Uniform convergence and transitivity

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Abstract

Let \((X, d)\) be a metric space and \(f_n : X \to X\) a sequence of continuous functions such that \((f_n)\) converges uniformly to a function \(f\). If \(f_n\) is transitive for all \(n \in \mathbb{N}\), then the purpose of this work is, on the one hand, to show that \(f\) is not necessarily transitive and, on the other, to give sufficient conditions for the transitivity of the limit function \(f\).

1. Introduction

A dynamical system may be defined [8] as a deterministic mathematical model for evolving the state of a system forward in time (the time here can be a continuous or discrete variable), and which can be represented by a set of functions (rules, equations) that specify how variables change over time. It is well known that a numerous class of real problems are modeled by a discrete dynamical system

\[ x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots, \] (1)

where \(X\) is a metric space and \(f : X \to X\) is a continuous function which represents the dynamics of the system (i.e., it describes the evolution of the system forward in time).

The basic goal in the study of the dynamical system (1) is to understand the nature of all orbits \(x, f(x), f^2(x), \ldots, f^n(x)\) as \(n\) becomes large and, in many cases, these orbits presents a chaotic structure, that is to say, the destiny of the orbits cannot be predictable.

Nevertheless, in most situations the dynamics of the system is unknown, but the empirical observation (experimental data) show us chaotic symptoms (for example, unpredictability, very small differences in starting values result in very different behavior). In this case, if we wish to predict the state of the system in future time we must approximate the dynamics of the system by an adequate function, in accordance with the available experimental data. The prediction obtained will be, obviously, an approximated prediction.

In this context, in order to model complex systems the two essential questions (and their respective answers) here are:

(Q1) When approximate the dynamics of a system by a chaotic function?
(A1) This approximation must be done whenever it presents chaotic symptoms.
(Q2) Why approximate the dynamics of a complex system by a chaotic function?
(A2) The reason is very simple: through it we can model unpredictable dynamical systems in a more real manner.

In order to exemplify the previous considerations, we will use some well-known population growth models. For instance, we know that the more simple discrete dynamical system for modeling the population growth is the Malthus’s growth model

\[ x_{n+1} = rx_n, \quad r > 0, \]  

where \( x_n \) is the size of the population at time \( n \) and \( r \) is the rate of growth of the population from one generation to another. If the initial condition \( x(0) = x_0 \), then by simple iteration we find that \( x_n = r^nx_0 \) is the solution of the system. In this case, the dynamics of the system is very simple, and it is described by the linear function \( f(x) = rx \) which is a non-chaotic function (finite-dimensional linear functions are non chaotic). If \( r > 1 \), then \( x_n \) increases indefinitely. If \( r = 1 \), then the size of the population is constant for the indefinite future. However, for \( r < 1 \), \( \lim_{n\to\infty}x_n = 0 \) and the population becomes eventually extinct.

Nevertheless, neither case is typically observed in genuine biological communities. What is often observed instead is that small populations often (though not always) increase in number while very large populations tend to decline in number. A good population growth model must therefore reproduce this behavior. Moreover, in the real world populations of fish, insects, people, etc., usually do not grow in a regular fashion. They grow and decline, grow some more, decline again, and so forth. So they are not modeled well by nice models like the Malthus’s model.

One of the chief factors controlling real-world populations is the availability of resources: environment for living, food, water, enough choices for mating, etc. Many of these factors are captured by the (simplified) Logistic growth model

\[ x_{n+1} = r x_n (1 - x_n), \quad 0 < r \leq 4, \]  

where \( x_{n+1} \) is population on day \( n + 1 \) (as a percentage of total capacity); \( x_n \) is population on day \( n \) (as a percentage of total capacity), \( r \) is a positive number (between 0 and 4) depending on the population, and which represents a combined rate for reproduction and starvation (of course, different populations have different \( r \)'s).

The Logistic model prevents unlimited growth by inhibiting growth whenever it achieves a high level, and this is gained through the additional factor \( 1 - x \). It is clear that \( (1 - x) \) term serves to inhibit growth because as \( x \) approaches 1, \( (1 - x) \) approaches 0 (see [3]).

Summarizing, the Malthus’s model (2) is a simple (non-chaotic) but not real model of population growth whereas Logistic model (3), which is a slight modification of (2), is a more real model of population growth but, in contraposition, it is often cited as an archetypal example of how complex, chaotic behaviour can arise from very simple non-linear dynamical equations.

In fact, with \( r \) slightly bigger than 3.57 we find the presence of a very complex and chaotic dynamics. Slight variations in the initial population yield dramatically different results over time, a prime characteristic of chaos. Actually, between 3.57 and 4 there is a rich interleaving of chaos and order and a small change in \( r \) can make a stable system chaotic, and vice versa (see [3,7]).

Also, some additional details on Logistic dynamics and other important examples arising in deterministic chaos can be found in [6].

Finally, let us say that the previous analysis suggests to us the importance of the approximation and convergence of chaotic functions in the applied context.

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2. Preliminary considerations

According to Devaney [5], if \((X,d)\) is a metric space and \( f: X \to X \) is a continuous mapping then \( f \) is chaotic (\(D\)-chaotic), if it satisfies the following three properties:

(D1) \( f \) is topologically transitive; that is, for all non-empty open subsets \( U \) and \( V \) of \( X \) there exists a natural number \( k \) such that \( f^k(U) \cap V \) is nonempty.

(D2) \( P(f) \), the set of periodic points of \( f \), is a dense subset of \( X \).

(D3) \( f \) is sensitively dependent (SD) on initial conditions; that is, there is a positive number \( \delta \) (a sensitivity constant) such that for every point \( x \in X \) and each \( \epsilon > 0 \) there are \( y \in X \) with \( d(x,y) < \epsilon \) and \( n \in \mathbb{N} \) such that \( d(f^n(x), f^n(y)) \geq \delta. \)
It is important to remark that properties (D1) and (D2) imply that \( f \) has sensitive dependence on \( X \) (see [2]).

Also, it should be remarked that for functions on intervals in \( \mathbb{R} \), it was shown by Vellekoop and Berglund [13] that transitivity implies chaos. Nevertheless, in metric spaces other than \( \mathbb{R} \), transitivity need not imply (D2) or (D3) in the definition of chaos (see [5,11,12]).

Also, as a consequence of the Birkhoff Transitivity Theorem (see [10] for details) we have the following remarkable characterization of transitive maps

**Proposition 1.** Let \( X \) be a complete and perfect metric space. If \( f : X \rightarrow X \) is a continuous function and \( X \) is invariant by \( f \), then \( f \) is transitive if and only if the orbit \( O(x,f) = \{f^n(x) : n = 0, 1, 2, \ldots \} \) of some point \( x \in X \) is dense in \( X \).

On the other hand, the convergence of chaotic functions has been studied by several authors some of which have contributed with interesting applications in physics and engineering. In fact, recently deLaubenfels et al. [4] considers a sequence \( (T_n(t)) \) of strongly continuous linear semigroups on Banach spaces \( X_n \) converging in the sense of Kato to a semigroup \( T(t) \) on the Banach space \( X \) and analyzes under what conditions the chaoticity of \( (T_n(t)) \) is inherited by \( T(t) \) and gives an interesting application to solve certain discrete parabolic equations, which are a class of partial differential equations arising in the mathematical analysis of diffusion phenomena, as the temperature distribution on a surface (for more details on parabolic and elliptic) equations and its applications see [9]).

Also, polynomials chaos approximation have been successfully employed to solve first order stochastic differential equations which are typical in transport problems (see [15]).

The problem of analyzing the convergence of chaotic functions has been recently retaken by Abu-Saris and Al-Hami in [1] on metric spaces and, in particular, they proved that under compactness conditions, the transitivity is preserved by uniform convergence. More precisely, they proved the following theorem:

**Theorem 1** [1, Theorem 3.1]. Let \( (X,d) \) be a compact metric space, and suppose that \( f_n : X \rightarrow X \) are continuous and topologically transitive functions. If \( f_n \) converges uniformly to \( f \), then \( f \) is topologically transitive.

Unfortunately, as is showed in the following counter-example, this result is wrong:

**Example 1.** Let \( S^1 \) be the unit circle and identify each point on the circle by the radian measure (in a counterclockwise direction) of the angle between the positive \( x \)-axis and the ray beginning at the origin and passing through the point. Then, the usual metric on \( S^1 \) is defined by letting \( d(\alpha, \beta) \) be the length of the shortest arc on the circle connecting \( \alpha \) and \( \beta \). More precisely,

\[
d(\alpha, \beta) = \begin{cases} 
|x - \beta| & \text{if } |x - \beta| \leq \pi, \\
|x - \beta| - \pi & \text{if } |x - \beta| > \pi,
\end{cases}
\]

which also can write as \( d(\alpha, \beta) = |x - \beta| \mod(\pi) \).

Now, consider the translation maps, \( T_\lambda(\theta) = \theta + 2\lambda\pi, \lambda \in \mathbb{R} \), on the unit circle \( S^1 \). It is well-known that if \( \lambda = \frac{p}{q} \) is a rational number, then all points are periodic of period \( q \) and, consequently, in this case \( T_\lambda \) is not transitive, whereas if \( \lambda \) is irrational then \( T_\lambda \) is transitive on \( S^1 \). Moreover, each orbit \( \{T_\lambda^n(\theta) : p \geq 1\} \) is dense in \( S^1 \) if \( \lambda \) is irrational (for details see [5,7,11]).

Thus, if \( \lambda \) is an irrational number and we define \( \lambda_n = \frac{p}{n} \), then \( (\lambda_n) \) is a strictly decreasing sequence of positive irrational numbers such that \( \lambda_n \searrow 0 \) and \( T_n = T_{\lambda_n} : S^1 \rightarrow S^1 \) is transitive for all \( n \in \mathbb{N} \). However,

\[
d(T_n(\theta), T_0(\theta)) = d(T_{\lambda_n}(\theta), \theta) = 2\lambda_n\pi = \frac{2\lambda\pi}{n}, \quad \text{for all } \theta \in S^1
\]

and, consequently, \( (T_{\lambda_n}) \) converges uniformly to a function \( T_0 = \text{Id} \) which is not transitive.

In the following section we will give sufficient conditions assuring the transitivity of the limit function for uniform convergent sequences of transitive functions. Also, several illustrative examples and applications are presented.

### 3. Uniform limit of transitive functions

In the sequel, as is usual, \( d_\infty(f,g) \) denotes the uniform metric on \( \mathcal{C}(X,X) \), that is to say:

\[
d_\infty(f,g) = \sup_{x \in X} d(f(x), g(x)).
\]

Also, it is important to remark that through this paper \( X \) denotes a perfect metric space (i.e., \( X \) is closed and has no isolated points). It is not difficult to see that in a perfect metric space \( X \), every open set in \( X \) also has no isolated points.
Now, in order to establish our main result in this section, we precise the following elementary previous result.

**Lemma 1.** Let $X$ be a perfect metric space and consider $U \subset X$ a nonempty open set. If $(x_n)$ is a dense sequence in $X$ and $x_{n_0} \in U$, then there exists $n_1 > n_0$ such that $x_{n_1} \in U$.

**Proof.** It is sufficient to observe that $U \setminus \{x_1, x_2, \ldots, x_{n_0}\}$ is a nonempty open set. □

**Theorem 2.** Let $(X,d)$ be a perfect metric space, and let $f_n : X \to X$ be a sequence of continuous and topologically transitive functions such that $(f_n)$ converges uniformly to a function $f$. Additionally, suppose that $\lim_{n \to \infty} (f_n(x))$ exists for all $x \in X$. Then $f$ is topologically transitive.

**Proof.** Let $U, V$ be two nonempty open subsets of $X$. Then, due to (T2), there exists $x_0 \in X$ such that $\{f_n(x_0)\}$ is dense in $X$. Thus, by Lemma 1 and condition (T1), we obtain that the sequence $\{f^n(x_0)\}$ is also dense in $X$.

Thus, there exists $p \in \mathbb{N}$ such that $z = f^p(x_0) \in U$.

Now, consider the set $G = V \setminus \{f(x_0), \ldots, f^p(x_0)\}$. Then, because $X$ is a perfect metric space, $G$ is a nonempty open set. Thus, due to denseness of $\{f_n(x_0)\}$, there exists $q > p$ such that $f^q(x_0) \in G \subset V$, which implies that

$$f^q(x_0) = f^{q-p}(f^p(x_0)) = f^{q-p}(z) \in f^{q-p}(U) \cap V$$

and, consequently, $f^{q-p}(U) \cap V$ is nonempty and $f$ is topologically transitive. This completes the proof. □

**Example 2.** Let $(T_{x+\frac{1}{2}})$ be a sequence of translations on $S^1$ with $\lambda$ a positive and irrational number. Then, $T_{x+\frac{1}{2}}$ is transitive for each $n \in \mathbb{N}$ and $(T_{x+\frac{1}{2}})$ is uniformly convergent to a transitive function $T_x$.

Also, in this case we note that:

1. $T_n^\theta = T_{x+\frac{1}{2}}^\theta = \theta + 2n\lambda \pi + 2\pi = T_\theta^\theta(\theta)$, for all $n \in \mathbb{N}$. Therefore $d_\infty(T_n^\theta, T_\theta^\theta) = 0$ for all $n$ and, consequently, property (T1) is verified.

2. $\{T_n^\theta(\theta)\} = \{T_\theta^\theta(\theta)\}$ is a dense subset of $S^1$, for each $\theta \in S^1$.

which implies that property (T2) is also verified. Consequently, in this case the conditions T1–T2 are verified and, as we know, the sequence $(T_{x+\frac{1}{2}})$ converges uniformly to a transitive function $T_x$.

**Remark 1.** In Example 1 the sequence $(T_{\theta})$ is uniformly convergent to a function $T_\theta = \text{Id}$, however $(T_n)$ does not verify property (T2) in above Theorem 2. In fact, for any arbitrary $\theta \in S^1$ we have

$$\{T_n^\theta(\theta) : n \in \mathbb{N}\} = \{\theta + 2n\frac{\lambda}{n} \pi : n \in \mathbb{N}\} = \{\theta + 2\lambda \pi\}$$

which, obviously, it is not dense in $S^1$.

**Example 3.** Let $(\lambda_n)$ be a strictly decreasing sequence of positive irrational numbers such that $\lambda_n \downarrow 0$ and $T_{\lambda_n} : S^1 \to S^1$. Then, we have $T_{\lambda_n}$ is transitive for all $n \in \mathbb{N}$ and

$$d(T_{\lambda_n}(\theta), T_\theta(\theta)) = d(T_{\lambda_n}(\theta), \theta) = 2\lambda_n \pi, \quad \forall \theta \in S^1$$

and, consequently, $(T_{\lambda_n})$ converges uniformly to a function $T_\theta = \text{Id}$ which is not transitive.

Why the limit function is not transitive?

The reason is the following: In this case, the sequence $T_{\lambda_n}$ cannot simultaneously verify conditions (T1) and (T2).

In fact, firstly we observe that for each $\theta \in S^1$:

$$T_n^\theta(\theta) = T_{\lambda_n}^\theta(\theta) = \theta + 2n\lambda_n \pi, \quad \forall n \in \mathbb{N}.$$
Thus, if \( n \lambda_n \to 0 \) then, because in this case \( T_n^\ast(\theta) \) remains (a.e.) in a neighborhood of \( \theta \), it is clear that the orbit \( \{T_n^\ast(\theta)\} \) cannot be dense in \( S^1 \) and, consequently, condition (T2) is not verified. On the other hand, if \( n \lambda_n \not\to 0 \) then

\[
d(T_n^\ast(\theta), T^\ast(\theta)) = 2n \lambda_n \pi \not\to 0
\]

and, consequently, condition (T1) is not verified.

**Remark 2.** We observe that Theorem 2 also improves Theorem 1 removing compactness conditions.

Also, it is interesting to observe that using Proposition 1 and condition (T1) in Theorem 2, we obtain the following characterization:

**Theorem 3.** Let \((X,d)\) be a complete and perfect metric space, and consider a sequence \( f_n : X \to X \) of continuous and topologically transitive functions such that \((f_n)\) converges uniformly to a function \( f \) and

\[
(T1) \quad d_\infty(f^n_n, f^n) \to 0 \quad \text{as} \quad n \to \infty.
\]

Then,

\[
f \text{ transitive } \iff \{f^n_n(x)\} \text{ is dense in } X, \text{ for some } x \in X.
\]

**Proof.** (\( \Rightarrow \)) If \( f \) is transitive then, due to Proposition 1, there exists \( x \in X \) such that the orbit \( \{f^n(x)\} \) is dense in \( X \) and, consequently, due to (T1) we have \( \{f^n_n(x)\} \) is also dense in \( X \).

(\( \Leftarrow \)) Conversely, if \( \{f^n_n(x)\} \) is dense for some \( x \in X \) then, due to (T1), \( \{f^n(x)\} \) is dense in \( X \) and, consequently, due to Proposition 1, \( f \) is a transitive function. \( \square \)

**Remark 3.** We note that the main role of the uniform convergence in this work (particularly in this last Theorem 3) it is to preserve the continuity of the limit function.

Finally, we know that the authors in [13] has proved that, on compact intervals in \( \mathbb{R} \), transitivity implies D-chaos. This fact implies that conditions (T1) and (T2) in our Theorem 2 are also sufficient conditions for assuring the chaotic behavior of the limit function under uniform convergence of D-chaotic functions. More precisely, as a direct consequence of Theorem 2 we obtain the following result:

**Corollary 1.** Let \( I \) be a compact interval in \( \mathbb{R} \), and let \( f_n : I \to I \) be a sequence of continuous and D-chaotic functions such that \((f_n)\) converges uniformly to a function \( f \). If, additionally, conditions (T1) and (T2) are verified, then \( f \) is a D-chaotic function.

### 4. Comments and discussion

The convergence of chaotic functions and the approximation of functions by chaotic polynomials have been studied by several authors, with interesting and successful applications in physics and engineering. For example, the convergence of chaotic linear semigroups has been used for solving certain parabolic equations arising in problems associated to diffusion phenomena such as the propagation of gas in the air or the temperature distribution on a surface (see [4,9]) and, on the other hand, the approximation of functions by chaotic polynomials has been used for solving certain stochastic differential equations arising in transport problems as well as in flow-structure interactions (see [15,16]).

Also, there are polynomial chaos algorithms to model certain dynamical systems (stochastic systems) where the complexity arising in the input uncertainty (stochastic input) and its future evolution (see [15,16]).

Actually, to find conditions assuring the preservation of any chaotic property under limit operations is an interesting problem.

In this context, the aim of this work is to give sufficient conditions for assuring the transitivity of the uniform limit for sequences of transitive functions (Theorem 2). These conditions (T1) and (T2) presents metric and topological characteristics, respectively.

In order to illustrate some applications of Theorem 2, some interesting examples are carefully presented and developed.

Also, Theorem 2 improves Theorem 3.1 in [1] removing compactness conditions.

Finally, let us say that Theorem 2 has an important application on compact intervals in \( \mathbb{R} \). In fact, conditions (T1) and (T2) in Theorem 2 are also sufficient conditions for preserving the chaotic behavior of the limit function under uni-
form convergence of D-chaotic functions (see Corollary 1). In a more general context of approximation of functions on intervals, and in connection with above results, we know that all continuous function \( f \in C(I, I) \) can be uniformly approximated by D-chaotic functions. More precisely, let \( I \) be a compact interval in \( \mathbb{R} \) and let \( \mathcal{D}(I, I) \) be the class of all continuous functions \( f: I \rightarrow I \) such that \( f \) is D-chaotic on some \( J \subseteq I \). Then, \( \mathcal{D}(I, I) \) is an open and dense subset of \( (C(I, I), d_\infty) \) (for details and comments the reader can see reference [14]).

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