A Jensen type inequality for fuzzy integrals

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Abstract

In this paper, we show a Jensen type inequality for the Sugeno integral. We also discuss some conditions assuring the satisfaction of opposite inequality (reverse Jensen inequality).

Keywords: Fuzzy measure; Fuzzy integral; Jensen’s inequality

1. Introduction

The theory of fuzzy measures and fuzzy integrals was introduced by Sugeno [18] as a tool for modeling non-deterministic problems.

The properties and applications of the Sugeno integral have been studied by many authors, including Ralescu and Adams [8], Román-Flores et al. [12,13] and Liu et al. [5]. For an overview on fuzzy measure and fuzzy integration theory, the reader is referred to Wang and Klir [20].

In connection with the topics that will be discussed in this article, there are several other theoretical and applied papers related to fuzzy measure theory on metric spaces, such as Li et al. [2,3], Narukawa et al. [6] and Song [17].

For an overview of general theory of non-additive set functions and its applications, the book Null-additive set functions by Pap [7] is an excellent resource. For some applications of fuzzy measures, we refer the reader to [4,9,16,19].

Not long ago, authors in [10] analyzed an interesting type of geometric inequalities for the Sugeno integral with some applications to convex geometry. More precisely, a Prékopa–Leindler type inequality for fuzzy integrals was proven, and subsequently used for the characterization of some convexity properties of fuzzy measures.

The purpose of this paper is to study a Jensen type inequality for the Sugeno integral. In this context, the research has two main aims. One is a Jensen type inequality for fuzzy integrals. The other one provides sufficient conditions assuring the reverse Jensen inequality.

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The paper is organized as follows. Some necessary preliminaries are presented in Section 2. We address the essential problems in Sections 3 and 4. Concluding remarks are in Section 5.

2. Preliminaries

2.1. Fuzzy measures and Sugeno integral

In the sequel, we present some definitions and basic properties of the Sugeno integral that will be used in the next sections.

**Definition 1.** Let $\Sigma$ be a $\sigma$-algebra of subsets of $X$ and let $\mu : \Sigma \to [0, \infty]$ be a nonnegative, extended real-valued set function. We say that $\mu$ is a fuzzy measure if:

- (a) $\mu(\emptyset) = 0$;
- (b) $E, F \in \Sigma$ and $E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity);
- (c) $\{E_p\} \subseteq \Sigma, E_1 \leq E_2 \leq \ldots,$ imply $\lim_{p \to \infty} \mu(E_p) = \mu(\bigcup_{p=1}^{\infty} E_p)$ (lower continuity);
- (d) $\{E_p\} \subseteq \Sigma, E_1 \geq E_2 \geq \ldots,$ $\mu(E_1) < \infty$, imply $\lim_{p \to \infty} \mu(E_p) = \mu(\bigcap_{p=1}^{\infty} E_p)$ (upper continuity).

For an overview on fuzzy measures and integrals and their applications refer to [20].

If $f$ is a nonnegative real-valued function defined on $X$, we will denote by $L_0 f = \{x \in X / f(x) \geq x\} = \{f \geq x\}$ the $x$-level of $f$, for $x > 0$, and $L_0 = \{x \in X / f(x) > 0\} = \text{supp}(f)$ is the support of $f$.

Obviously, $f(x) = 0$ for all $x \notin L_0 f$.

Also, we note that

$$x \leq y \Rightarrow \{f \geq y\} \subseteq \{f \geq x\}. \tag{1}$$

If $\mu$ is a fuzzy measure on $X$, we define the following

$$\mathcal{F}^\mu(X) = \{f : X \to [0, \infty] / f \text{ is } \mu \text{ measurable}\}.$$

**Definition 2.** Let $\mu$ be a fuzzy measure on $(X, \Sigma)$. If $f \in \mathcal{F}^\mu(X)$ and $A \in \Sigma$, then the Sugeno integral (or fuzzy integral) of $f$ on $A$, with respect to the fuzzy measure $\mu$, is defined (see [18,20]) by:

$$\int_A f \, d\mu = \bigvee_{x \geq 0} [x \land \mu(A \cap \{f \geq x\})], \quad A \in \Sigma, \tag{2}$$

where $\bigvee, \bigwedge$ denotes the operations sup and inf on $[0, \infty]$, respectively. In particular, if $A = X$ then

$$\int_X f \, d\mu = \int f \, d\mu = \bigvee_{x \geq 0} [x \land \mu\{f \geq x\}]$$

The following properties of the Sugeno integral are well known and can be found in [20]:

**Proposition 1.** If $\mu$ is a fuzzy measure on $X$ and $f, g \in \mathcal{F}^\mu(X)$, then

- (i) $\int f \, d\mu \leq \mu(A)$;
- (ii) If $\chi_A$ is the characteristic function of $A$, then $\int f \chi_A \, d\mu = \int f \, d\mu$;
- (iii) $\int f \, d\mu > x \iff$ there exists $\gamma > x$ such that $\mu(A \cap \{f \geq \gamma\}) > x$;
- (iv) If $\mu(A) < \infty$, then: $\int f \, d\mu > x \iff \mu(A \cap \{f \geq x\}) > x$.

**Remark 1.** From a numerical point of view, if the distribution function $F$ associated to $f, F(x) = \mu(\{f \geq x\})$ is continuous, then the fuzzy integral can be calculated solving the equation $F(x) = z$.

For further extensions of the Sugeno integral refer to [1]. In the sequel, we will assume that $\mu(X) < \infty$. 


2.2. Hausdorff metric and level-continuity

Let \((X,d)\) be a metric space and let \(\mathcal{K}(X)\) be the class of all non-empty and compact subsets of \(X\). If \(A \in \mathcal{K}(X)\) we define the “\(\varepsilon\)-neighbourhood of \(A\)” as the set:

\[
N(A, \varepsilon) = \{ x \in X \mid d(x, A) < \varepsilon \}
\]

where \(d(x, A) = \inf_{a \in A} d(x, a)\).

The Hausdorff metric on \(\mathcal{K}(X)\) induced by \(d\) is defined by

\[
H(A, B) = \inf \{ \varepsilon > 0 \mid A \subseteq N(B, \varepsilon) \text{ and } B \subseteq N(A, \varepsilon) \}\]

It is well known that \((\mathcal{K}(X), H)\) is a complete (separable, compact) metric space iff \(X\) is a complete (separable, compact) metric space (see [9,11]).

**Remark 2.** If \(A_p \xrightarrow{H} A\) then (see [9]) a topological characterization of \(A\) is the following:

\[
A = \bigcap_{p=1}^{\infty} \left( \bigcup_{q \geq p} A_q \right).
\]

Consider the space \(\mathcal{F}_k(X)\) of all real functions \(f : X \to [0, \infty)\) verifying:

(a) \(\text{supp}(f)\) is compact,
(b) \(f\) is upper semicontinuous.

If \(f \in \mathcal{F}_k(X)\), then it is clear that \(L_\alpha f\) is compact for all \(\alpha \geq 0\).

**Definition 3.** If \(f \in \mathcal{F}_k(X)\) then we say that \(f\) is level-continuous if and only if \(L_{\alpha_0} f \xrightarrow{H} L_{\alpha_0} f\) for all \(\alpha, \alpha_0\) such that \(\alpha \to \alpha_0\) and \(L_{\alpha_0} f, L_{\alpha_0} f \in \mathcal{K}(X)\).

We emphasize that strictly monotone real functions are an important class of level-continuous functions. Furthermore, there is an interesting connection between the level-continuity of functions and the existence of proper local maximum points. In fact, Román and Rojas showed in [14] the following geometrical and topological characterizations of level-continuity of functions:

**Proposition 2** [14]. Let \(f \in \mathcal{F}_k(X)\) and suppose that \(\sup_{x \in X} f(x) = M\). Then the following three properties are equivalent:

(i) \(f\) has no proper local maximum points
(ii) \(\{ f \geq x \} = \{ f > x \}, \forall x \in (0, M)\).
(iii) \(f\) is level-continuous.

Furthermore, the concept of \(H\)-continuity of fuzzy measures was discussed by the authors in [9] and subsequently used in the study of continuity of set defuzzification processes on \(\mathbb{R}^n\) defined by set-valued integration. Actually, this concept can be extended to any arbitrary metric space.

**Definition 4.** Let \(X\) be a metric space. If \(\mu\) is a Borelian fuzzy measure defined on \(X\), then we say that \(\mu\) is \(H\)-continuous iff \(K_p \xrightarrow{H} K\) in \(\mathcal{K}(X)\) implies \(\lim_{p \to \infty} \mu(K_p) = \mu(K)\).

The role of \(H\)-continuous fuzzy measures and level-continuous functions will be essential in Section 4 for obtaining a reverse Jensen type inequality for fuzzy integrals.

3. A Jensen type inequality for fuzzy integrals

The classical Jensen’s inequality (see [15]) is the following mathematical property of convex functions:

\[
\Phi \left( \int f \, d\mu \right) \leq \int \Phi(f) \, d\mu,
\]

where \(f\) is \(\mu\)-measurable and \(\Phi : [0, \infty) \to [0, \infty)\) is a convex function.
Remark 3. We know that inequality (4) is deeply connected with the vectorial nature of the concepts of the Lebesgue integral and the convexity of functions. However, the Sugeno integral is defined by using the reticular structure of \( \mathbb{R}^+ \) and, consequently, we might not expect the validity of (4), under the same conditions, in the fuzzy context.

In fact, the following example shows that the inequality (4) is not valid for the Sugeno integral under the same conditions discussed above.

Example 1. Let \( f : \mathbb{R}^+ \to [0, \infty) \) be defined by \( f(x) = \sqrt{x} \chi_{[0,6]}(x) \) and consider the convex function \( \Phi(x) = x^2 \). Then a straightforward calculus shows that:

\[
\int f \, dx = \bigvee_{0 \leq x \leq 6} [(x) \wedge (6 - x^2)] = 2
\]

and

\[
\int \Phi(f(x)) \, dx = \bigvee_{0 \leq x \leq 6} [(x) \wedge (6 - x)] = 3
\]

and, because \( \Phi(2) = 4 \), we conclude that the inequality (4) is not verified by the Sugeno integral.

In order to obtain a Jensen type inequality for the Sugeno integral, it is clear that the classical conditions must be changed. In this direction, and in connection with the order structure in \( \mathbb{R}^+ \), we have replaced the convexity condition of \( \Phi \) in (4) by a monotonic condition, obtaining the following new Jensen type inequality in the context of fuzzy sets.

Theorem 1. Let \( (X, \Sigma, \mu) \) be a fuzzy measure space and let \( f \in F^+(X) \) be such that \( \int f \, d\mu = p \). If \( \Phi : [0, \infty) \to [0, \infty) \) is a strictly increasing function such that \( \Phi(x) \leq x \), for every \( x \in [0, p] \), then:

\[
\Phi \left( \int f \, d\mu \right) \leq \int \Phi(f) \, d\mu.
\] (5)

Proof. Firstly, by Proposition 1 (iv) we have:

\[
\int f \, d\mu \geq p \Rightarrow \mu\{f \geq p\} \geq p.
\] (6)

On the other hand, because \( \Phi \) is a strictly increasing function, then \( \Phi^{-1} \) is also a strictly increasing function. Thus, it is not difficult to see that:

\[
\{f \geq p\} = \{\Phi(f) \geq \Phi(p)\}.
\] (7)

Therefore, from hypothesis, (6) and (7) we obtain:

\[
\mu\{\Phi(f) \geq \Phi(p)\} = \mu\{f \geq p\} \geq p \geq \Phi(p)
\] (8)

and, consequently, from (8) and Proposition 1 (iv) we conclude:

\[
\int \Phi(f) \, d\mu \geq \Phi(p) = \Phi \left( \int f \, d\mu \right),
\]

which completes the proof. □

Example 2. Let \( \mu \) be the Lebesgue measure on \( \mathbb{R} \) and consider the function \( f(x) = (1 - x) \chi_{[0,1]}(x) \). If we define:

\[
\Phi(x) = \begin{cases} 
\frac{x}{2} & \text{if } 0 \leq x < \frac{1}{2} \\
\frac{2x+1}{4} & \text{if } x \geq \frac{1}{2}
\end{cases}
\]

then:

\[
\int f \, d\mu = \bigvee_{0 \leq x \leq 1} [(x) \wedge (1 - x)] = \frac{1}{2}
\]
and, consequently,
\[
\Phi\left( \int f \, d\mu \right) = \Phi\left( \frac{1}{2} \right) = \frac{1}{2}.
\] (9)

On the other hand, a straightforward calculus shows that:
\[
\Phi(f)(x) = \begin{cases} 
\frac{1-x}{4} & \text{if } 0 \leq x \leq \frac{1}{2} \\
\frac{1-x}{2} & \text{if } \frac{1}{2} \leq x \leq 1 \\
0 & \text{if } x \geq 1
\end{cases}
\]

and, consequently,
\[
\mu\left( \left\{ \Phi(f) \geq \frac{1}{2} \right\} \right) = \mu\left( \left[ 0, \frac{1}{2} \right] \right) = \frac{1}{2}.
\] (10)

Thus, by (9) and (10) and Proposition 1 (iv) we have
\[
\int \Phi(f) \, d\mu \geq \frac{1}{2} = \Phi\left( \int f \, d\mu \right)
\] (11)

which implies that inequality (5) holds.

As a direct consequence of above Theorem 1 and Proposition 1 (i) we have

**Corollary 1.** Let \((X, \Sigma, \mu)\) be a fuzzy measure space and let \(\Phi : [0, \infty) \rightarrow [0, \infty)\) be a strictly increasing function such that \(\Phi(x) \leq x\), for every \(x \in [0, \mu(X)]\), then:
\[
\Phi\left( \int f \, d\mu \right) \leq \int \Phi(f) \, d\mu,
\]

for all \(f \in \mathcal{F}^\mu(X)\).

The following examples show some applications of the fuzzy Jensen’s inequality (5) when using Corollary 1.

**Example 3.** Let \((X, \Sigma, \mu)\) be a fuzzy probability space (that is to say, \(\mu(X) = 1\)) and consider \(f \in \mathcal{F}^\mu(X)\). Then, taking \(\Phi(x) = x^\lambda\) with \(\lambda \geq 1\), we have \(\Phi(x) = x^\lambda \leq x\) for all \(x \in [0, 1]\). Thus, due to Corollary 1 we obtain:
\[
\left( \int f \, d\mu \right)^\lambda \leq \int f^\lambda \, d\mu
\]

for all \(\lambda \geq 1\) and \(f \in \mathcal{F}^\mu(X)\).

**Example 4.** Let \((X, \Sigma, \mu)\) be a fuzzy measure space and consider \(f \in \mathcal{F}^\mu(X)\). Then, taking \(\Phi(x) = \lambda x\) with \(0 \leq \lambda \leq 1\), we have \(\Phi(x) = \lambda x \leq x\) for all \(x \in [0, \infty)\) and, consequently,
\[
\lambda \int f \, d\mu \leq \int \lambda f \, d\mu
\]

for all \(0 \leq \lambda \leq 1\) and \(f \in \mathcal{F}^\mu(X)\).

To finalize this section, we will show that conditions “\(\Phi\) strictly increasing” and “\(\Phi(x) \leq x\) for every \(x \in [0, \int f \, d\mu]\)” on the function \(\Phi\) in Theorem 1 above cannot be avoided, as we will illustrate in the following examples.

**Example 5.** We observe that “\(\Phi\) strictly increasing” condition of function \(\Phi\) in Theorem 1 above cannot be avoided. Let \(\mu\) be the Lebesgue measure on \(\mathbb{R}\) and consider \(f(x) = x\chi_{[0,1]}(x)\). If we define:
\[
\Phi(x) = \begin{cases} 
x & \text{if } 0 \leq x < \frac{1}{2} \\
(1-x)\chi_{[0,1]}(x) & \text{if } x \geq \frac{1}{2}
\end{cases}
\]
then:

\[
\int f \, d\mu = \sqrt{z \wedge (1 - z)} = \frac{1}{2}
\]

and, consequently,

\[
\phi \left( \int f \, d\mu \right) = \frac{1}{2}.
\]

In addition, because \( \Phi(f)(x) = \Phi(x)_{\Delta[0,1]}(x) \), then:

\[
\mu \left( \left\{ \Phi(f) \geq \frac{1}{2} \right\} \right) = \mu \left( \left\{ \frac{1}{2} \right\} \right) = 0.
\]

But, from Proposition 1 iv) we have:

\[
\int \Phi(f) \, d\mu \geq \frac{1}{2} \Rightarrow \mu \left( \left\{ \Phi(f) \geq \frac{1}{2} \right\} \right) \geq \frac{1}{2}
\]

in contradiction with (13), which implies that

\[
\int \Phi(f) \, d\mu < \frac{1}{2}
\]

and, consequently, inequality (5) is not verified.

**Example 6.** The condition “\( \Phi(x) \leq x \) for all \( 0 \leq x \leq \int f \, d\mu \)” in Theorem 1 above cannot be avoided. In fact, consider \( f(x) = x_{\Delta[0,1]}(x) \) and define \( \Phi(x) = \sqrt{x} \). Then:

\[
\int f \, d\mu = \sqrt{z \wedge \left( \frac{1}{2} - z \right)} = \frac{1}{4}
\]

and, consequently,

\[
\phi \left( \int f \, d\mu \right) = \frac{1}{2}
\]

Furthermore, because \( \Phi(f)(x) = \sqrt{x}_{\Delta[0,1]}(x) \), then:

\[
\int \Phi(f) \, d\mu = \sqrt{z \wedge \left( \frac{1}{2} - x^2 \right)} = -\frac{1 + \sqrt{3}}{2} < \frac{1}{2} = \phi \left( \int f \, d\mu \right)
\]

which implies that inequality (5) is not verified.

4. \( H \)-continuity and reverse Jensen type inequality

The aim of this section is to show some connections between \( H \)-continuity of fuzzy measures and the Jensen’s inequality. More precisely, we aim to find conditions assuring the satisfaction of the opposite inequality in (5) (reverse Jensen’s inequality). As we will see, a possible approximation to solving this problem can be done via \( H \)-continuity of fuzzy measures and level-continuity of functions. The concept of \( H \)-continuity of fuzzy measures was exhaustively studied by the authors in [9], and one of the most important results in [9] is Theorem 2 [8, pp. 235] where we have given sufficient conditions assuring the \( H \)-continuity of fuzzy measures defined on \( \mathbb{R}^n \). Summarizing, in this section we extend some results in [9] to an arbitrary compact metric space which will be subsequently used to solve the reverse problem.
First of all, we need the following previous result.

**Lemma 1.** Let $K_p, K \in \mathcal{K}(X)$ such that $K_p \xrightarrow{H} K$. If $\mu$ is an upper continuous Borel fuzzy measure on $X$, then

\[ \limsup_{p \to \infty} \mu(K_p) \leq \mu(K). \]

**Proof.** It follows directly from Theorem 1 in [9].

The next theorem gives sufficient conditions assuring the $H$-continuity of a Borelian fuzzy measure on a compact metric space.

**Theorem 2.** Let $X$ be a compact metric space and let $\mu$ be a Borel fuzzy measure on $X$. If we suppose that

(i) $\mu$ is upper continuous on $\mathcal{K}(X)$;
(ii) for each $K \in \mathcal{K}(X)$ and $\epsilon > 0$

\[ \mu\left( \overline{N(K, \epsilon)} \right) \leq \mu(K) + p(K, \epsilon), \]  

where $p : \mathcal{K}(X) \times [0, \infty) \to [0, \infty)$ is a nonnegative and monotone (in both $\epsilon$ and $K$) function such that $p(K, \epsilon) \to 0$ as $\epsilon \downarrow 0$. Then $\mu$ is $H$-continuous on $\mathcal{K}(X)$.

**Proof.** Let $K_p, K \in \mathcal{K}(X)$ such that $K_p \xrightarrow{H} K$. Then, due to **Lemma 1** and hypothesis (i), we have

\[ \limsup_{p \to \infty} \mu(K_p) \leq \mu(K). \]

Thus, it only remains to show that $\mu(K) \leq \liminf_{p \to \infty} \mu(K_p)$.

In fact, if $\epsilon > 0$ is given, then there exists $p_0 \in \mathbb{N}$ such that $H(K, K_p) < \epsilon$ for all $p \geq p_0$ and, by (3), we have

\[ \begin{cases} K & \subset \overline{N(K_p, \epsilon)} \subset \overline{N(K, \epsilon)}, & \forall p \geq p_0 \\ K_p & \subset \overline{N(K, \epsilon)} \subset \overline{N(K, \epsilon)}, & \forall p \geq p_0. \end{cases} \]

On the other hand, if $K^* = \bigcup_{q=1}^{p_0+1} K_q \bigcup \overline{N(K, \epsilon)}$, then $K^* \in \mathcal{K}(X)$ and, by (16), $K^*$ contain $K$ and $K_p$ for all $p$. So, from (16) and hypothesis (ii), we obtain:

\[ \mu(K) \leq \mu\left( \overline{N(K, \epsilon)} \right) \leq \mu(K_p) + p(K_p, \epsilon) \leq \mu(K_p) + p(K^*, \epsilon) \]

for all $p \geq p_0$.

Consequently,

\[ \mu(K) \leq \liminf_{p \to \infty} \mu(K_p) + p(K^*, \epsilon) \]

and, finally, from hypothesis (ii), taking limit $\epsilon \downarrow 0$, we conclude that $\mu(K) \leq \liminf_{p \to \infty} \mu(K_p)$, and the proof is complete. \(\square\)

**Example 7.** Let $X$ be a nonempty and compact subset of $\mathbb{R}^n$ and consider the monotone set function:

\[ \mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \|A\| & \text{if } A \neq \emptyset \end{cases} \]

for every $A \subseteq X$. Then $\mu$ is a fuzzy measure which verifies conditions of **Theorem 2** on $\mathcal{K}(X)$. In fact:

(i) It is not difficult to check that $\mu$ is an upper continuous fuzzy measure on $\mathcal{K}(X)$ (see Example 5 in [8, pp. 237]).

(ii) In addition, if $K \in \mathcal{K}(X)$ we have:

\[ \mu\left( \overline{N(K, \epsilon)} \right) = \|N(K, \epsilon)\| = \|x_0\| \]

for some $x_0 \in \overline{N(K, \epsilon)}$. Thus, there exists a sequence $(x_q) \subset N(K, \epsilon)$ such that $x_q \to x_0$ and, for each $q$, there exists $a_q \in K$ such that $\|x_q - a_q\| \leq \epsilon$, which implies that
\[ \|x_q\| \leq \|a_q\| + \varepsilon \leq \|K\| + \varepsilon \]

for every \( q \) and, consequently, \( \|x_0\| \leq \|K\| + \varepsilon \). So, in this case we can take \( p(K, \varepsilon) = \varepsilon \) verifying condition (ii) in Theorem 2 and, consequently, \( \mu \) is \( H \)-continuous on \( \mathcal{K}(\mathbb{R}) \).

Example 8. The Lebesgue measure \( \mu \) is not \( H \)-continuous on \( \mathcal{K}(\mathbb{R}^n) \), nevertheless \( \mu \) is \( H \)-continuous on \( \mathcal{K}_c(\mathbb{R}^n) \), the class of all nonempty compact and convex subsets of \( \mathbb{R}^n \) (for details see [9]).

In the following theorem sufficient conditions are given to assure the continuity of the distribution function \( F \) associated to a function \( f \):

**Theorem 3.** If \( f \in \mathcal{F}_k(X) \) is a level-continuous function and \( \mu \) is a \( H \)-continuous fuzzy measure on \( \mathcal{K}(X) \), then:

\[ F(x) = \mu(L_x f) \]

is a continuous function.

**Proof.** It is sufficient to observe that, by hypothesis, \( F \) is a composition of continuous functions. \( \square \)

**Remark 4.** In connection with Remark 1 and the calculus of fuzzy integrals, in the same conditions of Theorem 3 above, we have \( \int f \, d\mu = x_0 \), where \( x_0 \) is a solution of \( F(x) = x \).

The next theorem is the main result of this section, and it presents sufficient conditions to assure the opposite inequality in (5).

**Theorem 4** (Reverse fuzzy Jensen’s inequality). Let \( \mu \) be a \( H \)-continuous fuzzy measure on \( \mathcal{K}(X) \) and let \( f \in \mathcal{F}_p(X) \) a level-continuous function with \( \int f \, d\mu = p \). If \( \Phi: [0, \infty) \rightarrow [0, \infty) \) is a strictly increasing function such that \( \Phi(x) \geq x \), for every \( x \in [0, p] \), then:

\[ \Phi\left( \int f \, d\mu \right) \geq \int \Phi(f) \, d\mu. \]  

(17)

**Proof.** If we suppose that \( \int \Phi(f) \, d\mu > \Phi(p) \) then, due to Proposition 1 (iii), there exist \( \gamma > \Phi(p) \) such that

\[ \mu(\{f \geq \gamma\}) > \Phi(p), \]

(18)

which implies by (1) that

\[ \mu(\{f \geq \gamma\}) > \Phi(p). \]

But then, because \( \Phi \) is a strictly increasing function, we obtain:

\[ \mu(\{f \geq \gamma\}) = \mu(\{f \geq p\}) > \Phi(p) \geq p. \]  

(19)

Furthermore, considering the distribution function \( F \) associated to \( f \) then, from hypothesis and Theorem 3, \( F \) is a continuous function. Thus, because \( F(p) = \mu(\{f \geq p\}) > p \), there exists a neighbourhood \( V(p, \varepsilon) \) of \( p \) such that \( F(x) > p \) for every \( x \in V(p, \varepsilon) \). In particular, there exists \( \delta > p \) such that

\[ F(\delta) = \mu(\{f \geq \delta\}) > p \]  

(20)

and, consequently, by Proposition 1 (iii) we conclude

\[ \int f \, d\mu > p \]

This completes the proof. \( \square \)

**Example 9.** Let \( \mu \) be the Lebesgue measure on \( X = [0, 1] \). Then we know that \( \mu \) is \( H \)-continuous on \( \mathcal{K}_c(\mathbb{R}) \) (see Example 7). Additionally, suppose that \( f \in \mathcal{F}_p(X) \) is a level-continuous function with compact and convex level sets. Then, by Proposition 1 (i) we have \( \int f \, d\mu \leq 1 \). Thus:
(a) Taking $\Phi(x) = x^\lambda$ with $0 \leq \lambda \leq 1$, then $\Phi$ is a continuous and strictly increasing function such that $\Phi(x) = x^\lambda \geq x$ for all $x \in [0, 1]$ and, consequently, due to the above Theorem 4 we obtain:

$$\left( \int f \, d\mu \right)^\lambda \geq \int f^\lambda \, d\mu$$

for all $0 \leq \lambda \leq 1$.

(b) Taking $\Phi(x) = e^x$, then $\Phi(x) = e^x \geq x$ for all $x \in [0, 1]$ and, by using the same above arguments, we obtain:

$$e^{\int f \, d\mu} \geq \int e^f \, d\mu$$

5. Concluding remarks

The classical Jensen’s inequality is a useful result in several theoretical and applied fields. It provides a fundamental tool for understanding and predicting consequences of variance in any dynamical system (i.e., in terms of the classical expectation). However, the expectation associated to a non-deterministic phenomena is naturally the Sugeno integral (see [18,20]).

In this paper we have presented a Jensen’s type inequality for the Sugeno integral which is obtained by replacing the convexity requirement of classical Jensen’s inequality with new type of monotonic and order requirements.

In addition, by using $H$-continuity of fuzzy measures and level-continuity of functions, we have provided the necessary conditions for the reverse Jensen inequality.

Finally, some illustrative examples have been presented.

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References