The next proposition relates the parabolic type of a semigroup to the possible \( r \)-regular elements that exist inside the interior of the semigroup. In its statement we say that a matrix with real eigenvalues is of type \( r \) in case its semi-simple component is strictly \( r \)-regular.

**Proposition 3.4.** The following conditions are equivalent for a semigroup \( \Gamma \subset \text{Sl}(d) \) with \( \text{int} \Gamma \neq \emptyset \).

1. The parabolic type of \( \Gamma \) is \( r \).
2. \( r \) is maximal with the property: \( D(1) = \pi r^{-1} (D_r(1)) \).
3. There exists \( h \in \text{int} \Gamma \) of type \( r \), and conversely if \( g \in \text{int} \Gamma \) has real eigenvalues then its type is \( r' \supset r \).
4. \( \text{int} \Gamma \) contains a strictly \( r \)-regular element and if \( g \in \text{int} \Gamma \) is \( r' \)-regular then \( r' \supset r \).

**Proof.** See [13] for a proof which works for semigroups in general semi-simple Lie groups.

Afterwards we shall check that the parabolic type of both semigroups \( S_r \) and \( T_r \) is precisely \( r \).

If \( \Gamma_1 \subset \Gamma_2 \) and \( r \) is the parabolic type of \( \Gamma_2 \) then the parabolic type of \( \Gamma_1 \) contains \( r \).

For later reference we state the following properties of the invariant control sets related to the parabolic type of a semigroup. Let us say that a subset \( C \subset F(r) \) is \( \Gamma \)-admissible in case it is contained in \( \sigma(h) \) for all \( h \in \text{reg} \Gamma \).

**Proposition 3.5.** For a semigroup \( \Gamma \subset \text{Sl}(d) \) with non-empty interior the following properties hold.

1. Suppose that the parabolic type of \( \Gamma \) is \( r \), and let \( r_1 \) be a multi-index containing \( r \). Let \( \pi : F(r_1) \rightarrow F(r) \) be the projection. Then \( D_1 (r_1) = \pi^{-1} (D_1 (r)) \).
2. Suppose that the parabolic type of \( \Gamma \) is \( r \). Then its invariant control set \( D_1 (r) \) is \( \Gamma \)-admissible.
3. If \( D_1 (r) \) is \( \Gamma \)-admissible in \( F(r) \) then the parabolic type of \( \Gamma \) contains \( r \).

**Proof.** The first two properties are the contents of Theorem 4.3 and Proposition 4.8 in [13], respectively. The last statement is a consequence of previous two.

Note that the first property stated above allows the determination of the invariant control set of the semigroup \( \Gamma \) in any flag manifold as soon as one knows the invariant control set \( D_1 (r) \) if its parabolic type is \( r \). In fact, the
invariant control set in the full flag $F(r_c)$ is $\pi^{-1}(D_1(r))$. Hence the invariant control set in a flag manifold $F(r_1)$ is $\pi_{r_1}(\pi^{-1}(D_1(r)))$.

The third condition of Proposition 3.4 shows in particular that for a semigroup $\Gamma$ whose parabolic type is the complete multi-index $r_c$, any $h \in \text{int} \Gamma$ with real eigenvalues is diagonalizable. We can improve this fact by showing that any $g \in \text{int} \Gamma$ has real eigenvalues and is diagonalizable.

**Corollary 3.6.** Let $\Gamma$ be a semigroup with non-empty whose parabolic type is the complete multi-index $r_c$. Let $g \in \text{int} \Gamma$. Then $g$ has real distinct eigenvalues.

**Proof.** Let $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_s$ the decomposition of $\mathbb{R}^d$ into primary subspaces of $g$. We must show that $\dim V_i = 1$ for all $i = 1, \ldots, s$. First observe that we can perturb $g$ and get $g_1 \in \text{int} \Gamma$ which has the same primary components of $g$ and such that its eigenvalues are of the form $e^{a+ib}$ with $b = q \pi$, $q \in \mathbb{Q}$. Hence by taking some power of $g_1$ we arrive at $g_2 \in \text{int} \Gamma$, having positive eigenvalues and whose primary decomposition, say $\mathbb{R}^d = W_1 \oplus \cdots \oplus W_s$, satisfies $V_i \subset W_j$. However $\mathcal{W}(\Gamma) = \{1\}$, hence the eigenvalues of $g_2$ are simple. This implies that $\dim W_1 = 1$ and hence that $g$ is diagonalizable. 

In the next section we shall check that the semigroups $T$ of totally positive matrices and $S$ of sign regular matrices have parabolic type $r_c$, and hence satisfy the condition of the above corollary.

4. **Maximal semigroups**

In [11] the maximal semigroups in a semi-simple Lie group were described as compression semigroups of certain subsets in the flag manifolds of the group. We shall specialize here the results of [11] to the case of $\text{SL}(d)$ and the flag manifolds $F(r)$.

Recall that a subsemigroup $\Gamma$ of a group $G$ is said to be maximal provided $\Gamma$ is not a group and if $U$ is a semigroup satisfying $\Gamma \subset U \subset G$ then $\Gamma = U$ or $U = G$. In our context we have the following generalization of the notion of maximality (see [11]).

**Definition 4.1.** A semigroup $\Gamma \subset \text{SL}(d)$ with $\text{int} \Gamma \neq \emptyset$ is said to be $r$-maximal if $r$ is its parabolic type and $\Gamma$ is not properly contained in any semigroup of parabolic type $r$.

It is proved in [11] that a semigroup $\Gamma$ with $\text{int} \Gamma \neq \emptyset$ is maximal in case it is $r$-maximal with $r$ a singleton. We prove below that a sign-regular semigroup $S_T$ is $r$-maximal. In particular it will produce that $r$ is the parabolic type of $S_T$ which, in turn, implies that $r$ is also the parabolic type of $T_T$. This has some nice consequences about the eigenvalues of a matrix which belongs to the interior of one of these semigroups.

We start with the notion of duality between flag manifolds. Given a multi-index $r = \{r_1, \ldots, r_k\}$, with $1 \leq r_1 < \cdots < r_k < d$, let $r^* = \{s_1, \ldots, s_k\}$ be such that $s_1 = d - r_k$, $\ldots$, $s_k = d - r_1$. Following [11] (see Section 3.1) we
say that the flag manifolds $F(r)$ and $F(r^*)$ are dual to each other. The point about this duality is that the possible open cells in $F(r)$ (as well as in $F(r^*)$) are given by incidence as follows: Fix $b = (U_1, \ldots, U_k)$ in $F(r^*)$ and put

$$\sigma_b = \{(V_1, \ldots, V_k) \in F(r) : V_i \cap U_{i+1} = 0\}.$$

Then $\sigma_b$ is an open cell in $F(r)$ and every such cell is $\sigma_b$ for some $b \in F(r^*)$. Therefore, the flag manifold $F(r^*)$ identifies with the set of open cells in $F(r)$. Of course, since $r = (r^*)^*$, we have an analogous identification between $F(r)$ and the open cells in $F(r^*)$.

These identifications give rise to the following notion of duality between subsets of $F(r)$ and $F(r^*)$. Given $C \subset F(r)$ let $C^* \subset F(r^*)$ be defined by

$$C^* = \{b \in F(r^*) : C \subset \sigma_b\}.$$

Clearly, if $D \subset F(r^*)$ we can write the same way $D^* \subset F(r)$, so that it makes sense to write $(C^*)^* \subset F(r)$. According to [11] we say that a subset $C \subset F(r)$ is $B$-convex in case $C = (C^*)^*$. A subset $C \subset F(r)$ is said to be admissible if $C^* \neq \emptyset$.

The following result shows that the $r$-maximal semigroups are essentially the compression semigroups of $B$-convex sets.

**Theorem 4.2.** A semigroup $\Gamma \subset SL(d)$ with $int \Gamma \neq \emptyset$ is $r$-maximal if and only if there exists an admissible $B$-convex set $D \subset F(r)$ such that $\Gamma$ is the compression semigroup of $K = cl(intD)$, that is,

$$\Gamma = \{g \in SL(d) : gK \subset K\}.$$

**Proof.** See [11], Theorem 5.4. \hfill \Box

Our objective now is to prove that the subsets $C_{r_k}$ defined in (6) are $B$-convex. This will imply that the semigroup $S_{r_k}$ is $r$-maximal. This fact was claimed without proof in [11] (see Section 6.3) and a proof was offered for the case $r = \{k\}$ is a singleton. In this case the proof is based in the following lemma.

**Lemma 4.3.** Let $C_k \subset Gr_k$ be as before. Then

$$C_k^* = \{V^{\perp} : V \in int (C_k)\}$$

where $V^\perp$ stands for the ortho-complement in $\mathbb{R}^d$ of the $k$-dimensional subspace $V$.

**Proof.** See [11], Lemma 6.7 and comments after the proof. \hfill \Box

This lemma shows in particular that the subsets $C_k$ are admissible. Now we can check the $B$-convexity of $C_k$:
Corollary 4.4. For any $k$ we have $C^*_k = C_k^*$, that is $C_k$ is $B$-convex.

Proof. It follows from general facts that $C_k \subseteq C_k^*$. For the reverse inclusion, suppose that the $k$-dimensional subspace $V \subseteq C_k^*$ does not belong to $C_k$. By the lemma and the definition of duality, it follows that there exists $W \in \text{int} C_k$ such that $V \cap W^\perp \neq 0$. To see that this is a contradiction choose a basis $\{v_1, \ldots, v_k\}$ of $V$ and a basis $\{w_1, \ldots, w_k\}$ of $W$ such that $v_i \in W^\perp$. Then if we put $\xi = v_1 \wedge \cdots \wedge v_k$ and $\eta = w_1 \wedge \cdots \wedge w_k$ it follows that $\langle \xi, \eta \rangle = 0$ (c.f. formula (1)). But this contradicts the fact that $W \in \text{int} C_k$, that is, $\eta \in \pm \text{int} O_k$.

Now it is a matter of playing successively with the definitions to check $B$-convexity in general.

Proposition 4.5. Each set $C_R$ is $B$-convex in $F(r)$.

Proof. First recall that if $r$ is given by $1 \leq r_1 < \cdots < r_k < d$ then

$$C_R = (\pi_{r_1}^\Gamma)^{-1} (C_{r_1}) \cap \cdots \cap (\pi_{r_k}^\Gamma)^{-1} (C_{r_k}).$$

This definition implies immediately that $(V_1, \ldots, V_k) \in F(r)$ belongs to $C_R$ if and only if each $V_i \in C_{r_i}, i = 1, \ldots, k$. Now, take $b = (U_1, \ldots, U_k) \in F(r^*)$. By definition of the duality operator, it follows that $b \in C_R^*$ if and only if $V_i \cap U_{k-i+1} = 0$, for all $i = 1, \ldots, k$ and $(V_1, \ldots, V_k) \in C_R$. But this is equivalent to saying that each $U_{k-i+1}, i = 1, \ldots, k,$ belongs to $C_k^*$. This shows that

$$C_R^* = (\pi_{r_1}^\Gamma)^{-1} (C_{r_1}^*) \cap \cdots \cap (\pi_{r_k}^\Gamma)^{-1} (C_{r_k}^*).$$

If we repeat the same reasoning, we arrive that $C_R^{**}$ is given by the intersection of the pre-images of $C_{k_i}^{**}$ under the projections. Therefore, the above corollary implies that $C_R^{**} = C_R$, concluding the proof.

During the proof of this proposition we got the following description of $C_R^*$.

Corollary 4.6. Given $b = (V_1, \ldots, V_k) \in F(r)$ put $b^\perp = (V_1^\perp, \ldots, V_k^\perp) \in F(r^*)$. Then

$$C_R^* = \{b^\perp \in F(r^*) : b \in \text{int} (C_R)\}.$$

The $B$-convexity of $C_R$ together with the fact that $\text{cl} (\text{int} C_R) = C_R$ and Theorem 4.2 yield the desired results about the semigroups $S_R$ and $T_R$.

Theorem 4.7. The following facts hold true:

1. Given a multi-index, the parabolic type of $S_R$ is $r$ and $S_R$ is $r$-maximal. For each $k = 1, \ldots, d - 1$, the semigroup $S_k$ is maximal in $S_l(d)$.

2. The invariant control set of both semigroups $S_R$ and $T_R$ in $F(r)$ is $C_R$. 
3. If \( r' \subseteq r \) then the invariant control set of both semigroups \( S_r \) and \( T_r \) in \( \mathbb{F}(r') \) is \( (\pi^r_r)^{-1} C_r \).

4. If \( h \in \text{int}S_r \) or \( h \in \text{int}T_r \) has real eigenvalues then \( h \) has type \( r_1 \) with \( r_1 \subseteq r \).

5. If \( h \in \text{int}S \) then \( h \) is diagonalizable.

6. If \( h \in \text{int}T \) then its eigenvalues are \( > 0 \).

**Proof.**

1. Is a consequence of Theorem 4.2 and Proposition 4.5.

2. The invariant control set of \( S_r \) in \( \mathbb{F}(r) \) is \( C_r \) by Theorem 4.2. As to \( T_r \) we note that any \( x \in \text{int}C_r \) is the attractor fixed point of some regular element \( h \in \text{int}S_r \). Then \( h^2 \in \text{int}T_r \) also has \( x \) a attractor. This implies that \( x \) belongs to the interior of the invariant control set of \( T_r \). Taking closures it follows that \( C_r \) is the invariant control set of \( T_r \) as well.

3. Since \( r \) is the parabolic type of both \( S_r \) and \( T_r \), the result follows by Proposition 3.5.

4. Follows from Proposition 3.4.

5. Is a consequence of Corollary 3.6.

6. The subspace of \( \Lambda^k \) spanned by principal eigenvector of \( \Lambda^k g \) belongs to the invariant control set of \( T \) in \( \text{Gr}_k \). Hence, \( g \) has a principal eigenvector in \( O_k \). Since \( (\Lambda^k g) O_k \subset O_k \), it follows that the highest eigenvalue, say \( \mu_k \), of \( \Lambda^k g \) is positive. But \( \mu_k = \lambda_1 \cdots \lambda_k \) where \( \lambda_1, \ldots, \lambda_d \) are the eigenvalues of \( g \) ordered by \( |\lambda_1| > \cdots > |\lambda_d| \). Since \( \mu_k > 0 \) for every \( k = 1, \ldots, d \), it follows that \( \lambda_i > 0 \), \( i = 1, \ldots, d \).

**Remark.** The last two statements in the above theorem are well known results about sign-regular and totally positive matrices (see [1]).

### 5. Some lemmas

For the description of the control sets of the semigroups \( S, T \) and more generally \( S_r \) and \( T_r \) the following basic facts about sign changes of vectors in \( \mathbb{R}^d \) shall be required. By an orthant in \( \mathbb{R}^d \) we understand a closed set of the form \( O^+_{\varepsilon} = \{(x_1, \ldots, x_d) : \varepsilon_i x_i \geq 0\} \), where \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \) is a sign vector, that is, \( \varepsilon_i = \pm 1 \), \( i = 1, \ldots, d \). In our notation the superscript \( + \) is intended to distinguish the orthant in \( \mathbb{R}^d \) from the corresponding orthant \( O_{\varepsilon} \subset \mathbb{P} \), which is the set lines in \( \mathbb{R}^d \) contained in \( O^+_{\varepsilon} \cap O^{-}_{\varepsilon} \), \( O^{-}_{\varepsilon} = -O^+_{\varepsilon} \). Clearly, in \( \mathbb{P} \) the orthants satisfy \( O_{-\varepsilon} = O_{\varepsilon} \).

Given a sign vector \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \), we let \( \nu(\varepsilon) \) denote the number of sign changes of \( \varepsilon \), that is, \( \nu(\varepsilon) \) is the number of indices \( i = 1, \ldots, d - 1 \)
such that \( \varepsilon_1 \varepsilon_{i+1} = -1 \). The number of sign changes of the orthant \( O^+_\varepsilon \) is by definition \( \psi(\varepsilon) \). Note that \( \psi(-\varepsilon) = \psi(\varepsilon) \), so that it makes sense to define the number of sign changes of an orthant \( O_\varepsilon \) in \( \mathbb{P} \).

For \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) we say that \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \) is a sign sequence of \( x \) if \( x \in O_\varepsilon \). Let \( \psi_+ (x) \) [respectively \( \psi_- (x) \)] stand for the maximum [minimum] of \( \psi(\varepsilon) \) with \( \varepsilon \) running through the sign sequences of \( x \). Note that \( \psi_+ (x) = \psi_- (x) \) if and only if \( x \) belongs to the interior of an orthant (or equivalently if \( x \) has nonvanishing coordinates). In this case we put \( \psi(x) = \psi_+ (x) = \psi_- (x) \), which is the number of indices \( i \) such that \( x_i x_{i+1} < 0 \) (cf. [1]).

For \( k = 0, 1, \ldots, d-1 \), put

\[
O_{\leq k} = \bigcup_{\psi(\varepsilon) \leq k} O_\varepsilon, \quad \Sigma_k = O_{\leq k} \setminus O_{\leq (k-1)},
\]

where \( O_{\leq -1} = \emptyset \). The subspace \([x] \) spanned by a vector \( 0 \neq x \in \mathbb{R}^d \) belongs to \( O_{\leq k} \) if and only if some of its sign sequences have at most \( k \) sign changes, that is, if and only if \( \psi_-(x) \leq k \). It follows that \( [x] \in \Sigma_k \) if and only if \( \psi_-(x) = k \). For later reference we prove the following lemma which describes the interior and the boundary of these sets.

**Lemma 5.1.** Let \( 0 \neq x \in \mathbb{R}^d \) and denote by \([x] \in \mathbb{P}\) the subspace it spans. Then

1. \( [x] \in \text{int}(\Sigma_k) \) if and only if \( \psi_-(x) = \psi_+(x) = k \). Hence, \( [x] \in \Sigma_k \cap \partial \Sigma_k \) if and only if \( k = \psi_-(x) < \psi_+(x) \).

2. \( [x] \in \text{int}(O_{\leq k}) \) if and only if \( \psi_+(x) \leq k \). Hence, \( [x] \in O_{\leq k} \cap \partial O_{\leq k} \) if and only if \( \psi_-(x) \leq k < \psi_+(x) \).

**Proof.** Suppose that \( \psi_-(x) = \psi_+(x) = k \), and take a vector \( \delta = (\delta_1, \ldots, \delta_d) \) small enough so that \( (x_i + \delta_i) x_i > 0 \) if \( x_i \neq 0 \). Note that the only restriction for \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \) to be a sign sequence of \( y = (y_1, \ldots, y_d) \) is that \( \varepsilon_i = y_i / |y_i| \) if \( y_i \neq 0 \). Hence the sign sequences of \( x + \delta \) are sign sequences of \( x \). Hence \( \psi_-(x + \delta) = \psi_+(x + \delta) = k \), showing that \( x + \delta \in \Sigma_k \). Therefore \( x \in \text{int} \Sigma_k \).

Conversely, suppose that \( k = \psi_-(x) < \psi_+(x) \). Then \( x \) belongs to an orthant with \( \psi_+(x) \) sign changes, hence \( x \notin \text{int} \Sigma_k \).

For \( O_{\leq k} \) the proof is similar. If \( \psi_+(x) \leq k \) then the sign sequences of a neighboring point \( x + \delta \) have at most \( k \) sign changes, so that \( x + \delta \in O_{\leq k} \). Reciprocally, if \( k < \psi_+(x) \) then \( x \) belongs to an orthant with more than \( k \) sign changes.

We conclude this section with the following lemmas on \( k \)-positive subspaces. The notation \((1, \varepsilon)\) stands for a vector \((1, \varepsilon_2, \ldots, \varepsilon_d)\) with \( \varepsilon_i = \pm 1 \), \( i = 2, \ldots, d \).
Lemma 5.2. Let \( V \) be a \( k \)-positive (respectively, strictly \( k \)-positive) subspace containing a vector \((1, \varepsilon)\). Then \( V \cap e_1^+ \) is a \((k - 1)\)-positive (respectively strictly \((k - 1)\)-positive) subspace in the orthocomplement of \( e_1 \), with respect to the basis \( \{e_2, \ldots, e_d\} \).

Proof. The assumption that \((1, \varepsilon) \in V\) implies that \( V \) is not contained in \( e_1^- \), so that \( \dim(V \cap e_1^+) = k - 1 \). Hence, there are \( v_2, \ldots, v_k \in V \cap e_1^+ \) such that \( \{(1, \varepsilon), v_2, \ldots, v_k\} \) is a basis of \( V \). Since \( V \) is \( k \)-positive we can assume without loss of generality that

\[
\xi = (1, \varepsilon) \wedge v_2 \wedge \cdots \wedge v_k \in C_k^+.\]

Let \( r \) be a multi-index of \( \{2, \ldots, d\} \) and form the multi-index \((1, r)\), by adjoining \( 1 \). Since \( e_1 \) is orthogonal to \( v_2, \ldots, v_k \), it follows that

\[
\langle \xi, e_{(1, r)} \rangle = \langle v_2 \wedge \cdots \wedge v_k, 0_1 \rangle.
\]

Hence \( v_2 \wedge \cdots \wedge v_k, 0_1 \rangle \geq 0 \) (respectively \( > 0 \)) for every multi-index \( r \) if \( V \) is \( k \)-positive (respectively strictly \( k \)-positive), proving the lemma.

Clearly the subspace \( V \cap e_1^+ \) in this lemma is also \((k - 1)\)-positive in \( \mathbb{R}^d \) with respect to the standard basis. Later on this fact will be used to show that every \( k \)-positive subspace contains positive subspaces with smaller dimensions. On the other hand the next lemma shows how to extend \( k \)-positive subspaces to positive subspaces of higher dimensions.

Lemma 5.3. Let \( V \) be a \((k - 1)\)-positive subspace with \( 2 \leq k \leq d \). Then there exists a \( k \)-positive subspace \( W \supset V \).

Proof. Let \( \{v_2, \ldots, v_k\} \) be a basis of \( V \) such that \( v_2 \wedge \cdots \wedge v_k \in C_k^{(k - 1)} \). Then \( e_1 \wedge v_2 \wedge \cdots \wedge v_k \in C_k^+ \). In fact, the matrix whose columns are the coordinates of \( e_1, \ldots, v_k \) is

\[
\begin{pmatrix}
1 & a \\
0 & A
\end{pmatrix}
\]

with \( A \) a \((k - 1) \times (k - 1)\) matrix. Expanding determinants by the first column one sees that a \( k \)-minor of this matrix is either zero or a \((k - 1)\)-minor of \( A \). Thus the subspace spanned by \( V \) and \( e_1 \) is \( k \)-positive.

In the next lemmas we relate \( k \)-positive subspaces with the number of sign changes of its elements.

Lemma 5.4. Suppose that \((1, \varepsilon)\) belongs to a \( k \)-positive subspace \( V \). Then \( V(1, \varepsilon) \leq k - 1 \).

Proof. Suppose that there are \( k \) sign changes. Then there exists a multi-index

\[
r = (2 \leq i_1 \leq \cdots \leq i_k \leq d)
\]