcal{P}(\mathbb{R}^2) \text{ defined by } \mathcal{X}(a) = \mathcal{A}_a(x_0)$$

the accessible set of the system with range control Ω^a and where $\mathcal{P}(\mathbb{R}^2)$ denotes the family of subsets of $\mathbb{R}^2$.

Theorem 16 *(Ayala & San Martin, 1994) Assume* $a \geq 0$ *and* $\det[A,B] < 0$*. The controllability property of the restricted bilinear system*

$$\Sigma_a : \dot{x}(t) = Ax(t) + uBx(t),\ u \in \Omega^a$$

with range $\Omega^a = [-a,a]$ *is given by the relative position of the segment* $l_a = \{A + uB : u \in \Omega^a\}$ *as follows*

1. If $\det(A) \geq 0$ then $l_a \cap \mathcal{C}_{int} \neq \varnothing$ and $\mathcal{X}(a) = \mathbb{R}^2 - \{0\}$ for any $a > 0$
2. If $\det(A) < 0$, there are the possibilities

i) If $\det[A,B] < 0$ the line $A + uB$, $u \in \mathbb{R}$ cross the interior of $\mathcal{C}$ and the system Σ_a is controllable if and only if $l_a \cap \mathcal{C}_{int} \neq \varnothing$. The only bifurcation point of $\mathcal{X}$ is determined by $a^* = \inf\{a : l_a \cap \mathcal{C}_{int} = \varnothing\}$. In fact, for $a < a^*$, $\mathcal{X}(a)$ is strict contained in $\mathbb{R}^2$-$\{0\}$. For $a > a^*$, $\mathcal{X}(a) = \mathbb{R}^2$-$\{0\}$.

ii) If $\det[A,B] > 0$, the system Σ_a is not controllable for any $a \geq 0$ and $\mathcal{X}$ is a continuous map on $(0,\infty)$ with the Haudorff metric on $\mathcal{P}(\mathbb{R}^2)$.

Next, we show a controllable bilinear control system on the plane.

Example 17 *Let us consider the bilinear system*

$$\Sigma : \dot{x}(t) = Ax(t) + uHx(t),\ u \in \Omega$$

with range Ω *and where* A *and* H *are basis elements of* $\mathfrak{sl}(2,\mathbb{R})$ *as before. We have,* $det(A) = 1 > 0$ *and* $\det[A,H] = -4 < 0$*. According to the previous theorem, the bilinear system* Σ *is controllable for any positive real number* a*.*

It is worth point out that the class of bilinear systems with matrices in $\mathfrak{sl}(2,\mathbb{R})$ has served a model for an optimal compartment model for cancer chemotherapy with the quadratic objective, (Ledzewick & Shattler, 2006).

Finally, we mention that some effort was made to study the equivalence problem of this class of systems. See, (Ayala, Colonius & Kliemann, 2005), (Ayala, Colonius & Kliemann, 2007) and (Ayala & Kawan, 2014).

5 An elementary review on Lie theory

In this section we shortly introduce the main ingredients of the Lie theory we need for the next two sections which include invariant and linear control systems on Lie groups. We mention a couple of references on this subject, (Curtis, 1979), (Helgason, 1978), (San Martin, 1999) and (Warner, 1971).

Definition 18 *A Lie group G is an analytic manifold such that the group operations $\mu : G \times G \to G : (g, h) \to gh$ and $\iota : G \to G : g \to g^{-1}$ are analytic.*

Example 19 *The following sets are Lie groups under obvious multiplication*

1. The Euclidean space $\mathbb{R}^n$
2. The set $GL(d, \mathbb{R}) = \left[\det^{-1}(0)\right]^c$ and $GL^+(d, \mathbb{R})$ which contains the Id
3. The torus $T^d = S^1 \times .... \times S^1$ (d-times the circle)
4. The orthogonal group $O(d) = \{A \in GL(d, \mathbb{R}\ ) \mid AA^t = Id\}$
5. The special orthogonal group $SO(d) = \{A \in O(d) \mid \det(A) = 1\}$
6. The special linear group $SL(d, \mathbb{R}\ ) = \{A \in GL(d, \mathbb{R}\ ) \mid \det(A) = 1\}$.
7. The Heisenberg group $(\mathbb{R}^3, *)$, with

$$(x, y, z) * (a, b, c) = (x + a, y + b, z + c + xb).$$

The analytical maps R_g , $L_g : G \to G$ defined by $R_g(x) = xg$ and $L_g(x) = gx$ called the right and the left translations on G respectively, are diffeomorphisms. The Lie algebra of G comes from the notion of invariant vector fields as follows. We denote by $\mathcal{X}^\infty(G)$ the set of C^∞-vector fields on G.

Definition 20 *We say that $X \in \mathcal{X}^\infty(G)$ is a right invariant vector field if*

$$X \circ R_g = (R_g)_* \ (X) \text{ for every } g \in G.$$

Here, $(R_g)_*$ or $(dR_g)_e$ denotes the differential of R_g at the identity $e = Id$.

Remark 21 *It is not difficult to prove that given two right invariant vector fields X, Y the Lie bracket $[X, Y]$ between them is also a right invariant vector field. Furthermore, observe that $X \in \mathfrak{g}$ is wholly determined by its value at the identity. In other words, $\mathfrak{g}$ is isomorphic to the tangent space T_eG.The set of right invariant vector fields on G is called the Lie algebra $\mathfrak{g}$ of G which satisfy*

i) $[X, Y] = -[Y, X]$ (skew-symmetric)

ii) $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$, (Jacobi identity).

A subspace $V \subset \mathfrak{g}$ is a subalgebra if $[V, V] \subset V$ and it is an ideal if $[V, \mathfrak{g}] \subset V$.

Example 22 *In few cases we show the Lie algebra* $\mathfrak{g} \cong T_eG$

1. $T_0\mathbb{R}^d = \mathbb{R}^d$
2. $T_{Id}\ GL^+(d,\mathbb{R}) = \mathfrak{gl}(d,\mathbb{R})$, the set of real matrices of order d
3. $T_{Id}\ S^d = \mathbb{R}^d$
4. $T_{Id}\ O(d,\mathbb{R}) = \mathfrak{o}(d) = \{A \in GL(d,\mathbb{R}\ ) \mid A + A^t = 0\}$, skew symmetric
5. $T_{Id}\ SO(d,\mathbb{R}) = \mathfrak{so}(d) = \mathfrak{o}(d)$
6. The trace zero matrices $\mathfrak{sl}(d,\mathbb{R}\ ) = \{A \in \mathfrak{gl}(d,\mathbb{R}) \mid tr(A) = 0\}$
7. The Heisenberg Lie algebra $(\mathbb{R}^3, +, [,])$, has the basis $\left\{X^1, X^2, X^3\right\}$ such that $\left[X^1, X^2\right] = X^3$ is the only non vanish bracket. In fact, the Heisenberg group has the matrix representation

$$G = \left\{ g = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\} \overset{\phi:\ g\ \to\ (x_1,x_2,x_3)}{\longrightarrow} \mathbb{R}^3.$$

   The derivative of $\gamma_i : \mathbb{R} \to \mathbb{R}^{\ 3}$, $\gamma_i(t) = \phi^{-1}(te_i)$ at $t = 0$, determines X^i.

In this paper we just consider matrix groups, so the exponential map is

$$\exp : \mathfrak{gl}(d,\mathbb{R}) \to GL^+(d,\mathbb{R}),\ \ \exp A = \sum_{k\ =\ 0}^{\infty} \frac{1}{k!} A^k \text{ with } A^0 = Id.$$

Since $d(\exp)_0 = Id$ there exists a neighborhood $V \subset G$ of e such that exp is a local diffeomorphism. Furthermore, for nilpotent and simply connected Lie groups exp is a global diffeomorphism, which means $V = G$.

A C^∞ homomorphism between two Lie groups G and H is called a Lie group homomorphism. A bijective Lie group homomorphism of G with itself is called a Lie group automorphism. If G is connected, the set $\mathrm{Aut}(G)$ of G-automorphisms is a Lie group with Lie algebra $\mathfrak{aut}(G)$, (Warner, 1971).

Remark 23 *An important relation between a Lie group homomorphism* $\varphi : G \to H$*, and its derivative* $(d\varphi)_e : T_{e_G} \longrightarrow T_{e_H}$ *is given by* $\varphi(\exp X) = \exp(d\varphi(X))$*, which comes from the commutative diagram*

$$\begin{array}{ccc} \mathfrak{g} & \overset{(d\varphi)_e}{\longrightarrow} & \mathfrak{h} \\ \exp_{\mathfrak{g}} \downarrow & \hookrightarrow & \downarrow \exp_{\mathfrak{h}} \\ G & \underset{\varphi}{\longrightarrow} & H \end{array}$$

Since $(d\det)_e = tr$ it follows that $e^{trA} = \det(\exp A)$, $A \in \mathfrak{gl}(d,\mathbb{R}\ )$.

Definition 24 *A Lie algebra* $\mathfrak{g}$ *is say to be*

1. *Abelian* if for any $X, Y \in \mathfrak{g}$ the bracket $[X, Y]$ is zero

2. *Nilpotent* if $\exists k \geq 1$: its descendent central series stabilizes at 0

$$0 = ad^k(\mathfrak{g}) = \left[ad^{k-1}(\mathfrak{g}), ad^1(\mathfrak{g})\right] \subset \ldots \subset ad^1(\mathfrak{g})$$

3. *Solvable* if $\exists k \geq 1$: such that its derivative series stabilizes at 0

$$0 = ad^{(k)}(\mathfrak{g}) = \left[ad^{(k-1)}(\mathfrak{g}), ad^{(k-1)}(\mathfrak{g})\right] \subset \ldots \subset ad^1(\mathfrak{g})$$

4. *Completely solvable* if it is solvable and $Spec(\mathrm{ad}(Y)) \subset \mathbb{R}$, for every $Y \in \mathfrak{g}$

4. *Simple* if $\mathfrak{g}$ it is not Abelian and contains non proper ideals

5. *Semisimple* if the largest solvable subalgebra $\mathfrak{r}(\mathfrak{g})$ of $\mathfrak{g}$ is null.

A Lie group is said to be Abelian, nilpotent, solvable, completely solvable, simple, semisimple, if its Lie algebra satisfy the same property.

Example 25 *Here, we mention the type of some Lie groups*

1. The Euclidean space $\mathbb{R}^d$ is Abelian

2. The torus $T^n = S^1 \times \ldots\times S^1$ (n-times) is Abelian and compact. Furthermore, any Abelian group has the form $\mathbb{R}^d \times T^n$ for some $d, n \in \mathbb{N}$.

3. The Heisenberg group $(\mathbb{R}^3, *)$ is nilpotent

4. The Affine group $\left\{ \begin{pmatrix} A & y \\ 0 & 1 \end{pmatrix} : A \in GL(d, \mathbb{R}),\ y \in \mathbb{R}^d \right\}$ is solvable

5. The upper triangular group $T(d)$ is completely solvable

$$T(d) = \{A = (a_{ij}) \in GL(d, \mathbb{R}) : a_{ij} < 0 \text{ for } j < i\}$$

6. The orthogonal group $SO(d, \mathbb{R})$ is compact and simple for $d \neq 4$

$$SO(d, \mathbb{R}) = \{A \in GL(d, \mathbb{R}) \mid AA^t = Id\}$$

7. The orthogonal group $SO(4, \mathbb{R})$ is compact and semisimple

8. The special linear group $SL(d, \mathbb{R})$ is non bounded and semisimple

$$SL(d, \mathbb{R}) = \{A \in GL(d, \mathbb{R}) \mid \det(A) = 1\}.$$

6 Invariant control systems on Lie groups

An invariant control system $\Sigma_{Inv}(G)$ on a connected Lie group G is determined by a family $\mathcal{D}$ of differential equations given by

$$\mathcal{D}=\left\{X+\sum_{j=1}^{m}u_jY^j : u\in\mathcal{U}\right\}.$$

The drift vector field X and the control vectors $Y^1,\ldots,Y^m$ here are elements of the Lie algebra $\mathfrak{g}$ of G which we think as the set of right invariant vector fields. We consider $\mathcal{U}$ as before as the set of the admissible class of control.

It is well known that the class of invariant control systems is relevant both from the theoretical and practical point of view. In fact, since the beginning of the 1970s, many people have been working in this kind of systems. We mention the first work in the subject by (Brockett, 1972). Then, several mathematicians started to study this system on different classes of Lie groups: Abelian, compact, nilpotent, solvable, completely solvable, simple, semisimple, etc. We mention some of them (Jurdjevic & Sussmann 1972-a), (Hilgert et al., 1985), (Ayala, 1995) (Sachkov, 1999), (Jurdjevic & Kupka, 1981), (Gauthier, Kupka & Sallet, 1984), (Dos Santos & San Martin, 2013) For an excellent survey on the topic see references (Sachkov, 1999, 2000 and 2006). Recently in (Dos Santos & San Martin, 2013), the authors obtain a very general controllability result for complex Lie groups.

For the first and third categories of systems, the accessible set from the identity $\mathcal{A}(e)$ is a semigroup. Here, $\mathcal{A}(e)$ stands for the set of points that can be reached from e through all admissible trajectories in positive time. Unfortunately, for the class of linear systems on Lie groups, this is not longer true. In fact, as we mention in (Ayala & San Martin, 2001) the authors give an example on $G=SL(2,\mathbb{R})$ where the linear system is locally controllable from the identity but not controllable at all. Since the group is connected, it follows immediately that the accessibility $\mathcal{A}(e)$ can not be a semigroup.

As appointed by Professor Jurdjevic, optimal control on Lie groups is a natural setting for geometry and mechanics, see (Jurdjevic, 1997-a), (Jurdjevic, 1997-b) and (Sachkov, 2006). As a consequence, differential systems on Lie groups and their homogeneous spaces deserve to be developed. For instance, the Dubin´s problem (Dubins, 1957), the brachistochrone problem (Sussmann & Willems, 1997), and the control of the altitude of a satellite in orbit (Isidori, 1998), are invariant systems on a particular class of Lie groups.

As usual, through this section, we assume that any invariant system satisfies the Lie algebra rank condition $Span_{\mathcal{LA}}\left\{X,\ Y^1,...,\ Y^m\right\}=\mathfrak{g}$. By the Sussmann Theorem, the orbit of e is the group, .i.e., $G_{\Sigma_{Inv}(G)}(e)=G$. The system has an associated semigroup, in fact

Theorem 26 $S_{\Sigma_{Inv}(G)}(e)=\mathcal{A}(e)$ *is a semigroup of* G

Proof. We just sketch the prove. Since the vector fields of the system generate the Lie algebra it turns out that

$$\mathcal{A}(e) = \{\exp(t_1 Z_1)\exp(t_2 Z_2)...\exp(t_j Z_j) : t_j \geq 0,\ Z_j \in \mathcal{D},\ j = 1, ..., k\}.$$

Thus, given $g_1, g_2 \in \mathcal{A}(e)$ the product $g_1 g_2$ is an element of $\mathcal{A}(e)$. ∎

In the sequel, we establish some controllability results for $\Sigma = \Sigma_{Inv}(G)$. We start on a nilpotent Lie group. In (Ayala & Vergara, 1992) the authors state:

"if there exists a strictly increasing real function f on the positive trajectories of Σ, then the system can not be controllable".

Assuming controllability, it turns out that there exists a loop-trajectory of the system starting and ending on the identity element, indeed at any element of G. In particular, a strictly increasing function f cannot exist along the loop. To develop this idea, they introduce the notion of symplectic vectors via the co-adjoint representation. Next, in (Ayala, 1995) the author search for finding algebraic conditions to determine the existence of the symplectic vectors on nilpotent Lie algebras. They state the following results

Proposition 27 *Let G be a nilpotent simply connected Lie group with lie algebra $\mathfrak{g}$ and let $\mathfrak{h}$ be an ideal such that $\mathfrak{g}/\mathfrak{h}$ is not an Abelian Lie algebra. If $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical projection and there exists $Z \in \mathfrak{g}$ such that $\pi(Z) \in \mathfrak{z}(\mathfrak{g}/\mathfrak{h})$ belongs to the center of $\mathfrak{g}/\mathfrak{h}$ is a nonnull vector field, therefore there exists a symplectic vector λ for Z.*

It is possible to characterize the controllability property of this system effortlessly. In fact, let us define the Lie algebra of the control vectors by $\mathfrak{h} = Span_{\mathcal{LA}}\left\{Y^1, Y^2, ..., Y^m\right\}$. Then, we get

Theorem 28 *(Ayala, 1995) Let $\Sigma_{Inv}(G)$ be an invariant control system on a nilpotent connected and simply connected Lie group G. Therefore, Σ is controllable if and only if $\mathfrak{g} = \mathfrak{h}$.*

The main idea in the proof uses the co-adjoint representation to build several homogeneous spaces and then to find algebraic conditions on these spaces to define symplectic vectors. For the class of invariant control systems on a solvable Lie group G the controllability property can be characterized, as follows:

Theorem 29 *(Hilgert et al., 1985) Σ is controllable if and only if*

i) $Span_{\mathcal{LA}}(\mathcal{D}) = \mathfrak{g}$, and

ii) $\mathcal{D}$ is not contained in a half space of $\mathfrak{g}$ bounded by a subalgebra.

To have a more direct way to check controllability for connected and simply connected solvable Lie groups, in (Sachkov, 1997) the author introduces the notion of a completely solvable Lie algebra.

Theorem 30 *(Sachkov, 1997) Let Σ be an invariant control system on a completely solvable Lie group G. Then, Σ is controllable if and only if $\mathfrak{h} = \mathfrak{g}$.*

If $\mathfrak{g}$ is nilpotent, for any $Z \in \mathfrak{g}$ the set $Spec(\text{ad}(Z))$ reduces to zero.

Example 31 *On the Heisenberg group G, $\mathcal{D} = \left\{X^3 + u_1 X^1 + u_2 X^2 : u \in \mathcal{U}\right\}$ is controllable. In fact, $X^3 = \left[X^1, X^2\right]$.*

Example 32 *Let G be the Lie group $T(d)$ of all $d \times d$ upper triangular matrices with positive diagonal entries. $T(d)$ is connected, simply connected and completely solvable Lie group with Lie algebra $\mathfrak{t}(d)$ consisting of all $d \times d$ upper-triangular matrices. By Theorem 30 any transitive invariant system Σ is controllable on G if and only if $\mathfrak{h} = \mathfrak{g}$.*

Remark 33 *We could continue with many relevant theorems on the controllability property for invariant systems $\Sigma = (G, \mathcal{D})$ for different classes of groups. However, as we mentioned, on this subject there exists a review by Sachkov, Y. (Sachkov, 1999). Instead of that, we concentrate on a special kind of controllability problem and a more theoretical result, and applications we send the readers to the references given at the beginning of the chapter.*

Definition 34 *For a system $\Sigma = (G, \mathcal{D})$, a set $A \subset G$ is said to be* Isochronous *if there exist $T > 0$ such that for any two arbitrary elements $x, y \in A$ there exits $\varphi = Z_{t_1} \circ Z_{t_2} \circ ... \circ Z_{t_r} \in S_\Sigma$ with $\Sigma_{i=1}^r t_i = T$ and $\varphi(x) = y$. In this case, T is said to be an isochronal time to A. The system Σ is said to be controllable at uniform time if G is isochronous.*

In (Jurdjevic & Sussmann, 1972-b) the authors prove

Theorem 35 *Let Σ be an invariant control system on a connected, compact and semisimple Lie group G. If Σ satisfy the Lie algebra rank condition, then Σ is controllable at uniform time.*

In (Ayala, Kliemann & Vera, 2011) we give an alternative proof. In fact, we show the existence of time s_+ such that the accessibility set $\mathcal{A}(e, s_+)$ at exact time s_+ coincides with G. Next, we sketch the proof.

Let H be the normal Lie subgroup of G with Lie algebra given by the ideal $\mathfrak{h} = ideal_{\mathfrak{g}} \left\{Y^1, Y^2, ..., Y^m\right\}$. Since the group is compact, for any positive time t, $A(e, t) \subset \exp(tX)H$. Furthermore, related to the topology of this submanifold $intA(e, t_+) \neq \varnothing$. It follows that for every positive time t, the right translation of H by $\exp(tX)$ is contained in a submanifold of co-dimension 0 or 1, (Jurdjevic & Sussmann, 1972-a).

On the other hand, the Lie algebra is semisimple. Thus $\mathfrak{g}$ does not contain ideals of co-dimension one, (San Martin, 1999). So, for any positive time $t, \exp(tX)H = G$. Therefore, we already proved the existence of time s_+ for Σ such that starting from e it is possible to reach any point of G at exact time s_+. Now, we consider the invariant transitive control system $-\Sigma$ on G. Again, $\exists s_- > 0$ such that through $-\Sigma$ it is possible to reach e from any point of the Lie group G at the exact time s_-. Take $s_\Sigma = s_+ + s_-$. Therefore, any two arbitrary points of G can be connected in exactly time s_Σ and Σ is controllable at uniform time. In a more general set up, we obtain

Theorem 36 *Let G a connected semisimple Lie group. Let $\Sigma = (G, \mathcal{D})$ be a right invariant control system. If Σ is controllable, then G can be covered by a sequence of isochronous subsets V_n of G with a nonvoid interior. Also if G is compact G itself is an isochronous set.*

Remark 37 *If Σ is controllable at uniform time it doesn´t imply that you can reach any point of G from e in arbitrary time. But, in some particular cases, the uniform time could be arbitrary. For instance, it happens in the homogeneous case, i.e., when the system does not has drift vector field, i. e., $X = 0$.*

Example 38 *Consider on $T^2 = S^1 \times S^1$ the invariant system $\Sigma = (T^2, \mathcal{D})$ with*

$$\mathcal{D} = \{X^u = X + uY : u \in \mathcal{U}\}.$$

Here, $X = (\frac{\partial}{\partial x}, 0)$ and $Y = (0, \frac{\partial}{\partial y})$. The nonnull component of X and Y are invariant vector fields on the sphere S^1. Since T is commutative, $\mathfrak{h}$ has dimension 1. So, the subgroup H has co-dimension 1. Thus, for any $t > 0$ the accessible set $\mathcal{E}(t, e)$ at exact time t is contained in the one-dimensional submanifold $\exp(tX)H$ of T^2. Therefore, we cannot expect uniform controllability.

Example 39 *From the general theory of Lie groups we know that any Abelian Lie group G has the form $G = T^d \times \mathbb{R}^n$ for some nonnegative integers d, n. Then, for an invariant system Σ on an Abelian Lie group, we can not expect the uniform time property.*

Remark 40 *The proof of Theorem 35 strongly depends on the existence of a state in the interior of an accessible set of Σ and $-\Sigma$ simultaneously. In particular, the proof doesn´t show that you can reach any point of G from e in arbitrary time, see an example in our paper (Ayala et al. 2011). But, the uniform time could be arbitrary. For instance, when the Lie algebra $\mathfrak{h} = Span_{\mathcal{LA}}\left\{Y^1, Y^2, ..., Y^m\right\}$ coincides with $\mathfrak{g}$. In fact, in this particular case, $A(t, e) = G$, for every $t > 0$, see (Kunita, 1978).*

Next, we use a relationship between $\Sigma_{Inv}(G)$ with $\Sigma_{Bil}(\mathbb{R}^d)$ to apply the previous results for bilinear systems.

Example 41 *Let us consider a bilinear control system*

$$\Sigma_{Bil}(\mathbb{R}^d) : \dot{x}(t) = (A + \sum_{j=1}^{m} u_j(t) B_j)\ x(t)$$

determined by the matrices A and $B_j \in \mathfrak{gl}(d, \mathbb{R})$, $j = 0, 1, ..., m$, and $u \in \mathcal{U}$. Let G be the connected Lie group with Lie subalgebra

$$\mathfrak{g} = Span_{\mathcal{LA}}\{A, B_1, ..., B_m\} \subset \mathfrak{gl}(d, \mathbb{R}).$$

Therefore, the control system

$$\dot{X}(t) = (A + \sum_{j=1}^{m} u_j(t)B_j)X(t),\ X(t) \in G,\ u \in \mathcal{U}$$

induced by the bilinear system is invariant on G. Assume that G is compact and semisimple. Thus, the affine system

$$\dot{x}(t) = (A + \sum_{j=1}^{m} u_j(t)B_j)\ x(t),\ x(0) = x_0$$

defined on the orbit $G(x_0) = \left\{x \in \mathbb{R}^d : x = gx_o \text{ with } g \in G\right\}$ is controllable at uniform time on $G(x_0)$.

Example 42 *We consider the bilinear control system on* $\mathbb{R}^d$

$$\dot{x}(t) = (A + u(t)B)x(t),\ u \in \mathcal{U}$$

where A_0 *and* A_1 *generate the Lie algebra* $\mathfrak{g} = Span_{\mathcal{LA}}\{A, B\} = \mathfrak{so}(d, \mathbb{R})$ *of the skew-symmetric matrices of order d. The associated compact Lie group is* $G = SO(d, \mathbb{R})$*. Recall that for each natural number d the group* $SO(d, \mathbb{R})$ *is simple, except the semisimple case* $SO(4, \mathbb{R})$*. It turns out that the affine system is controllable at uniform time on the manifold* $M = S^{d-1}$*.*

7 Linear control systems on Lie groups

A generalization of the notion of a linear system from the Euclidean space $\mathbb{R}^d$ to a specific matrix group was given by (Markus, 1980). After that, the authors in (Ayala & Tirao, 1999) comes for a definitive definition of the subject on an arbitrary connected Lie group G. The authors involve the concept of normalizer of a Lie algebra $\mathfrak{g}$ (considered as the set of right invariant vector fields), in the Lie algebra $\mathfrak{X}^\infty(G)$ of all smooth vector fields on G, as follows

$$\mathfrak{n} = norm_{\mathfrak{X}^\infty(G)}(\mathfrak{g}) = \{X : [X, Y] \in \mathfrak{g} \text{ for every } Y \in \mathfrak{g}\}.$$

After characterize $\mathfrak{n} = \mathfrak{g} \otimes_s \mathfrak{aut}(G)$ as the semi-product between $\mathfrak{g}$ and the Lie algebra $\mathfrak{aut}(G)$ of the Lie group of all automorphisms $Aut(G)$ of G, the authors introduce the following notion

Definition 43 *A linear control system* $\Sigma_{Lin}(G)$ *on* G *is determined by the data*

$$\dot{x}(t) = \mathcal{X}(x(t)) + \sum_{j=1}^{m} u_j(t)Y^j(x(t)),\ x(t) \in G,\ u \in \mathcal{U}$$

where, $\mathcal{X}$ *is a linear vector field, by definition* $\mathcal{X} \in \mathfrak{n}$ *with* $\mathcal{X}(e) = 0$*. Furthermore,* $X^j \in \mathfrak{g}$ *and* $\mathcal{U} = L^1_{loc}(\mathbb{R}, \Omega \subset \mathbb{R}^m)$ *with* Ω *is a closed and convex subset* Ω *of* $\mathbb{R}^m$ *with* $0 \in int(\Omega)$*.*

Remark 44 *Why this class of control is relevant? First, it contains the following categories of systems* Σ *on a Lie group* G *with Lie algebra* $\mathfrak{g}$

$$\dot{x}(t) = X(x(t)) + \sum_{j=1}^{m} u_j(t)Y^j(x(t)) + Y,\ x(t) \in G,\ u \in \mathcal{U}$$

1. Linear system on Euclidean spaces,

$$G = \mathbb{R}^d, X = A \in \mathfrak{aut}(\mathbb{R}^d) = gl(d, \mathbb{R}), Y^j = b^j \in \mathfrak{g} = \mathbb{R}^d, Y = 0 \in \mathbb{R}^d$$

2. Affine systems on Euclidean spaces,

$$G = \mathbb{R}^d, A \in \mathfrak{aut}(\mathbb{R}^d), Y^j = b^j \in \mathfrak{g} = \mathbb{R}^d \text{ and } Y \in \mathbb{R}^d$$

3. Bilinear systems on Euclidean spaces $G = \mathbb{R}^d$ and $A, B^j \in \mathfrak{aut}(\mathbb{R}^d)$

4. Invariant systems on Lie groups, $X,\ X^j \in \mathfrak{g}$

And, by definition includes the linear systems on G. Just observe that the drift and the control vectors of any system in the list are elements of the normalizer. For instance,

$$\text{in } (2) : A = 0 + A \in \mathfrak{g} \otimes_s \mathfrak{aut}(\mathbb{R}^d) \text{ and in } (4)\ X = X + 0 \in \mathfrak{g} \otimes_s \mathfrak{aut}(G).$$

Finally, we would like to introduce the notion of a normalizer system $\Sigma_{\mathfrak{n}}$

$$\dot{x}(t) = X(x(t)) + \sum_{j=1}^{m} u_j(t)Y^j(x(t)) + Y,\ x(t) \in G,\ X, Y^j, Y \in \mathfrak{n} \text{ and } u \in \mathcal{U}.$$

Here, any dynamic of $\Sigma_{\mathfrak{n}}$ is an element of the normalizer. In particular, all the categories mentioned before are special cases. This control system is a very general algebraic system and, it is a challenge even to start to work on it. On the other side, in (Jouan, 2010) the author shows that linear control systems on Lie groups are also necessary for a different reason. He proved an equivalence theorem as follows

Theorem 45 *Let* Σ *be a transitive affine control system*

$$\dot{x}(t) = X(x(t)) + \sum_{j=1}^{m} u_j(t)Y^j(x(t)),\ u \in \mathcal{U}$$

on a manifold M*. Then,* Σ *is diffeomorphic to a linear system on a Lie group or a homogeneous space if and only if the vector fields are completed and generate a finite dimensional Lie algebra.*

This result allows applying the theory of linear systems on concrete applications. In fact, giving a system which satisfies the Jouan´s condition, it is possible to obtain information of the system on a manifold by looking at the linear one in the group, or in some homogeneous space of the group.

The main aim of this section is to show some basic properties of $\Sigma_{Lin}(G)$. We prove some results and we illustrated through examples the fundamental theorems of the theory, (Ayala & San Martin, 2001), (Jouan, 2011), (Ayala & Da Silva, 2016-b). See also more recent results relatives to the restricted case and the existence and uniqueness of control sets, (Ayala & Da Silva, 2016-a).

As we explained, for invariant control systems Jurdjevic and Sussmann proved that the positive orbit of e of G is a semigroup. Hence, for a connected Lie group controllability on G is equivalent to the local controllability from e. For linear systems the situation is totally different. In fact, if Σ satisfies the ad-rank condition then it is locally controllable at e, (Ayala & Tirao, 1999). However, the positive orbit of the identity is not a semigroup. We start by showing a characterization of the drift vector field $\mathcal{X}$, see (Jouan, 2011).

Theorem 46 *Let $\mathcal{X}$ be a vector field on a connected Lie group G. Are equivalent*

1. *$\mathcal{X}$ is a linear vector field*
2. *$\mathcal{X}$ is an infinitesimal automorphism*
3. $\mathcal{X}(gh) = (dL_g)_h\mathcal{X}(h) + (dR_h)_g\mathcal{X}(g)$, *for all* $g, h \in G$.

If we denote by $(\varphi_t)_{t\in\mathbb{R}}$ the flow associated to the drift $\mathcal{X}$, by definition an infinitesimal automorphism is a vector field such that $\{\varphi_t : t \in \mathbb{R}\}$ is a subgroup of $Aut(G)$. Certainly, the vector field $\mathcal{X}$ is complete. Furthermore, one can associate to $\mathcal{X}$ a derivation $\mathcal{D}$ of $\mathfrak{g}$ defined by $\mathcal{D}Y = -[\mathcal{X}, Y](e)$, for all $Y \in \mathfrak{g}$.In fact, the Jacobi identity assures that $\mathcal{D}[X,Y] = [\mathcal{D}X, Y] + [X, \mathcal{D}Y]$.The relation between φ_t and $\mathcal{D}$ is given by the formula, (Warner, 1971) $(d\varphi_t)_e = \mathrm{e}^{t\mathcal{D}}$ for all $t \in \mathbb{R}$, which implies that $\varphi_t(\exp Y) = \exp(\mathrm{e}^{t\mathcal{D}}Y)$, for all $t \in \mathbb{R}, Y \in \mathfrak{g}$. On the other hand, if the group is simply connected any derivation $\mathcal{D} \in \partial\mathfrak{g}$ has an associated linear vector field $\mathcal{X} = \mathcal{X}^D$ through the same formula above. For connected Lie groups, the same is true when $\mathcal{D} \in \mathfrak{aut}(G)$ the Lie algebra of $Aut(G)$ the Lie group of G-automorphism, see (Ayala & Tirao, 1999). As a mater of fact,

$$\mathfrak{aut}(G) \subset \partial\mathfrak{g} \ \text{ and } \mathfrak{aut}(G) = \partial\mathfrak{g} \Leftrightarrow G \text{ is simply connected.}$$

A particular class of such dynamics comes from inner automorphisms. More precisely, consider an element $W \in \mathfrak{g}$. Since W is complete its flow $W_t(z) = \exp_G(tW)z$, $z \in G$ defines by conjugation a 1-parameter group of inner automorphisms on G, $\varphi_t(x) = W_t(e)\,x\,W_{-t}(e)$, $x \in G$. Therefore, $\varphi_t \in \mathrm{Aut}(G)$ for any $t \in \mathbb{R}$. In this case, the associated derivation $D : \mathfrak{g} \to \mathfrak{g}$ defined by $D(Y) = -\mathrm{ad}(\mathcal{X})(Y) = -[W, Y]$ for any Y in $\mathfrak{g}$.

Example 47 *Consider the solvable not complete solvable group $G = E(2)$ of the plane Euclidean motions*

$$G = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & a & b \\ y & -b & a \end{pmatrix} : (x, y) \in \mathbb{R}^2 \text{ and } a^2 + b^2 = 1 \right\}, \text{ with Lie algebra}$$

$$\mathfrak{g} = Span \left\{ Y^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Y^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; Y^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \right\}$$

For $Y^1 \in \mathfrak{g}$ we obtain the linear vector field

$$\begin{aligned} \mathcal{X}(z) &= \left.\frac{d}{dt}\right|_{t=0} \exp(tY^1) z \exp(-tY^1) \\ &= \left.\frac{d}{dt}\right|_{t=0} \begin{pmatrix} 1 & 0 & 0 \\ x + t - at & a & b \\ y + bt & -b & -a \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 - a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}, \; z \in G. \end{aligned}$$

Remark 48 *The family of inner derivations is a subclass which is far from determining all the elements in $\partial\mathfrak{g}$. In fact, there could exist a significant difference of cardinality between derivations and inner derivations. In fact, for $G = \mathbb{R}^d$ any real matrix of order d is a derivation, so $\dim \partial\mathbb{R}^d = d^2$. However, for a semisimple Lie group G, any derivation is inner, which means that $\dim \partial\mathfrak{g} = d$. To show a case between these extremum, we give the following example*

Example 49 *Consider the simply connected Heisenberg Lie group with Lie Algebra $\mathfrak{g}$ generated by*

$$X = \frac{\partial}{\partial x}, \; Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \; Z = \frac{\partial}{\partial z} \text{ with } [X, Y] = Z$$

Any $\mathfrak{g}$-derivation D written in the basis $\{X, Y, Z\}$ has the form

$$D = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & a + d \end{pmatrix}.$$

In this case $\dim(\mathfrak{h}) = 6$ but there are just two independent inner derivations: $ad(X)$ and $ad(Y)$. Furthermore, for a derivation D the associated linear vector field $\mathcal{X} = \mathcal{X}^D$ is explicitly given in (Jouan, 2011) by:

$$\mathcal{X}(g) = (ax + by)\frac{\partial}{\partial x} + (cx + dy)\frac{\partial}{\partial y} + (ex + fy + (a + d)z + \frac{1}{2}cx^2 + \frac{1}{2}by^2)\frac{\partial}{\partial z}.$$

7.1 Local controllability

In this section, we show some controllability results, some of them without proofs. We follow the reference (Ayala & Tirao, 1995) and we start with the natural transitivity property.

Let $\Sigma_{Lin}(G)$ be a linear control system. By the orbit theorem, $\Sigma_{Lin}(G)$ is transitive if and only if LARC is satisfied. i.e.,

$$\dim Span_{\mathcal{LA}}\{Y^j, ad^i(\mathcal{X})(Y^j) \ \mid \ 1 \leq j \leq m \text{ and } 0 \leq i \leq \dim(G)\} = \dim(\mathfrak{g}).$$

Denotes by $\mathfrak{h} = Span\left\{Y^1, ..., Y^m\right\}$, the Lie algebra generated by the control vectors and by $< X \mid \mathfrak{h} >$ the smallest $ad(\mathcal{X})$-invariant Lie subalgebra containing $\mathfrak{h}$. Just observe that the LARC condition is not equivalent to $< X \mid \mathfrak{h} > = \mathfrak{g}$. But, for Abelian groups they agree.

For the classical unrestricted linear system, the algebraic object associated to controllability is a subspace, (in particular a semigroup) defined by the smallest A-invariant subspace of $\mathbb{R}^d$ containing the subspace generated by the columns vectors. And we know that controllability is equivalent to

$$\dim Span\{b^1, ..., b^m, \ Ab^1, ..., Ab^m, \ ..., \ A^{d-1}b^1, ..., A^{d-1}b^m\} = n$$

In fact, $Y^j = b^j$, $ad^1(\mathcal{X})(Y^j) = \left[A, b^j\right] = -Ab^j, ad^2(\mathcal{X})(Y^j) = \left[A, \left[A, b^j\right]\right] = A^2b^j$, etc., for any $b^j \in \mathbb{R}^m, 1 \leq j \leq m$.

As usual, we assume LARC for $\Sigma_{Lin}(G)$. In order to go further, we need the *ad-rank* condition which is determined by the requirement

$$\dim Span\{Y^j, ad^i(\mathcal{X})(Y^j) : 1 \leq j \leq m \text{ and } 0 \leq i \leq n-1\} = \dim(\mathfrak{g}).$$

In order to reach the dimension of G the following brackets are forbidden

$$\left[Y^j, ad^i(\mathcal{X})(Y^j)\right] \text{ with } 1 \leq j \leq m \text{ and } 0 \leq i \leq n-1$$

Theorem 50 *With the ad-rank condition Σ_{Lin} is locally controllable from e.*

Proof**.** We denote by $x(t,u)$, the solution through the identity element

$$\dot{x} \ = \mathcal{X}(x) + \sum_{j=1}^{m} u_j Y^j(x), \ x \in G$$

associated to the control $u \in \mathcal{U}$. For every non-negative t consider the infinitely differential *endpoint map*

$$E_t : u \in \mathcal{U} \rightarrow E_t(u) = x(t,u) \in G.$$

In a neighborhood B of the control $u \equiv 0$ its differential $d(E_t)_0$ at the control zero is given by (Agrachev, Gamkrelidze & Sarychev, 1989)

$$d(E_t)_0\ (u(\cdot)) = \int_0^t e^{(t-s)ad(\mathcal{X})}\ (\sum_{j=1}^m u_j(s)\ Y_e^j)\ ds,$$

Suppose this linear map is not surjective. There exists a co-vector ω in the dual space $T_e^*G \cong \mathfrak{g}^*$ of the tangent space $T_eG \cong \mathfrak{g}$, such that

$$< \omega, d(E_t)_0\ (u(\cdot)) > \ = 0 \text{ for every } u(\cdot) \in B.$$

By the bilinearity property of $< , >$ we get

$$< \omega, \int_0^t e^{(t-s)ad(\mathcal{X})}\ (\sum_{j=1}^m u_j(s)\ Y_e^j)\ ds > \ = \int_0^t \sum_{j=1}^m < \omega,\ e^{(t-s)ad(\mathcal{X})}\ (Y_e^j) >\ u_j(s)\ ds.$$

Since this expression is true for every piecewise constant function $u : [0,t] \to \mathbb{R}^m$ we can conclude that

$$< \omega, e^{(t-s)ad(\mathcal{X})}\ (Y_e^j) > \ = 0,\ \forall s \in [0,t].$$

Differentiating the last expression with respect to t at $t = 0$ we obtain

$$< \omega, ad^i(\mathcal{X})(Y_e^j) > \ = 0 \text{ for each } i \geq 0 \text{ and } j = 1, 2, ..., m.$$

which is in contradiction with the *ad-rank* condition assumption.

Hence, the linear map $d(E_t)_0$ is surjective. By the implicit function theorem, the map $d(E_t)_0$ is locally onto on G. That is, there exists a neighborhood V of the identity element e in G such that $d(E_t)_0$ is onto on $V \subset G$. As a consequence, $\Sigma_{Lin}(G)$ is locally controllable at e.

Next, we show an example of a not locally controllable transitive linear system.

Example 51 *Consider $\Sigma_{Lin}(G)$ on the Heisenberg Lie group with dynamic $\dot{x}(t) = \mathcal{X}(x(t)) + uY^2(x(t))$, $u \in \mathbb{R}$, where the drift $\mathcal{X}$ comes from the derivation $D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \partial\mathfrak{g}$. For $u \in \mathbb{R}$ in coordinates the system reads*

$$\dot{x}_1(t) = x_2(t) + \frac{1}{2}x_3(t) + \frac{1}{2}x_1(t) - \frac{1}{4}x_2(t)x_3(t),\ \dot{x}_2(t) = x_1(t) - \frac{1}{2}x_2(t)x_3(t) + u,\ \dot{x}_3(t) = 0$$

A computation shows that $[\mathcal{X}, Y^1] = Y^2$ and $[\mathcal{X}, Y^2] = Y^1$. So, $ad(\mathcal{X})(\{Y^1\}) = Span\{Y^1, Y^2\} \subsetneq \mathfrak{g}$. While the Lie algebra rank condition is satisfied. i.e., $Span_{\mathcal{LA}}\{Y^1, Y^2\} = Span\{Y^1, Y^2, Y^3\} = \mathfrak{g}$, the system is not locally controllable although it is transitive. Geometrically speaking, no one integral curve of the system can leave the x_1x_2-plane.

In the sequel, we establish some controllability results according to the group and the admissible class of control under consideration.

7.2 Controllability: the unrestricted case

In general, controllability property is a very exceptional issue. We can not expect something different from the class in study. However, like in the classical linear systems on Euclidean spaces, the same Kalman rank condition determines controllability for any unrestricted linear systems on an Abelian or on a compact semisimple Lie group.

Theorem 52 *Let $\Sigma_{Lin}(G)$ be linear control system. Assume*

1. the group G is Abelian, then

$$\Sigma_{Lin}(G) \text{ is controllable} \Leftrightarrow \Sigma_{Lin}(G) \text{ is transitive}$$

2. the Lie algebra $\mathfrak{g}$ of G is semisimple, and G is compact, then

$$\Sigma_{Lin}(G) \text{ is controllable} \Leftrightarrow \Sigma_{Lin}(G) \text{ is transitive}$$

Proof. To prove the first claim, we observe that in this very particular case the positive orbit of the identity $\mathcal{A}(e)$ is a semigroup. On the other hand, since the system is transitive and the group is Abelian, the rank condition and the *ad-rank* condition coincides, which implies local controllability from e on some neighborhood V_e. Since G is connected and $\mathcal{A}(e)$ is a semigroup we obtain

$$V_e \subset \mathcal{A}(e) \Rightarrow G = \cup_{n\in\mathbb{N}}(V_e)^n \subset \mathcal{A}(e)$$

and $\Sigma_{Lin}(G)$ is controllable from e. The same argument is possible to apply to the negative system $\Sigma^-_{Lin}(G)$ which is precisely $\Sigma_{Lin}(G)$ but with the drift $-\mathcal{X}$. Thus, $\Sigma^-_{Lin}(G)$ is also controllable from e. So, for any element $x \in G$, $\exists u$ and a positive time t_u such that x can be achieve from e through $\Sigma^-_{Lin}(G)$ in t_u units of time. Equivalently, x can be transferred to e from $\Sigma_{Lin}(G)$ with the same control and at the same time. Let us consider an arbitrary $y \in G$. There are controls u and v such that y can be reached from x. In fact, after to reach the identity from x we continue with a control v transferring e to y at t_v units of time. So, y can be reached by x at $t_u + t_v$ units of time. Thus, the system is controllable from x for each $x \in G$.

We just show the main ideas of the proof of 2. Since G is compact, we note that the Haar measure is finite. On the other hand, since the group is semisimple any derivation is inner, which implies that the linear vector field has the form $\varphi_t(x) = \exp(tW)\, x \exp(-tW)$, for some $W \in \mathfrak{g}$ and any $x \in G$. In this case, for any constant control the dynamic of the system combines conjugation with invariant, both preserving the Haar measure. Since G is compact, it is closed. So for more arbitrary control, the argument works for the limit. Furthermore, according to a well-known fact, the system is controllable if and only if satisfy the LARC condition. The invariance argument together with the Poincaré recurrence theorem, finish the proof, (Lobry, 1970). ■

Remark 53 *Next we show the first example in the literature of a noncontrollable system which is locally controllable from the identity, (Ayala & San Martin, 2001). Since the group is connected, it turns out that* "for a linear system on groups the reachable set from the identity $A(e)$ cannot be a semigroup".

Example 54 *On* $G = SL(2,\mathbb{R})$ *consider the linear vector field* $\mathcal{X}$ *associated to the derivation* $D = ad(W)$, *where* $W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *Define the linear control system* $\Sigma_{Lin}(SL(2,\mathbb{R}))$ *by* $\dot{x} = \mathcal{X}(x) + uY(x)$, $x \in G$, *where* $Y = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. *Since* $Span\{Y, [\mathcal{X}, Y], [\mathcal{X}, [\mathcal{X}, Y]]\} = \mathfrak{g}$, *the system is locally controllable at* e. *For an argument of reversibility (which is far from the scope of the paper), the authors show that the system can not be controllable.*

As we saw in general, the reachable set is not a semigroup. Actually, under the LARC condition, Jouan prove the following result

Theorem 55 *Let* $\Sigma_{Lin}(G)$ *be a transitive linear control system on* G. *Hence,*

$$\mathcal{A}(e) \text{ is a semigroup } \Leftrightarrow \mathcal{A}(e) = G.$$

In this section we follow the reference (Ayala & Da Silva, 2016-a). The next proposition state the main properties of the reachable sets.

Proposition 56 *It holds:*

1. $0 \leq \tau_1 \leq \tau_2 \Rightarrow \mathcal{A}_{\tau_1} \subset \mathcal{A}_{\tau_2}$
2. $\mathcal{A}_\tau(g) = \mathcal{A}_\tau \varphi_\tau(g)$, *for any* $g \in G$
3. $\tau, \tau' \geq 0 \Rightarrow$

a) $\mathcal{A}_{\tau+\tau'} = \mathcal{A}_\tau \varphi_\tau(\mathcal{A}_{\tau'}) = \mathcal{A}_{\tau'} \varphi_{\tau'}(\mathcal{A}_\tau)$, *and inductively*

b) $\mathcal{A}_{\tau_1} \varphi_{\tau_1}(\mathcal{A}_{\tau_2}) \varphi_{\tau_1+\tau_2}(\mathcal{A}_{\tau_3}) \cdots \varphi_{\sum_{i=1}^{n-1} \tau_i}(\mathcal{A}_{\tau_n}) = \mathcal{A}_{\sum_{i=1}^{n} \tau_i}$, *for any positive real numbers* $\tau_1, \ldots, \tau_n$

4. $u \in \mathcal{U}$, $g \in G$ *and* $t \geq 0 \Rightarrow \phi_{t,u}(\mathcal{A}(g)) \subset \mathcal{A}(g)$
5. $e \in \operatorname{int} \mathcal{A} \Leftrightarrow \mathcal{A}$ *is open.*

The proof of items 1. to 3. can be found in (Jouan, 2011), Proposition 2. The items 4. and 5. in (Da Silva, 2015) Proposition 2.13.

Remark 57 *We notice that the item 4. of the above proposition together with the fact that* $0 \in \operatorname{int} \Omega$ *shows us in particular that* $\mathcal{A}$ *is invariant by the flow* φ_t *in positive time, that is,* $\varphi_t(\mathcal{A}) \subset \mathcal{A}$ *for any* $t \geq 0$.

The positive orbit $\mathcal{A}$ of a linear control system $\Sigma_{Lin}(G)$ is in general, not a semigroup. In this section, we associate to $\Sigma = \Sigma_{Lin}(G)$ a new algebraic object $\mathcal{S}_\Sigma$ which turns out to be a semigroup. In particular, $\mathcal{S}_\Sigma$ enable us to pass from the control theory of linear systems to the semigroup theory. Furthermore, controllability of Σ is equivalent to $\mathcal{S}_\Sigma = G$.

As before, we denote by $(\varphi_t)_{t\in\mathbb{R}}$ the 1-parameter group of automorphisms associated to $\mathcal{X}$. We define in (Ayala & Da Silva, 2016-a)

$$\mathcal{S}_\Sigma = \cap_{t\in\mathbb{R}} \varphi_t(\mathcal{A})$$

Since $\varphi_t(e) = e$ for all $t \in \mathbb{R}$ and $e \in \mathcal{A}$ it follows that $\mathcal{S}_\Sigma \neq \emptyset$.

Proposition 58 *With the previous notations it holds*

1. $\mathcal{S}_\Sigma$ *is the greatest* φ*-invariant subset of* $\mathcal{A}$
2. *For any* $\tau_0 \geq 0$, $\mathcal{S}_\Sigma = \cap_{t \geq 0}\varphi_t(\mathcal{A})$
3. $x \in \mathcal{S}_\Sigma$ *if and only if* $\varphi_t(x) \in \mathcal{A}$ *for all* $t \leq 0$
4. $\mathcal{S}_\Sigma$ *is a semigroup*

Proof.

1. We start by proving the φ-invariance of $\mathcal{S}_\Sigma$. Let $\tau \in \mathbb{R}$, then

$$\varphi_\tau(\mathcal{S}_\Sigma) = \varphi_\tau\left(\cap_{t\in\mathbb{R}}\right) = \cap_{t\in\mathbb{R}}\varphi_\tau(\varphi_t(\mathcal{A})) = \cap_{t\in\mathbb{R}}\varphi_{\tau+t}(\mathcal{A}) = \mathcal{S}_\Sigma$$

Now, let C be a φ-invariant subset of $\mathcal{A}$. It holds that

$$C = \varphi_t(C) \subset \varphi_t(\mathcal{A}), \text{ for all } t \in \mathbb{R} \Leftrightarrow C \subset \cap_{t\in\mathbb{R}} = \mathcal{S}_\Sigma$$

showing that $\mathcal{S}_\Sigma$ is the greatest φ-invariant subset of $\mathcal{A}$.

2. By the φ-invariance of $\mathcal{A}$ in positive time, we get

$$\tau_0 - t \geq 0 \Rightarrow \varphi_{\tau_0 - t}(\mathcal{A}) \subset \mathcal{A} \Leftrightarrow \varphi_{\tau_0}(\mathcal{A}) \subset \varphi_t(\mathcal{A})$$

and consequently $\varphi_{\tau_0}(\mathcal{A}) = \cap_{t \leq t_0}\varphi_t(\mathcal{A})$. Therefore,

$$\mathcal{S}_\Sigma = \cap_{t > t_0}\varphi_t(\mathcal{A}) \cap \cap_{t \leq t_0}\varphi_t(\mathcal{A}) = \cap_{t > t_0}\varphi_t(\mathcal{A}) \cap \varphi_{\tau_0}(\mathcal{A}) = \cap_{t \geq t_0}\varphi_t(\mathcal{A})$$

3. We have

$$x \in \mathcal{S}_\Sigma \Leftrightarrow x \in \varphi_t(\mathcal{A}) \text{ for all } t \geq 0 \Leftrightarrow \varphi_t(x) \in \mathcal{A} \text{ for all } t \leq 0.$$

4. Let $x, y \in \mathcal{S}_\Sigma$ with $x_t = \varphi_{-t}(x)$ and $y_t = \varphi_{-t}(y)$. From 3., we just need to show that

$$x_t y_t = \varphi_{-t}(x)\varphi_{-t}(y) = \varphi_{-t}(xy) \in \mathcal{A} \text{ for any } t \geq 0.$$

By hypothesis $x_t \in \mathcal{A}$, $\exists\ s_t > 0 : x_t \in \mathcal{A}_{s_t}$. But $y \in \mathcal{S}_\Sigma$ so $\varphi_{-s_t}(y_t) = \varphi_{-s_t - t}(y) \in \mathcal{A}$ and $\exists\ s'_t > 0$ with $\varphi_{-s_t}(y_t) \in \mathcal{A}_{s'_t}$ showing that S_Σ is a semigroup. In fact,

$$x_t y_t = x_t \varphi_{s_t}(\varphi_{-s_t}(y_t)) \in \mathcal{A}_{s_t}\varphi_{s_t}(\mathcal{A}_{s'_t}) = \mathcal{A}_{s_t + s'_t} \subset \mathcal{A}.$$

■

Definition 59 *$\mathcal{S}_\Sigma$ is called the semigroup of the system $\Sigma = \Sigma_{Lin}(G)$.*

The controllability property of Σ depends on the semigroup $\mathcal{S}_\Sigma$.

Theorem 60 *$\mathcal{A} = G$ if and only if $\mathcal{S}_\Sigma = \mathcal{A}$.*

Proof. If $\mathcal{A} = G$ then $\varphi_t(\mathcal{A}) = \varphi_t(G) = G$ for all $t \in \mathbb{R}$. Hence, $G = \cap_{t>t_0}\varphi_t(\mathcal{A}) = \mathcal{S}_\Sigma$. Conversely, if $\mathcal{S}_\Sigma = \mathcal{A}$ then $\mathcal{A}$ is a semigroup. However, Proposition 7 of (Jouan, 2011) assures that $\mathcal{A} = G$. ■

Corollary 61 *$\mathcal{A} = G \Leftrightarrow \mathcal{A}$ is φ-invariant*

Proof. If $\mathcal{A} = G$, $\mathcal{A}$ is certainly φ-invariant. Conversely, if $\varphi_t(\mathcal{A}) = \mathcal{A}$ for any $t \in \mathbb{R}$ we get $\mathcal{A} \subset \mathcal{S}_\Sigma$ which implies $\mathcal{S}_\Sigma = \mathcal{A}$. So, $\mathcal{A} = G$. ■

In (Da Silva, 2011) the author prove a general controllability result

Theorem 62 *Assume the reachable set $\mathcal{A}$ is open. Then $G^{+,0} \subset \mathcal{S}_\Sigma$. Moreover,*

$$\Sigma \text{ is controllable} \Leftrightarrow G^- \subset \mathcal{S}_\Sigma.$$

Based in this theorem, Da Silva extends Theorem 8 at follows

Theorem 63 *Let $\Sigma_{Lin}(G)$ a linear control systems on G. Therefore,*

1. If G is solvable

$$e \in int(\mathcal{A}) \text{ and } Spec(A)_{Ly} = \{0\} \Rightarrow \Sigma_{Lin}(G) \text{ is controllable.}$$

2. If G is nilpotent

$$e \in int(\mathcal{A}) \text{ and } Spec(A)_{Ly} = \{0\} \Leftrightarrow \Sigma_{Lin}(G) \text{ is controllable.}$$

To extend the previous theorem, in (Ayala & Da Silva, 2017) the authors introduce the following notion

Definition 64 *Let G be a connected Lie group. The Lie group G has a finite semisimple center if all semisimple Lie subgroups of G have a finite center.*

Several classes of Lie groups satisfy Definition 64. For example, any Abelian, nilpotent and solvable Lie group has the finite semisimple center property. Furthermore, any semisimple Lie group with finite center and any direct or semi-direct product of these classes of groups have a semisimple finite center.

Theorem 65 *Let G be a Lie group with finite semisimple center. Hence,*

$$e \in \operatorname{int} \mathcal{A}_{\tau_0} \text{ and } Spec(A)_{Ly} = \{0\} \Rightarrow \Sigma_{Lin}(G) \text{ is controllable.}$$

7.3 Controllability: the restricted case

In this section, we follow our reference (Ayala & Da Silva, 2016-b). Considering the controllability behavior of an unrestricted linear system $\Sigma_{Lin}(G)$, we now approach the problem more realistically way. We go back to the notion of a control set, a maximal subset $\mathcal{C}$ where approximately controllability holds, i.e.,

$$x \in \text{int}(\mathcal{C}) \Rightarrow \text{int}(\mathcal{C}) \subset \mathcal{A}(x) \text{ and for } x \in \partial\mathcal{C},\ \exists x_n \in \text{int}(\mathcal{C}) : x_n \to x.$$

Like in the classical linear system it is possible to characterize the existence of a control set with nonempty interior around the identity. Of course, many topological properties of $\mathcal{C}$ are intrinsically connected with the eigenvalues of the associated derivation $\mathcal{D}$ to the drift $\mathcal{X}$.

The following decomposition is crucial in the study of controllability of restricted linear systems. It was used for the first time by Da Silva in (Da Silva, 2015). For a derivation $\mathcal{D} : \mathfrak{g} \to \mathfrak{g}$ there exists a special decomposition through its Lyapunov exponents, (San Martin, 1999). For any eigenvalue α of $\mathcal{D}$ there exists the α-generalized eigenspace determined by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : (\mathcal{D} - \alpha)^n X = 0 \text{ for some } n \geq 1\}.$$

It turns out that for if β is also an eigenvalue of $\mathcal{D}$ then

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \text{ when } \alpha + \beta \text{ is an eigenvalue of } \mathcal{D}$$

and zero otherwise. Hence, $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$.

The subspaces $\mathfrak{g}^+$, $\mathfrak{g}^0$ and $\mathfrak{g}^-$ defined by

$$\mathfrak{g}^+ = \bigoplus_{\alpha\,:\,\text{Re}(\alpha)>0} \mathfrak{g}_\alpha, \qquad \mathfrak{g}^0 = \bigoplus_{\alpha\,:\,\text{Re}(\alpha)=0} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{g}^- = \bigoplus_{\alpha\,:\,\text{Re}(\alpha)<0} \mathfrak{g}_\alpha.$$

are Lie algebras. And, $\mathfrak{g}^+$, $\mathfrak{g}^-$ are nilpotent. Denote by G^+, G^0, G^-, $G^{+,0}$ and $G^{-,0}$ the connected Lie subgroups of G with Lie algebras $\mathfrak{g}^+$, $\mathfrak{g}^0$, $\mathfrak{g}^-$, $\mathfrak{g}^{+,0}$ and $\mathfrak{g}^{-,0}$ respectively. By Proposition 2.9 of (Da Silva, 2015), the subsets G^+, G^0, G^-, $G^{+,0}$ and $G^{-,0}$ are $(\varphi_t)_{t\in\mathbb{R}}$-invariant closed subgroups of G.

Theorem 66 *Let $\Sigma_{Lin}(G)$ be a linear system where G is a solvable or $G^0 \subset G$ is compact. Then,*

1. The only control set with nonempty interior is given by $\mathcal{C} = \text{cl}(\mathcal{A}) \cap \mathcal{A}^*$
2. Furthermore,

$$\mathcal{C} \text{ is bounded} \Rightarrow G^0, \text{cl}(\mathcal{A}_{G^-}) \text{ and } \text{cl}(\mathcal{A}^*_{G^+}) \text{ are compact sets, where}$$

$$\mathcal{A}_{G^-} = \mathcal{A} \cap G^- \text{ and } \mathcal{A}^*_{G^+} = \mathcal{A}^* \cap G^+.$$

3. For a nilpotent Lie group G more is true

$$\mathcal{C} \text{ is bounded } \Leftrightarrow \text{cl}(\mathcal{A}_{G^-}),\ \text{and } \text{cl}(\mathcal{A}^*_{G^+}) \text{ are compacts and } \mathcal{D} \text{ is hyperbolic.}$$

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