Robinson’s chaos in set-valued discrete systems

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Abstract

Let \((X, d)\) be a compact metric space and \(f: X \rightarrow X\) a continuous function. If we consider the space \(\mathcal{K}(X), H\) of all non-empty compact subsets of \(X\) endowed with the Hausdorff metric induced by \(d\) and \(\hat{f}: \mathcal{K}(X) \rightarrow \mathcal{K}(X), \hat{f}(A) = \{f(a)/a \in A\}\), then the aim of this work is to show that Robinson’s chaos in \(f\) implies Robinson’s chaos in \(\hat{f}\). Also, we give an example showing that R-chaos in \(f\) does not implies R-chaos in \(\hat{f}\).

1. Introduction

Consider the following discrete dynamical systems:

\[ x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots \]  \hspace{1cm} (1)

where \(f: X \rightarrow X\) is a continuous function and \((X, d)\) is a compact metric space.

According to Devaney [2], if \(f: X \rightarrow X\) is a continuous mapping then \(f\) is chaotic (D-chaotic), if it satisfies the following three properties:

(D1) \(f\) is topologically transitive; that is, for all non-empty open subsets \(U\) and \(V\) of \(X\) there exists a natural number \(k\) such that \(f^k(U) \cap V\) is non-empty.

(D2) \(P(f)\), the set of periodic points of \(f\), is a dense subset of \(X\).

(D3) \(f\) is sensitively dependent (SD) on initial conditions; that is, there is a positive number \(\delta\) (a sensitivity constant) such that for every point \(x \in X\) and each \(\epsilon > 0\) there are \(y \in X\) with \(d(x, y) < \epsilon\) and \(n \in \mathbb{N}\) such that \(d(f^n(x), f^n(y)) \geq \delta\).

It is important to remark that properties (D1) and (D2) imply that \(f\) has sensitive dependence on \(X\) and, consequently, (D3) is redundant in the above definition (see [1]).
It should be remarked that for functions on intervals in \( \mathbb{R} \), it was shown by Vellekoop and Berglund \cite{12} that transitivity implies chaos. Nevertheless, in metric spaces other than \( \mathbb{R} \), transitivity need not imply (D2) or (D3) in the definition of chaos (see \cite[Example 4.1]{9}).

See also \cite{11} where some new criteria of chaos in complete metric spaces are obtained.

On the other hand, some similar definitions of chaos have been explored by many authors, including Knudsen \cite{5} (K-chaos) and Robinson \cite{6} (R-chaos). More precisely

- **K-chaos**: A continuous map \( f : X \rightarrow X \) is called chaotic in the sense of Knudsen ([5]) if and only if \( f \) has a dense orbit in \( X \) and \( f \) exhibits sensitive dependence on initial conditions.

- **R-chaos**: A continuous map \( f : X \rightarrow X \) is called chaotic in the sense of Robinson \cite{6} if it has sensitive dependence and it is topologically transitive.

This last definition of R-chaos does not require periodic density, but as argued by Robinson \cite{6}, this does not seem to be intrinsic to the phenomenon of chaos, so we might leave out.

Also, we remark that K-chaos and R-chaos are equivalent definitions on a compact metric space.

We observe that above definitions describe the complexity of the discrete dynamical system (1) using the behaviors of points under iterations. Actually, the basic goal of the theory of discrete dynamical systems is to understand the nature of all orbits (for instance, in migration phenomena), what carries us to the problem of analyzing the dynamics of set-valued discrete systems.

In this direction, given a discrete system (1) we consider the set-valued discrete system associated to \( f \)

\[
A_{n+1} = \tilde{f}(A_n), \quad n = 0, 1, 2, \ldots \tag{2}
\]

where \( \tilde{f} \) is the natural extension of \( f \) to \( \mathcal{K}(X) \).

We remark that in \cite{9} the author proves that transitivity in (2) implies transitivity in (1) and, moreover, the converse implication is not necessarily true (see \cite{9} and examples there in).

Also, by using the Vietoris topology on \( \mathcal{K}(X) \), Fedeli \cite{3} investigate the chaoticity of some set-valued discrete systems associated to (1) obtaining several interesting results.

The aims of this paper is, on the one hand, to prove that R-chaos in (2) implies R-chaos in (1) (also we construct an example showing that the converse implication is not necessarily true), and on the other, to develop some tools for analyzing the chaoticity of certain set-valued discrete systems via extension of conjugations. Likewise, several illustrative examples are presented.

2. Preliminaries and basic results

Let \( (X,d) \) be a compact metric space and let \( \mathcal{K}(X) \) be the class of all non-empty and compact subset of \( X \). If \( A \in \mathcal{K}(X) \) we define the “\( \varepsilon \)-neighbourhood of \( A \)” as the set

\[
N(A, \varepsilon) = \{ x \in X : d(x, A) < \varepsilon \},
\]

where \( d(x, A) = \inf_{a \in A} d(x, a) \).

The Hausdorff metric on \( \mathcal{K}(X) \) is defined as

\[
H(A, B) = \inf \{ \varepsilon > 0 : A \subseteq N(B, \varepsilon) \text{ and } B \subseteq N(A, \varepsilon) \}
\]

and it is well known that \( (\mathcal{K}(X), H) \) is a compact metric space (see \cite{4, 7, 8}).

If \( A \in \mathcal{K}(X) \) we denote by \( B(A, \varepsilon) \) the ball centered in \( A \) and radius \( \varepsilon \) in \( H \)-metric.

**Remark 1.** An equivalent formula for \( H \) is given by

\[
H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}.
\]

If \( f : X \rightarrow X \) is a continuous function then we define the extension \( \tilde{f} \) of \( f \) to \( K(X) \) as \( \tilde{f}(A) = f(A) = \{ f(a) : a \in A \} \).
Proposition 1. Let \((X,d_1), (Y,d_2)\) be two compact metric spaces and suppose that \(f:X \to Y\) is a function. Then, \(f\) is continuous if and only if \(\bar{f}:\mathcal{H}(X) \to \mathcal{H}(Y)\) is continuous in \(H\)-metric.

Proof. (\(\Rightarrow\)) If we suppose that \(f\) is continuous then, due compactness of \(X, \bar{f}\) is uniformly continuous on \(X\). Thus, given \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[
d_1(x,y) < \delta \Rightarrow d_2(f(x),f(y)) < \varepsilon.
\]

(3)

Now, if \(A, B \in \mathcal{H}(X)\) and \(H(A,B) < \delta\) then \(A \subseteq N(B,\delta)\) and \(B \subseteq N(A,\delta)\).

Therefore

\[
d_1(a,B) < \delta, \quad \forall a \in A \quad \text{and} \quad d_1(b,A) < \delta, \quad \forall b \in B.
\]

So, given \(a \in A\) there exists \(b \in B\) such that \(d_1(a,b) < \delta\) and, by (3), must be \(d_2(f(a),f(b)) < \varepsilon\). Consequently, \(d_2(f(a),f(B)) < \varepsilon, \forall a \in A,\) i.e., \(f(A) \subseteq N(f(B),\varepsilon)\).

By using an analogous argument we have \(f(B) \subseteq N(f(A),\varepsilon)\), concluding that \(H(f(A),f(B)) < \varepsilon\) and, consequently, \(\bar{f}\) is uniformly continuous on \(\mathcal{H}(X)\).

(\(\Leftarrow\)) Conversely, if we suppose that \(\bar{f}\) is continuous then, due compactness of \(\mathcal{H}(X)\), \(\bar{f}\) is uniformly continuous on \(\mathcal{H}(X)\). Thus, given \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[
H(A,B) < \delta \Rightarrow H(\bar{f}(A),\bar{f}(B)) < \varepsilon.
\]

(4)

Now, if \(d_1(x,y) < \delta\) then \(H((x),\{y\}) < \delta\) and, by (4), must be

\[
H(\bar{f}(\{x\}),\bar{f}(\{y\})) = H(\{f(x),\{f(y)\}) = d_2(f(x),f(y)) < \varepsilon
\]

and, consequently, \(f\) is uniformly continuous on \(X\). \(\square\)

Corollary 1. Suppose that \(f:X \to Y\) is a function. Then, \(f\) is continuous if and only if \(\bar{f}\) is continuous in \(H\)-metric.

In connection with the idea of R-chaos, there exits an interesting class of sensitive dependent functions: the class of expansive mappings.

Definition 1. Let \(f:X \to Y\) be a function. We say that \(f\) is expansive if and only if there exists a real constant \(\lambda > 1\) such that

\[
d(f(x),f(y)) \geq \lambda d(x,y) \quad \forall x,y \in X.
\]

In this case we say that \(f\) is \(\lambda\)-expansive.

Proposition 2. Let \(f:X \to X\) be an expansive function. Then \(f\) is sensitively dependent.

Proof. It is sufficient to observe that if \(x \neq y\) then

\[
d((f^n(x),f^n(y)) \geq \lambda^n d(x,y) \to \infty \text{ as } n \to \infty.
\]

Consequently, \(f\) has sensitive dependence. \(\square\)

Proposition 3. Let \(f:X \to X\) be a continuous function. Then \(f\) is \(\lambda\)-expansive if and only if \(\bar{f}\) is \(\lambda\)-expansive.

Proof. (\(\Rightarrow\)) Suppose that \(f\) is \(\lambda\)-expansive and consider \(A, B \in \mathcal{H}(X)\) and \(a \in A\), then

\[
d(f(a),f(B)) = \inf_{b \in B} d(f(a),f(b)) \geq \inf_{b \in B} \lambda d(a,b) = \lambda d(a,B).
\]

Therefore

\[
\sup_{a \in A} d(f(a),f(B)) \geq \lambda \sup_{a \in A} d(a,B)
\]

(5)

Analogously,

\[
\sup_{b \in B} d(f(b),f(A)) \geq \lambda \sup_{b \in B} d(b,A)
\]

(6)
Thus, due Remark 1 and (5) and (6), we have

\[ H(f(A), f(B)) = \max \left\{ \sup_{a \in A} d(f(a), f(B)), \sup_{b \in B} d(f(b), f(A)) \right\} \geq \max \left\{ \lambda \sup_{a \in A} d(a, B), \lambda \sup_{b \in B} d(b, A) \right\} = \lambda H(A, B) \]

and, consequently, \( \tilde{f} \) is \( \lambda \)-expansive.

\( (\leftarrow) \) If \( \tilde{f} \) is \( \lambda \)-expansive and \( x, y \in X \) then

\[ d(f(x), f(y)) = H(f(\{x\}), f(\{y\})) \geq \lambda H(\{x\}, \{y\}) = \lambda d(x, y) \]

which implies that \( f \) is \( \lambda \)-expansive. \( \square \)

**Corollary 2.** Let \( f : X \to X \) be an expansive function. Then \( \tilde{f} \) has sensitive dependence on \( \mathcal{H}(X) \).

### 3. R-chaos in (2) implies R-chaos in (1)

A natural question is the following: chaos in (1) implies chaos in (2) (and conversely)?

As a partial response to this question, we will show that R-chaos in (2) implies R-chaos in (1).

**Theorem 1.** Let \( f : X \to X \) be a continuous function. Then

(i) \( \tilde{f} \) transitive implies \( f \) transitive;
(ii) \( \tilde{f} \) sensitively dependent implies \( f \) sensitively dependent.

**Proof**

(i) Let \( U, V \) non-empty open subsets of \( X \). Thus, we can choose \( x \in X \), \( y \in Y \) and \( \epsilon > 0 \) such that \( B(x, \epsilon) \subset U \) and \( B(y, \epsilon) \subset V \). Now, in \( \mathcal{H}(X) \) consider the open sets \( B(\{x\}, \epsilon) \) and \( B(\{y\}, \epsilon) \) (balls in \( \mathcal{H} \)-metric). Then, due transitivity of \( \tilde{f} \), there exists \( n \in N \) such that \( \tilde{f}^n(B(\{x\}, \epsilon)) \cap B(\{y\}, \epsilon) \neq \emptyset \).

Therefore, there exists \( G \in B(\{x\}, \epsilon) \) such that \( \tilde{f}^n(G) = f^n(G) \in B(\{y\}, \epsilon) \). But then \( G \subset B(x, \epsilon) \) and, analogously, \( f^n(G) \subset B(y, \epsilon) \), which implies that \( f^n(B(x, \epsilon)) \cap B(y, \epsilon) \neq \emptyset \) and, consequently, \( f^n(U) \cap V \neq \emptyset \). That is, \( f \) is a transitive function.

(ii) Because \( \tilde{f} \) has sensitive dependence then there exists a constant \( \delta > 0 \) such that for every \( K \in \mathcal{H}(X) \) and every \( \epsilon > 0 \) there exists \( G \in B(K, \epsilon) \) and \( n \in \mathbb{N} \) such that \( H(f^n(K), f^n(G)) \geq \delta \).

Now, let \( x \in X \) be and \( \epsilon > 0 \). Then, taking \( K = \{x\} \in \mathcal{H}(X) \) we have that there exists \( G \in B(\{x\}, \epsilon) \) and \( n \in \mathbb{N} \) such that

\[ H(f^n(\{x\}), f^n(G)) = H(\{f^n(x)\}, f^n(G)) \geq \delta. \]

Thus, \( H(f^n(\{x\}), f^n(G)) = \sup_{y \in G} d(f^n(x), f^n(y)) \geq \delta \) and, due compactness of \( G \) and continuity of \( f \), there exists \( y_0 \in G \) such that

\[ H(\{f^n(x)\}, f^n(G)) = d(f^n(x), f^n(y_0)) \geq \delta. \]

But, \( G \in B(\{x\}, \epsilon) \) implies \( G \subset B(x, \epsilon) \) and, consequently, \( y_0 \in B(x, \epsilon) \). This proves that \( f \) is sensitively dependent (with sensitivity constant \( \delta \)). \( \square \)

**Corollary 3.** If \( \tilde{f} : \mathcal{H}(X) \to \mathcal{H}(X) \) is R-chaotic, then \( f : X \to X \) is R-chaotic.

**Theorem 2.** If \( f \) has periodic density, then \( \tilde{f} \) has periodic density.

**Proof.** Let \( K \in \mathcal{H}(X) \) and \( \epsilon > 0 \). Then there exists a \( \epsilon/2 \)-net covering \( K \), That is to say, there are \( x_1, \ldots, x_p \) in \( K \) such that \( K \subset B(x_1, \epsilon/2) \cup \cdots \cup B(x_p, \epsilon/2) \) Because \( f \) has periodic density, there are \( y_i \in X \) and \( n_i \in N \) such that

\[ y_i \in B(x_i, \epsilon/2), \quad \forall i = 1, \ldots, p \quad \text{and} \quad f^n(y_i) = y_i, \forall i = 1, \ldots, p. \]
Now, taking \( G = \{ y_1, \ldots, y_p \} \) then by construction we have \( H(K, G) < \epsilon \) and, moreover, \( f^{\lfloor n \mu_{y_p} \rfloor}(y_i) = y_i \), for all \( i = 1, \ldots, p \). Therefore,
\[
f^{\lfloor n \mu_{y_p} \rfloor}(G) = G
\]
which implies that \( \tilde{f} \) has periodic density. \( \square \)

**Remark 2.** The above Theorem 2 was presented by the authors in the 2nd International Conference on Soft methods in Probability and Statistics (SMPS 2004) in Oviedo, Spain, September 2004 (see [10, Theorem 3]). Recently, via an interesting and totally different technique, Fedeli ([3, Theorem 3.1]) obtain this result by using the Vietoris topology on \( \mathcal{K}(X) \).

**Corollary 4** (see also [3]). If \( f \) has periodic density and \( \tilde{f} \) is topologically transitive then \( \tilde{f} \) is \( R \)-chaotic.

**Proof.** It was shown by Banks et al. [1] that properties (D1) and (D2) imply property (D3). Therefore, if \( f \) has periodic density and \( \tilde{f} \) is topologically transitive then, due to Theorem 2, \( \tilde{f} \) has periodic density and, consequently, \( \tilde{f} \) is \( R \)-chaotic on \( \mathcal{K}(X) \). \( \square \)

Also, by using Propositions 2 and 3, we obtain

**Corollary 5.** If \( f : X \rightarrow X \) is an expansive function and \( \tilde{f} \) is transitive, then \( \tilde{f} \) is \( R \)-chaotic.

To finalize this section, and in the converse direction of Theorem 2, we will give sufficient conditions on \( \tilde{f} \) for the periodic density of \( f \). For this, we recall the Schauder’s fixed-point theorem on normed space.

Let \( C \) be a non-empty compact convex subset of a normed space, and let \( f : C \rightarrow C \) be continuous. Then \( f \) has a fixed point.

**Theorem 3.** Let \( X \) be a non-empty compact convex subset of a normed space \( E \) and let \( f : X \rightarrow X \) be a continuous function. If we suppose that \( f \) has periodic density on \( \mathcal{K}(X) \), then \( f \) has periodic density on \( X \).

**Proof.** If \( x_0 \in X \) and \( \epsilon > 0 \) then \( \{ x_0 \} \in \mathcal{K}(X) \) and, consequently, there exists \( K \in \mathcal{H}(X) \) and \( n \in \mathbb{N} \) such that
\[
\begin{align*}
(a) & \quad H(\{x_0\}, K) < \epsilon; \\
(b) & \quad f^n(K) = K.
\end{align*}
\]
Thus, combining (a) and (b) we have
\[
d(x_0, f^n(x)) < \epsilon \quad \text{for all } x \in K. \tag{7}
\]
Now, because
\[
\tilde{f}^n(K) = f^n(K) = K
\]
and \( f^n \) is continuous on \( K \) then, due Schauder’s fixed-point theorem, there exists \( x_p \in K \) such that \( f^n(x_p) = x_p \). Thus, \( x_p \) is a periodic point of \( f \) and, due (7), we obtain \( d(x_0, x_p) < \epsilon \). Consequently, \( f \) has periodic density on \( X \). \( \square \)

4. A counter example

To describe our main example in this section showing that, in general, the converse of Corollary 3 is not necessarily true, we shall use two previous examples given by Devaney [2]: the tent map on \([0,1]\) and an irrational rotation on the circle \( S^1 \).

**Example 1** (The tent map). Let \( X = [0,1] \) be and consider the function
\[
f(x) = \begin{cases} 
2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\
2(1-x) & \text{if } \frac{1}{2} \leq x \leq 1.
\end{cases}
\]
It is well known that \( f \) is \( D \)-chaotic on \([0,1]\) (see [2]). Also, in [9, Example 4.3], we prove that \( \tilde{f} \) is transitive.
Now, we will prove that $\tilde{f}$ has sensitive dependence and, to see this, we observe that a straightforward calculus shows that $f^n$ consist of $2^n$ linear segments defined on the intervals $[p/2^n, (p + 1)/2^n]$, $p = 0, 1, \ldots, 2^n - 1$, whose length is $1/2^n$ and its extremal points form a dense subset of $X = [0, 1]$ as $n \to \infty$ (see Fig. 1).

Furthermore, $\bigcup_{p=0}^{2^n-1} [p/2^n, (p + 1)/2^n] = [0, 1]$ and $f^n$ has a fixed point on every one of these intervals and, consequently, the periodic points of $f$ form a dense subset of $X$. On the other hand, given $K \in \mathcal{H}(X)$ and $\epsilon > 0$, there are $x_1, \ldots, x_q \in K$ such that $K \subset B(x_1, \epsilon/2) \cup \cdots \cup B(x_q, \epsilon/2)$. Let $n \in \mathbb{N}$ be such that

$$1/2^n < \min \left\{ \epsilon/2, \min_{j \neq i} |x_i - x_j| \right\}$$

then, due (8), for each $i = 1, \ldots, q$, we have that $x_i \in [p/2^n, (p + 1)/2^n] = J_i$ for some unique $p_i \in \{0, 1, \ldots, 2^n - 1\}$ and $J_i \neq J_k$ for $i \neq k$. Thus, choosing a fixed point $y_i \in J_i$ of $f^n$ and taking $G = \{y_1, \ldots, y_q\}$, we have that $H(K, G) < \epsilon$ and $f^n(G) = G$ and, consequently, $P(f)$ (the set of periodic points of $f$) is a dense subset of $\mathcal{H}(X)$. So, due transitivity and periodic density of $f$, we obtain that $\tilde{f}$ is sensitively dependent (see [1]). Consequently, $\tilde{f}$ is $D$-chaotic on $\mathcal{H}(X)$. In particular, $\tilde{f}$ is $R$-chaotic on $\mathcal{H}(X)$.

**Remark 3.** In relation to the above Example 1, by using Corollary 4 we can obtain a more simple proof of the $R$-chaoticity of $\tilde{f}$. In fact, because $f$ is periodically dense [2] and $\tilde{f}$ is transitive [9], then $\tilde{f}$ is $R$-chaotic.

**Example 2** (Rotations of the circle). On the circle $S^1$, we consider $T$ the rotation of angle $1$ rad, that is, $T: S^1 \to S^1$ defined by $T(e^{i\theta}) = e^{i(\theta + 1)}$. Then, it was shown by Devaney [2] that each orbit $\{T^n(e^{i\theta})/n \in \mathbb{N}\}$ is dense in $S^1$ and, consequently, $T$ is transitive. Nevertheless, $T$ has no periodic points and, because $T$ is isometric, it does not exhibit sensitive dependence on initial conditions either. Also, in [9] the author proves that $T$ is not transitive on $\mathcal{H}(S^1)$ and, because $T$ has no sensitive dependence, $T$ does not exhibit sensitive dependence either.

The following example shows that, in general, $R$-chaos in (1) does not implies $R$-chaos in (2).

**Example 3** (The rotation tent on a cylinder). Let $C = [0, 1] \times S^1$ equipped with the usual metric $p((x, e^{i\theta}),(y, e^{i\phi})) = \max\{|x - y|, |e^{ix} - e^{i\phi}|\}$ and let $h$ be the function defined on it by $h(x, e^{i\phi}) = (f(x), T(e^{i\phi}))$, where $f$ and $T$ are the functions considered in above examples.

First, we shall see that $h$ is a transitive function. In fact, let $U = I \times J$ and $V = I' \times J'$ be two open and non-empty subsets of $C$. Then, there is an integer $n$ such that $f^n(I) = [0, 1]$.

On the other hand, it is easy to see that there are $e^{i\lambda}$ in $J$ and $m > n$ such that $T^m(e^{i\lambda}) \in J'$. As $f^n(I) = [0, 1]$, there is a $x \in I$ such that $f^m(x) \in I'$. Thus, $(x, e^{i\lambda}) \in I \times J = U$ and $h^m(x, e^{i\lambda}) \in I' \times J' = V$. Therefore, $h$ is topologically transitive. But it is not $D$-chaotic as it has no periodic point as well as $T$. Nevertheless, it exhibits sensitive dependence on initial conditions because $f$ does. Consequently, $h$ is $R$-chaotic on $C$.

Now, we shall see that $h$ is not transitive on $\mathcal{H}(C)$. We observe that if $K \in \mathcal{H}(S^1)$ and $\operatorname{diam}(K) \geq 1$, then $[0, 1] \times K \in \mathcal{H}(C)$ and $\operatorname{diam}(h([0, 1] \times K)) = \operatorname{diam}(K)$. In fact, due compactness of $K$ there are $k_1$ and $k_2$ in $K$ such that $\operatorname{diam}(K) = \|k_1 - k_2\| \geq 1$ and, therefore

$$\operatorname{diam}(h([0, 1] \times K)) = \max_{x, x' \in [0, 1], k, k' \in K} \{|f(x) - f(x'), |T(k) - T(k')|\} = \max_{k, k' \in K} \|k - k'\| = \|k_1 - k_2\| = \operatorname{diam}(K).$$

Consequently, $\operatorname{diam}(h([0, 1] \times K)) = \operatorname{diam}(K)$ for every $K \in \mathcal{H}(S^1)$ such that $\operatorname{diam}(K) \geq 1$. Now, let $K \in \mathcal{H}(S^1)$ be with $\operatorname{diam}(K) = 2$ and let $\epsilon > 0$ be such that $2 - \epsilon > 1 + \epsilon$ and consider the open balls $U = B([0, 1] \times K, \epsilon/2)$ and $V = B([0, 1] \times \{1\}, \epsilon/2)$ in $\mathcal{H}(C)$. Then, it is clear that:

$$F \in U = B([0, 1] \times G, \epsilon/2) \Rightarrow \operatorname{diam}(F) \geq 2 - \epsilon$$
whereas

\[ G \in V = B([0, 1] \times \{1\}, \epsilon/2) \Rightarrow \text{diam}(G) \leq 1 - \epsilon. \]

Thus, \( \text{diam}(\hat{h}^n(F)) \geq 2 - \epsilon > 1 - \epsilon, \forall n \in \mathbb{N} \) and, consequently, \( \hat{h}^n(U) \cap V = \emptyset \) for all \( n \in \mathbb{N} \), which implies that \( \hat{h} \) is not transitive on \( \mathcal{X}(C) \). Therefore, \( \hat{h} \) is not R-chaotic.

**Remark 4.** We wish to remark the following strange situation which say us that, sometimes, whereas the total extension to \( \mathcal{X}(X) \) is R-chaotic we can have non-R-chaotic partial extensions. For instance, if we consider the tent function \( f \) on \( X = [0, 1] \) then

(i) According to above Example 1, \( \tilde{f} \) is R-chaotic on \( \mathcal{X}(X) \);

(ii) as it was shown in [9], if we consider the restriction \( \tilde{f} : \mathcal{X}_c(X) \to \mathcal{X}_c(X) \), then \( \tilde{f} \) is not transitive.

Consequently, \( \tilde{f} \) is not R-chaotic on \( \mathcal{X}_c(X) \).

**Remark 5.** The Example 3 also provides a function \( h \) showing that, in general, R-chaos does not implies D-chaos.

### 5. Extension of conjugations and D-chaos

The concept of conjugation is a very useful tool in chaos analysis. Actually, mappings which are topologically conjugate are completely equivalent in terms of their dynamics and, consequently, it is possible to know the dynamics of a function \( p \) throughout of another topologically conjugate function \( q \). In this direction, the aim of this section is to show that conjugation can be extended to the set-valued context in a natural way.

The following diagram

\[ X \xrightarrow{f} X \\
Y \xrightarrow{g} Y \]

is commutative if \( h \circ f = g \circ h \).

**Definition 2.** Let \( f : X \to X \) and \( g : Y \to Y \) be two maps. Then, \( f \) and \( g \) are said to be topologically conjugate if there exists a homeomorphism \( h : X \to Y \) such that \( h \circ f = g \circ h \). The homeomorphism \( h \) is called a topological conjugation and we write \( f \sim g \).

Also, \( h : X \to Y \) is called a semiconjugation (or \( g \) is a factor of \( f \)) if (9) is commutative and \( h \) is a continuous, surjective but non-injective function.

In connection with chaos, the next proposition contain the principal results associated to topologically conjugate functions, and these may be summarized as follows (see [1,2]).

**Proposition 4.** If we consider the diagram (9) then

(a) The equality \( h \circ f = g \circ h \) implies \( h \circ f^n = g^n \circ h \), for all \( n \in \mathbb{N} \).

(b) If \( f \sim g \), then \( f \) is transitive on \( X \) if and only if \( g \) is transitive on \( Y \).

(c) If \( f \sim g \), then \( f \) has periodic density on \( X \) if and only if \( g \) has periodic density on \( Y \). Consequently, if \( f \sim g \), then \( f \) is chaotic on \( X \) if and only if \( g \) is chaotic \( Y \).

(d) In (9), suppose that \( X, Y \) are compact metric spaces and \( f \) has sensitive dependence on \( X \). Then \( f \sim g \) implies that \( g \) has sensitive dependence on \( Y \).

(e) If \( g \) is a factor of \( f \) and \( f \) is chaotic on \( X \), then \( g \) is chaotic \( Y \).

This results can be extended to the set-valued context. More precisely
Theorem 4. Consider the diagram (9) and suppose that \( f \sim g \). Then the following diagram:

\[
\begin{array}{ccc}
\mathcal{K}(X) & \xrightarrow{f} & \mathcal{K}(X) \\
\downarrow \hbar & & \downarrow \hbar \\
\mathcal{K}(Y) & \xrightarrow{g} & \mathcal{K}(Y)
\end{array}
\]

is also commutative and \( \tilde{f} \sim \tilde{g} \).

Also, if \( g \) is a factor of \( f \) then \( \tilde{g} \) is a factor of \( \tilde{f} \).

Proof

(i) Because \( h \circ f = g \circ h \), it is obvious that \( \hbar \circ \tilde{f} = \tilde{g} \circ \hbar \).

(ii) If \( h \) is a homeomorphism then \( \hbar \) is a homeomorphism. In fact, we know that \( h \) is a bijective and bicontinuous functions (i.e., \( h \) and \( h^{-1} \) are continuous). Thus, if \( B \in \mathcal{K}(Y) \) we can take \( A = h^{-1}(B) \) and it is clear that \( A \in \mathcal{K}(X) \) and \( \hbar(A) = B \), which proves that \( \hbar \) is a surjective function.

On the other hand, if \( A, B \in \mathcal{K}(X) \) with \( A \neq B \) then, without loss of generality, we can suppose that there exists \( a \in A \setminus B \). Now, we suppose \( \hbar(A) = \hbar(B) \) then

\[ h(a) \in h(A) = h(B) \Rightarrow h(a) \in h(B) \]

which implies that there exists \( b \in B \) such that \( h(a) = h(b) \) and, consequently \( a = b \), contradicting our hypothesis.

This proves that \( \hbar \) is an injective function.

Finally, due bicontinuity of \( h \), by using Proposition 1 we obtain that \( \hbar \) is also a bicontinuous function and, consequently, \( \hbar \) is a homeomorphism.

(iii) From (i) and (ii) we conclude that \( f \sim g \) implies \( \tilde{f} \sim \tilde{g} \).

(iv) If we suppose that \( g \) is a factor of \( f \), then \( \tilde{h} : X \to Y \) is a continuous function and, due Proposition 1, we have \( h : \mathcal{K}(X) \to \mathcal{K}(Y) \) is also continuous. On the other hand, if \( B \in K(Y) \) we can define \( A = h^{-1}(B) = \{ x \in X | h(x) \in B \} \) and, due surjectivity of \( h \), it is clear that \( A \neq \emptyset \) and \( h(A) = B \). Now, we claim of \( A \in \mathcal{K}(X) \). In fact, if \( (x_p) \) is a sequence in \( A \) such that \( x_p \to x_0 \) as \( p \to \infty \) then \( (h(x_p)) \) is a sequence in \( B \) and, by continuity of \( h \) and compactness of \( B \), we obtain \( h(x_p) \to h(x_0) \) and \( h(x_0) \in B \). Thus, \( x_0 \in A \) and \( A \) is a closed subset of \( X \) and, due compactness of \( X \), we conclude that \( A \in \mathcal{K}(X) \) proving that \( h \) is a surjective function.

Finally, suppose that \( \hbar \) is an injective function and let \( a, b \in X \) with \( a \neq b \). Then

\[ \hbar(\{a\}) = \{h(a)\} \neq \{h(b)\} = \hbar(\{b\}) \]

implying the injectivity of \( h \), which contradicts our hypothesis. So, if \( g \) is a factor of \( f \) then \( \tilde{g} \) is a factor of \( \tilde{f} \). In this form, the proof is now complete. \( \square \)

The following examples show some applications of extension of conjugations.

Example 4. The “tent” function \( f(x) = 1 - 2|x - \frac{1}{2}| \), \( x \in [0, 1] \) (see above Example 1 and Fig. 1), is topologically conjugate to the “logistic” map \( g(x) = 4x(1 - x) \) on \([0, 1]\). Moreover, the homeomorphism \( h : [0, 1] \to [0, 1] \) defined by \( h(x) = \frac{\sin^2(\pi x)}{\pi^2} \) is a topological conjugation between \( f \) and \( g \), that is to say, \( h \circ f = g \circ h \). Graphically, the following diagram is commutative:

\[
\begin{array}{ccc}
[0, 1] & \xrightarrow{f} & [0, 1] \\
\downarrow h & & \downarrow h \\
[0, 1] & \xrightarrow{g} & [0, 1]
\end{array}
\]

Thus, according with Theorem 4, we have \( \tilde{f} \sim \tilde{g} \). Now, because \( \tilde{f} \) is D-chaotic on \( \mathcal{K}([0, 1]) \) (see Example 4.3 in [9]), then \( \tilde{g} \) is D-chaotic on \( \mathcal{K}([0, 1]) \).

Example 5. Consider the map \( p(x) = 2|x| - 2 \) on \([-2, 2]\) (see Fig. 2, where some iterated of \( p \) are showed). Then, it was shown by Devaney in [2] that \( p \) is D-chaotic on \([-2, 2]\).

Now, by using a similar reasoning that in Example 4.3 in [9] and above Example 1 (also see [9, Remark 4.4]), we can prove that \( p \) is D-chaotic on \( \mathcal{K}([-2, 2]) \). On the other hand, if \( q(x) = x^2 - 2 \), \( x \in [-2, 2] \), then \( p \sim q \) on \([-2, 2]\). In fact, taking \( h : [-2, 2] \to [-2, 2] \) defined by \( h(x) = -2\cos(\pi x/\pi) \), then \( h \) is a homeomorphism and a straightforward calculus shows that
is a commutative diagram. Thus, by Theorem 4, we conclude that \( p \sim q \) and, consequently, \( q \) is also D-chaotic on \( \mathcal{H}([-2, 2]) \).

Example 6. In this example we show that the “doubling” map \( D: S^1 \to S^1 \), defined by \( D(e^{i\theta}) = e^{2i\theta} \), is chaotic on \( S^1 \). For this, it is sufficient to prove that \( D \) is a factor of the baker-map

\[
B(x) = \begin{cases} 
2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\
2x - 1 & \text{if } \frac{1}{2} < x \leq 1.
\end{cases}
\]

which is chaotic on \([0, 1]\) (see [2]). In fact, taking \( h: [0, 1] \to S^1 \) defined by \( h(x) = e^{2\pi i x} \), we have the following diagram

\[
\begin{array}{ccc}
[0, 1] & \xrightarrow{h} & [0, 1] \\
\downarrow & & \downarrow h \\
S^1 & \xrightarrow{D} & S^1
\end{array}
\]

is commutative. Therefore, \( D \) is factor of \( B \).

Because \( B \) is a “doubling-period” like functions, Remark 4.4 in [9] say us that \( \overline{B} \) is transitive on \( \mathcal{H}([0, 1]) \). Also, we observe that, in general, the graph of \( B^n \) consists of \( 2^n \) pieces, each of which is a straight line defined on an interval \( J \) of length \( 1/2^n \) and, observing the iterated of \( B \) (see Fig. 3), we have that \( B^n \) has a fixed point in \( J \), we can conclude that periodic points are dense in \([0, 1]\).

Thus, by Theorem 2, \( \overline{B} \) has periodic density in \( \mathcal{H}([0, 1]) \). Consequently, \( \overline{B} \) is D-chaotic on \( \mathcal{H}([0, 1]) \).

Now, because \( D \) is factor of \( B \) and \( \overline{B} \) is D-chaotic on \( \mathcal{H}([0, 1]) \) then, by Theorem 4, we have that \( \overline{D} \) is also D-chaotic on \( \mathcal{H}([0, 1]) \).

Remark 6. It is interesting to observe that in above Example 6 we have used the exponential map \( h: x \to e^{2\pi i x} \) to connect the (behavior) dynamics of a discontinuous map (\( B \)) on the unit interval with a continuous map (\( D \)) on the circle. Actually, this is a special case of something quite general:

If \( T: X \to X \) is transitive (not necessarily continuous) then so is every factor of \( T \).
6. Conclusions

Let \( f \) be a continuous function defined on a compact metric space \( X \) and \( \tilde{f} \) its natural extension to \( \mathcal{K}(X) \). Then, the fundamental question here is: R-chaos in \( f \) implies R-chaos in \( \tilde{f} \)? (and conversely?).

In this context, and as a partial response to this question, Theorem 1 shows that: a) \( \tilde{f} \) transitive implies \( f \) transitive, and b) \( \tilde{f} \) sensitively dependent implies \( f \) sensitively dependent and, consequently, R-chaos in \( \tilde{f} \) implies R-chaos in \( f \).

However, in general, the converse implication is not true. In fact, Example 3 shows a R-chaotic function \( h \) (the rotation tent on a cylinder) which has a non-R-chaotic extension \( \tilde{h} \).

A curious situation is observed in Remark 4, where is showed that a partial extension of \( f \) (for instance, to \( \mathcal{K}_c(X) \)) can be not R-chaotic, whereas the total extension of \( f \) to \( \mathcal{K}(X) \) is R-chaotic.

Also, in relation to periodic density of \( f \) and \( \tilde{f} \), two interesting result are presented: in Theorem 2 we prove that \( \tilde{f} \) inherits the periodic density of \( f \) (under compactness condition) and, conversely, in Theorem 3 we give sufficient conditions on \( \tilde{f} \) for the periodic density of \( f \).

Likewise, as a contribution to study of set-valued discrete systems, we develop some tools for analyzing the chaoticity of certain extended set-valued discrete systems via extension of conjugations.

Finally is interesting to remark that, in connection with the above theoretical results, an interesting applied open question can be explored: individual chaos implies collective chaos?, and conversely?

References