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CONTROLLABILITY ON SL(2, ℂ) WITH RESTRICTED CONTROLS*

VICTOR AYALA†, RODOLFO RIBEIRO, JR.‡, AND LUIZ A. B. SAN MARTIN‡

Abstract. In this paper we study controllability of affine invariant control systems on the group $\mathrm{Sl}(2,\mathbb{C})$ with restricted controls. We develop a method based on the action of $\mathrm{Sl}(2,\mathbb{C})$ on the sphere $S^2 \approx \mathbb{C} \cup \{\infty\}$ by Möbius functions. Some controllability results are proved. It is proved also that controllability with restricted controls is not a generic property, contrary to the case of unrestricted controls, as proved in the classic paper by Jurdjevic and Kupka [*J. Differential Equations,* 39 (1981), pp. 186–211].

Key words. controllability, complex special linear group, restricted controls

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1. Introduction.

In this paper we study controllability of control systems

$$\dot{g} = X(g) + uY(g) \tag{1.1}$$

on the Lie group $\mathrm{Sl}(2,\mathbb{C})$, where X and Y are right invariant vector fields.

The problem of getting necessary and sufficient conditions for controllability of invariant control systems on semisimple Lie groups is a hard and long-standing open question in control theory. In the literature there are sufficient conditions obtained in the seminal papers by Jurdjevic and Kupka [9], [10] that were improved in several papers. See Gauthier, Kupka, and Sallet [6], El Assoudi, Gauthier, and Kupka [2], and references therein. See the recent results of Santos and San Martin [17] and also Braga and San Martin [5], San Martin [14], [15] and El Assoudi [3]. In these papers the control system is taken with unrestricted controls $u \in \mathbb{R}$.

To the best of our knowledge closed necessary and sufficient conditions were obtained only for the group $\mathrm{Sl}(2,\mathbb{R})$ in Braga Barros et al. [4], Ayala and San Martin [1], and Joó and Tuan [8]. (See also Mittenhuber [11] for a different point of view.) For $\mathrm{Sl}(2,\mathbb{R})$ the conditions work also for control systems with restricted controls $|u| < \rho$.

In this paper we consider the group $\mathrm{Sl}(2,\mathbb{C})$ and look at control systems with restricted controls. The case with unrestricted controls is solved by the Jurdjevic–Kupka conditions since these conditions are generic in a complex Lie group. Hence they imply that there is an open and dense set of pairs (X,Y) such that the control system (1.1) with unrestricted controls $u \in \mathbb{R}$ is controllable.

For restricted controls the controllability problem reduces to finding pairs (A,B) in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ such that the semigroup S generated by $\exp tA$ and $\exp tB$, $t \geq 0$, coincides with $\mathrm{Sl}(2,\mathbb{C})$. We assume throughout that the Lie algebra rank condition holds, so that S has nonempty interior. We approach the controllability

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†Departamento de Matemática, Universidad Católica del Norte, Casilla 1280, Antofagasta, Chile (vayala@ucn.cl). This author was partially supported by Proyecto Fondecyt 1100375 and FIAC2 UCN 1102.

‡Institute of Mathematics, Universidade Estadual de Campinas, Campinas, SP, Brazil (rodolfoirpr@gmail.com, smartin@ime.unicamp.br). The second author was supported by CNPq. The third author was supported by CNPq grant 303755/09-1, FAPESP grant 2012/18780-0, and CNPq/Universal grant 476024/2012-9.

problem via the transitive action of $\mathrm{Sl}(2,\mathbb{C})$ on the complex projective line $\mathbb{C}P^1 \approx S^2 \approx \mathbb{C} \cup \{\infty\}$. Using this action we can prove noncontrollability by exhibiting subsets of $\mathbb{C}P^1$ that are forward invariant by $\exp tA$ and $\exp tB$. On the other hand it follows by the general results of [12], [13], and [16] about semigroup actions on homogeneous spaces that S acts transitively on $\mathbb{C}P^1$ if and only if $S = \mathrm{Sl}(2,\mathbb{C})$. Thus we get controllability of the invariant system by proving transitivity of S on $\mathbb{C}P^1$, or what is the same by proving controllability of the induced system on $\mathbb{C}P^1$. Put another way, via the action on $\mathbb{C}P^1$ we can reduce the analysis of controllability in the six-dimensional manifold $\mathrm{Sl}(2,\mathbb{C})$ to the two-dimensional manifold $\mathbb{C}P^1 \approx S^2$.

In $\mathbb{C}P^1$ we can look at controllability by pursuing the geometry of the trajectories of the vector fields defined by A and B. This way we exhibit an open set of noncontrollable pairs $(A,B) \in \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ showing an essential difference from the case with unrestricted controls.

Finally we mention that as a real Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is isomorphic to $\mathfrak{so}(1,3)$. Hence our results also hold for connected Lie groups whose Lie algebra is $\mathfrak{so}(1,3)$, like, e.g., $\mathrm{SO}(1,3)_0$, the identity component of the isometry group of the three-dimensional hyperbolic space.

2. Controllability. Let us consider the invariant control system

$$\dot{g} = (X + uY)g : |u| \leq 1$$

on the Lie group $G = \mathrm{Sl}(2,\mathbb{C})$ of the 2×2 complex matrices of determinant 1. Here X and Y belong to the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ of G, the 2×2 complex matrices of trace zero, and the restricted control u belongs to $\mathcal{U}$: the admissible class of the real piecewise constant functions. The system semigroup is generated by the exponential of elements in the cone

$$\{\lambda(X + uY) : \lambda \geq 0, |u| \leq 1\}.$$

It is well known that the semigroup of the system is generated just by the boundary of the cone, i.e., by $\exp tA$ and $\exp tB$, $t \geq 0$, $A = X + Y$, and $B = X - Y$.

Therefore, the controllability problem reduces to finding the pairs A, $B \in \mathfrak{g}$ such that the semigroup S generated by $\exp tA$ and $\exp tB$, $t \geq 0$, is equal to G.

We assume the Lie algebra rank condition, which means that $\mathfrak{sl}(2,\mathbb{C})$ is the Lie algebra generated (over $\mathbb{R}$) by the pair A, B. This implies that S has nonempty interior.

The following result is a special case of either Theorem 4.2 of [12] or Theorem 6.1 of [16]. (The point is that $\mathbb{C}P^1$ is the only flag manifold of $\mathrm{Sl}(2,\mathbb{C})$.)

THEOREM 2.1. *Let $S \subset \mathrm{Sl}(2,\mathbb{C})$ be a semigroup with nonempty interior. Then $S = \mathrm{Sl}(2,\mathbb{C})$ if and only if S acts transitively on $\mathbb{C}P^1 = S^2 = \mathbb{C} \cup \{\infty\}$. (Here S acts by restricting the natural action of $\mathrm{Sl}(2,\mathbb{C})$ on the projective line $\mathbb{C}P^1$.)*

Related to this result is the notion of *control set* for S, which in our context is a subset $D \subset \mathbb{C}P^1$ with nonempty interior such that D is contained in the closure of Sx for any $x \in D$ and D is maximal with this property. By the main results of [16], specialized to this case, we have at most two control sets, one that is closed and S-invariant and another one that is open and S^{-1}-invariant. The two control sets merge to a unique one if and only if they coincide with the whole $\mathbb{C}P^1$, in which case S acts transitively. By the above theorem this holds only if $S = \mathrm{Sl}(2,\mathbb{C})$.

We note also that S acts transitively on $\mathbb{C}P^1$ if and only if there is no proper subset which is S-invariant. For the semigroup generated by $\exp tA$ and $\exp tB$, $t \geq 0$, this invariance become forward invariance under $\exp tA$ and $\exp tB$.

Given A, $B \in \mathfrak{sl}(2,\mathbb{C})$ and $g \in \mathrm{Sl}(2,\mathbb{C})$ the pair A, B is controllable if and only if the pair gAg^{-1}, gBg^{-1} is controllable. This allows us to select convenient pairs inside an adjoint orbit of $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$. Now, there are only two Jordan canonical forms for elements in $\mathfrak{sl}(2,\mathbb{C})$, namely,

$$\begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In what follows we work mainly with diagonalizable elements.

3. Möbius functions: Lines and circles. The complex projective line $\mathbb{C}P^1$ is diffeomorphic to S^2 which in turn can be seen as $\mathbb{C} \cup \{\infty\}$. In the last case an element $z \in \mathbb{C}$ is associated to the complex line in $\mathbb{C}^2$ spanned by the vector $(z, 1)$. It follows that the action of

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{C})$$

on $z \in \mathbb{C}$ is given by the Möbius function

$$g(z) = \frac{az+b}{cz+d}$$

because $(az+b, cz+d)$ and $((az+b)/(cz+d), 1)$ span the same line. This function extends continuously to $\mathbb{C} \cup \{\infty\}$ by $g(\infty) = a/c$. The action defines the infinitesimal action of $\mathfrak{sl}(2,\mathbb{C})$: A 2×2 complex matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C})$$

induces a vector field, also denoted by A, on $\mathbb{C}P^1$, whose flow is $\exp tA$. By taking derivatives in the Möbius function with respect to a, b, c, and d we get the quadratic vector field

$$A(z) = -\gamma z^2 + 2\alpha z + \beta$$

which is associated to a Ricatti differential equation.

In what follows we will need the following remarks about the Möbius functions. They are obtained by composing two types of functions:

1. affine $z \mapsto az + b$;
2. inverse $\iota : z \mapsto 1/z$.

These two types preserve the set of circles of $\mathbb{C} \cup \{\infty\}$, where the lines of $\mathbb{C}$ can be seen as circles through ∞. In fact, the affine case is clear. As to the inverse consider the circle $C = \{|z - c| = r\}$ with center c and radius r. If $z \in \iota(C)$, then $1/z$ belongs to the circle and $|\frac{1}{z} - c| = r$. Thus

$$|cz - 1|^2 = r^2 |z|^2,$$

which is the same as

$$(cz - 1)(\overline{cz} - 1) = r^2 |z|^2.$$

Therefore,

$$1 - (cz + \overline{cz}) + (|c|^2 - r^2)|z|^2 = 0.$$

Now there are the following possibilities:

1. If $|c| = r$, that is, if the circle contains the origin, then z belongs to the line

$$2\,\mathrm{Re}(cz) = cz + \overline{cz} = 1.$$

   This line contains the point $1/2c$ and is perpendicular to the line through the origin and $\overline{c}$. Thus, it is parallel to $i\overline{c}$. Actually, any line in $\mathbb{C}$ can be obtained in this way, which means that the image of a line by the inverse map ι is a circle through the origin.

2. If $|c| \neq r$, then $\iota(C)$ has the equation

$$(|c|^2 - r^2)|z|^2 - (cz + \overline{cz}) + 1 = 0,$$

   which is a circle with center

$$\frac{\overline{c}}{|c|^2 - r^2}$$

   and radius

$$\frac{r}{||c|^2 - r^2|}.$$

   We have

$$\frac{\overline{c}}{|c|^2 - r^2} - \frac{1}{c + re^{i\theta}} = \frac{|c|^2 + r\overline{c}e^{i\theta} - |c|^2 + r^2}{(|c|^2 - r^2)(c + re^{i\theta})} = \frac{r\overline{c}e^{i\theta} + r^2}{(|c|^2 - r^2)(c + re^{i\theta})}$$
$$= \frac{r}{|c|^2 - r^2} \cdot \frac{r + \overline{c}e^{i\theta}}{c + re^{i\theta}}.$$

   It follows that

$$\left|\frac{r + \overline{c}e^{i\theta}}{c + re^{i\theta}}\right|^2 = \frac{(r + \overline{c}e^{i\theta})(r + ce^{-i\theta})}{(c + re^{i\theta})(\overline{c} + re^{-i\theta})} = \frac{|c|^2 + \mathrm{Re}(cre^{-i\theta}) + r^2}{|c|^2 + \mathrm{Re}(cre^{-i\theta}) + r^2} = 1.$$

   Thus,

$$\left|\frac{\overline{c}}{|c|^2 - r^2} - \frac{1}{c + re^{i\theta}}\right| = \frac{r}{||c|^2 - r^2|}.$$

   The previous computations show that the inverse of $c + re^{i\theta}$, which is a generic point of C, belongs to the circle with center $\frac{\overline{c}}{|c|^2 - r^2}$ and radius $\frac{r}{|c|^2 - r^2}$.

Therefore, we get the following statement.

PROPOSITION 3.1. *Denote by $C(c, r)$ a circle with center c and radius r. If $|c| \neq r$, then*

$$\iota(C(c, r)) = C\left(\frac{\overline{c}}{|c|^2 - r^2}, \frac{r}{||c|^2 - r^2|}\right).$$

Alternatively,

$$\frac{\overline{c}}{|c|^2 - r^2} = \frac{1}{\left(1 - \frac{r^2}{|c|^2}\right)c} = \frac{|c|^2}{(|c|^2 - r^2)c}.$$

4. Trajectories of diagonalizable elements. The study of the controllability properties of the system would require the analysis of the trajectories of $(\exp tA)x$ with $A \in \mathfrak{sl}(2,\mathbb{C})$ and $x \in \mathbb{C} \cup \{\infty\}$. The matrix

$$A = \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix}, \qquad \alpha \in \mathbb{C},$$

induces the linear vector field $A(z) = -2\alpha z$ on $\mathbb{C}$, which has two singularities 0 and ∞.

If α is purely imaginary, then the A-trajectories are circles with the origin as the common center. In case α is real, then the trajectories are the lines through the origin, having 0 as an attractor and ∞ as repeller if $\alpha > 0$. In both cases the A-trajectories starting in a circle centered at the origin remain inside the circle. If $\alpha < 0$, then the origin 0 is a repeller and ∞ an attractor.

For a general complex α we have again that 0 is an attractor if $\operatorname{Re}\alpha > 0$ and a repeller if $\operatorname{Re}\alpha < 0$. Since

$$e^{t\alpha} = e^{t\operatorname{Im}\alpha} e^{t\operatorname{Re}\alpha} = e^{t\operatorname{Re}\alpha} e^{t\operatorname{Im}\alpha}$$

the trajectory at time t, starting at $z \in \mathbb{C}$, is given geometrically by going radially through the line between 0 and z and then applying a rotation, or alternatively, first apply a rotation and then go radially. Hence, if $\operatorname{Re}\alpha \geq 0$ the r-balls $|z| \leq r$ are forward A-invariant and the r-sets defined by $|z| \geq r$ are forward A-invariant when $\operatorname{Re}\alpha \leq 0$.

Now take $g \in \mathrm{Sl}(2,\mathbb{C})$ and

$$B = g \begin{pmatrix} -\gamma & 0 \\ 0 & \gamma \end{pmatrix} g^{-1}.$$

Then a trajectory of B is the g-image of a trajectory of the diagonal matrix. Therefore, there are two fixed points $g(0)$ and $g(\infty)$ and the B-trajectories are as follows:

1. If the eigenvalues $\pm\gamma$ are imaginary, then the orbits are the circles $gC(0,r)$. These circles are not necessarily concentric.
2. If the eigenvalues $\pm\gamma$ are real, the orbits remain inside the circles through the fixed points $g(0)$ and $g(\infty)$. Inside any of these circles, there are four orbits: the fixed points and the connected components separated by the fixed points. The trajectories run from $g(\infty)$ to $g(0)$ if $\gamma < 0$ and in the other direction if $\gamma > 0$.

In any case, we call the sets $gC(0,r)$ *level circles of* B. The corresponding balls $gB[0,r]$ are forward B-invariant if $\operatorname{Re}\gamma \geq 0$. Also, just one of these circles contains ∞ and is a line in $\mathbb{C}$ if ∞ is not a singularity of B. This line is called the *separatrix* (see Figure 2 below).

5. A normal form of diagonalizable pairs. Let $0 \neq A,\ B \in \mathfrak{sl}(2,\mathbb{C})$ be diagonalizable elements that generate $\mathfrak{sl}(2,\mathbb{C})$ as a Lie algebra. In this section we get a convenient representative in the conjugacy orbit of (A,B) in $\mathfrak{sl}(2,\mathbb{C})^2$.

First we assume that A is indeed diagonal,

$$A = \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix}, \qquad \operatorname{Re}\alpha \geq 0,$$

so that its fixed points are 0 and ∞, where 0 is the attractor and ∞ the repeller if $\operatorname{Re}\alpha > 0$.

If $g \in \mathrm{Sl}(2,\mathbb{C})$ is a diagonal matrix we have $gAg^{-1} = A$. Therefore, the pairs (A,B) and (A, gBg^{-1}) are conjugate to each other. On the other hand, the group of the diagonal matrices in $\mathrm{Sl}(2,\mathbb{C})$ acts transitively on $\mathbb{C}\backslash\{0\}$. Thus, by Möbius functions it is possible to choose one of the fixed points of gBg^{-1} in a convenient way. Then, we take B such that $1 \in \mathbb{C}$ is one of its fixed points: the attractor one if $\mathrm{Re}\,\gamma \neq 0$, where $\pm\gamma$ are the eigenvalues of B.

In what follows we show that the set of the matrices B which satisfy this condition depend on two complex parameters. One of them corresponds to its eigenvalues $\pm\gamma$ and the other one to the position of the repeller.

To this purpose take $x \in \mathbb{C}$ and write

$$t_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad s_x = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

The corresponding Möbius functions are

$$t_x(z) = z + x \qquad s_x(z) = \frac{z}{xz+1}.$$

These functions satisfy $s_x = \iota \circ t_x \circ \iota$, where as before $\iota(z) = 1/z$. Furthermore, we have

1. $t_x(0) = x$ and $t_x(\infty) = \infty$,
2. $s_x(0) = 0$ and $s_x(\infty) = 1/x$.

Thus,

$$t_1 \circ s_x(0) = 1, \qquad t_1 \circ s_x(\infty) = \frac{1}{x} + 1.$$

PROPOSITION 5.1. *Let B be a diagonalizable matrix such that $1 \in \mathbb{C} \subset \mathbb{C}P^1$ is one of its singularities. Then, B has the form*

$$B = \begin{pmatrix} -\gamma - 2x\gamma & 2(\gamma + x\gamma) \\ -2x\gamma & \gamma + 2x\gamma \end{pmatrix},$$

where $\pm\gamma$ are the eigenvalues of B and $x = 1/(w-1)$ with $w \in \mathbb{C} \subset \mathbb{C}P^1$ the other singularity ($w \neq 1$). Besides, 1 is the attractor fixed point if $\mathrm{Re}\,\gamma > 0$.

Proof. According to the previous computations, if B has has singularities at 1 and $w = \frac{1}{x} + 1$, then $D = (t_1 s_x)^{-1} B t_1 s_x$ has singularities on 0 and ∞. That is, D is a diagonal matrix. Hence $B = t_1 s_x D (t_1 s_x)^{-1}$, that is,

$$\begin{aligned} B &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} -\gamma & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\gamma - 2x\gamma & 2(\gamma + x\gamma) \\ -2x\gamma & \gamma + 2x\gamma \end{pmatrix}. \end{aligned}$$

If $\mathrm{Re}\,\gamma > 0$, then 0 is an attractor of

$$\begin{pmatrix} -\gamma & 0 \\ 0 & \gamma \end{pmatrix}.$$

It follows that $t_1 \circ s_x(0) = 1$ is an attractor of B. ∎

DEFINITION 5.2. *We say that the diagonalizable pair (A, B) is in the standard form if*

$$A = \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix}, \qquad B = \begin{pmatrix} -\gamma - 2x\gamma & 2(\gamma + x\gamma) \\ -2x\gamma & \gamma + 2x\gamma \end{pmatrix}$$

and $\operatorname{Re}\alpha \geq 0$, $\operatorname{Re}\gamma \geq 0$, $\alpha \neq 0 \neq \gamma$, *and* $x \neq 0, -1$.

The condition $x \neq 0$, -1 is to ensure that the singularity of B given by $w = (1+x)/x$ is different from 0 and ∞.

PROPOSITION 5.3. *Let (C, D) be a pair of diagonalizable matrices satisfying the Lie algebra rank condition. Then (C, D) is conjugate to a pair (A, B) in standard form.*

Proof. Take first a conjugation $gCg^{-1} = A$, $gDg^{-1} = B'$ such that A is diagonal. By the Lie algebra rank condition the singularities of B' are in $\mathbb{C}\backslash\{0\}$. In fact, A and B' do not have a common singularity z_0, for otherwise z_0 would be a singularity of any element of the Lie algebra generated by them. Hence there is a diagonal matrix h such that 1 is the attractor singularity of $B = hB'h^{-1}$. By Proposition 5.1 the pair $(A, B) = (hgCg^{-1}h^{-1}, hgDg^{-1}h^{-1})$ is in standard form. ∎

The next proposition describes the level circles of a matrix B in a standard pair.

PROPOSITION 5.4. *Take (A, B) in standard form. Then the level circles of B are given by*

1. *the separatrix which is the line orthogonal to $\overline{x}$ (with direction $i\overline{x}$) and through the point $1 + 1/(2x) = 1 + \overline{x}/2|x|^2$, and*
2. *the circles of the form $1 + C(\overline{x}/(|x|^2 - r^2), r/(||x|^2 - r^2|))$, $r > 0$.*

Proof. The level circles are $t_1 \circ s_x(C(0, r))$, $r \geq 0$, because $B = (t_1 s_x)D(t_1 s_x)^{-1}$ with D diagonal. As observed, $s_x = \iota \circ t_x \circ \iota$. Now $\iota C(0, r) = C(0, 1\backslash r)$ and $t_x C(0, r) = C(x, r)$. So, the level circles are

$$t_1(\iota C(x, r)) = 1 + \iota C(x, r), \qquad r \geq 0.$$

If $r = |x|$ we obtain that $\iota C(x, r)$ is the line through $1/(2x)$ which is parallel to $i\overline{x}$. When we translate this line by 1, we get a line which is parallel to $\iota C(x, r)$ through the point $1 + 1/(2x)$.

On the other hand, if $r \neq |x|$, then $\iota C(x, r) = C(\overline{x}/(|x|^2 - r^2), r/(||x|^2 - r^2|))$. Thus, $1 + \iota C(x, r)$, $r \geq 0$ are the translated circles as claimed. ∎

Next we list properties of a pair (A, B) in standard form and their level circles.

1. The center of every level circle belongs to the line which contains 1 and is parallel to $\overline{x}$, since the centers are $1 + \overline{x}/(|x|^2 - r^2)$. We call this line the *centers line*.
2. The separatrix is the line through $1 + \overline{x}/2|x|^2$ which is perpendicular to the centers line. In fact, since $i\overline{x}$ is perpendicular to $\overline{x}$, a parametrization of the separatrix is given by

$$f(t) = 1 + \frac{\overline{x}}{2|x|^2} + ti\overline{x}, \qquad t \in \mathbb{R}.$$

This lines intersect the centers line when $f(t) - 1$ is a multiple of $\overline{x}$, i.e., when

$$\frac{f(t) - 1}{\overline{x}} = \frac{1}{2|x|^2} + it \in \mathbb{R}.$$

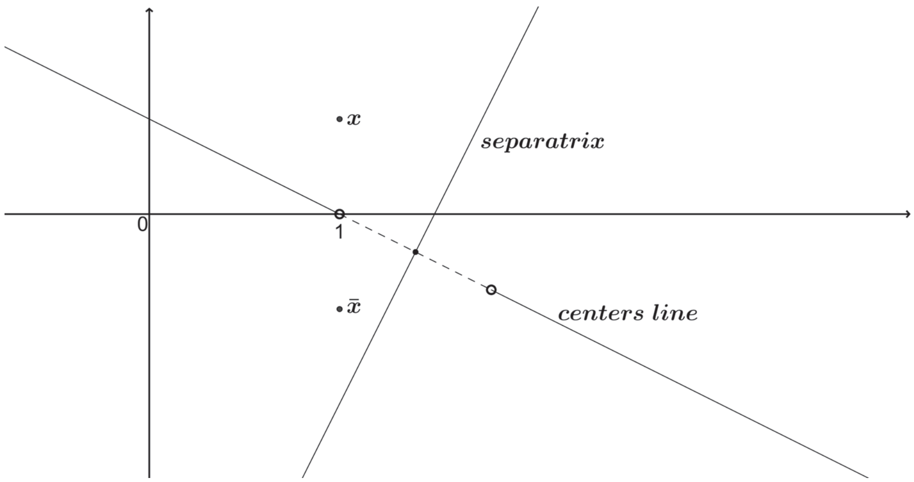

FIG. 1. *Centers line and separatrix for* $x = (1, 0.5)$.

But this happens when $t = 0$. Therefore, the intersection point of the separatrix and the centers line is

$$1 + \frac{1}{2|x|^2}\overline{x}.$$

See Figure 1.

3. Analogously, the separatrix intersect the real axis when $f(t)$ is real, which means that

$$\operatorname{Im} f(t) = t \operatorname{Re} x - \frac{\operatorname{Im} x}{2|x|^2} = 0,$$

i.e., $t = \operatorname{Im} x/2|x|^2 \operatorname{Re} x$. Furthermore, the intersection point is

$$f(\operatorname{Im} x/2|x|^2 \operatorname{Re} x) = 1 + \frac{1}{2 \operatorname{Re} x}.$$

If $\operatorname{Re} x = 0$ (i.e., if x and $\overline{x}$ are imaginary), the separatrix is parallel to the real axis and contains the point $-\frac{\operatorname{Im} x}{2|x|^2}$, which is different from zero when $x \neq 0$. In any case if $x \neq 0$, the separatrix does not contain 1.

4. The centers of the level circles are $1 + \overline{x}/(|x|^2 - r^2)$. Geometrically they are as follows:
   (a) If $r \in [0, |x|)$ (with $x \neq 0$), then the center belongs to the ray of centers line that is bounded by the singularity $1 + \overline{x}/|x|^2 = 1 + 1/x$ and does not contain 1. As r changes from 0 to $|x|$ the center moves form $1 + 1/x$ (with $r = 0$) up to ∞ when the level circle degenerates to the separatrix.
   (b) If $r \in (|x|, \infty)$, then the center belongs to the ray of centers line that is bounded by 1 and does not contain the other singularity $1 + 1/x$. The radii change from ∞ to 0 and the circles degenerate to point 1 when $r \to \infty$.
   (c) There is no center in the segment of the centers line between the singularities 1 and $1 + 1/x$.
5. If $x \notin \mathbb{R}$, then the centers line of B and the real line intercept at 1. Hence, the origin is not the center of a level circle of B. Since the level circles of A

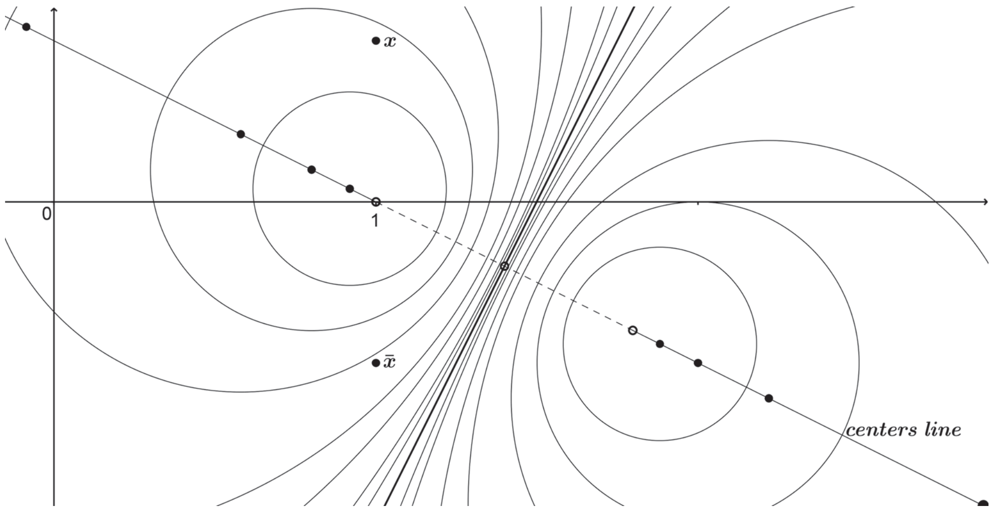

FIG. 2. *Some level circles for* $x = (1, 0.5)$.

are centered at 0 it follows that the level circles of B are different from those of A if $x \notin \mathbb{R}$.

On the other hand if $x \in \mathbb{R}$, then the level circles of B are centered in the rays of the real line that complement the interval between 1 and $1 + 1/x$. Hence 0 is a center if and only if $1 + 1/x > 0$. In this case there is just one level circle of B equal to a level circle of A, namely, $C(0, r)$ with $r^2 = x + |x|^2$. See Figure 2.

6. Suppose that $C(0, r)$, $r > 0$, is not a level circle of B, which is the case if $x \notin \mathbb{R}$ or if $1 + 1/x < 0$. Then there exists a level circle of B which intercept $C(0, r)$ in exactly two points. In fact, for any $y \in C(0, r)$ there exits a level circle C_y of B containing y, because these circles cover $\mathbb{C} \cup \{\infty\}$. If C_y is tangent to $C(0, r)$, then its center is contained in the line through 0 and y. Now, if $x \in \mathbb{R}$, then the center of C_y is real as well. Hence C_y is not tangent to $C(0, r)$ if $y \notin \mathbb{R}$. Such C_y is the required level circle of B. If $x \notin \mathbb{R}$, then C_y and C_{-y}, $\pm y \in C(0, r)$, must have the same center if both are tangent to $C(0, r)$, namely, the intersection of the line through 0 and $\pm y$ with the centers line of B. It follows that one of the level circles C_y or C_{-y} is not tangent to $C(0, r)$.

To conclude this section we look at the Lie algebra rank condition of a pair (A, B) in standard form. For this we must look at $\mathfrak{sl}(2, \mathbb{C})$ as a six-dimensional real Lie algebra, which is isomorphic to $\mathfrak{so}(1, 3)$. This is because the complexification of $\mathfrak{so}(1, 3)$ is $\mathfrak{so}(4, \mathbb{C})$, which splits into two simple components isomorphic to $\mathfrak{so}(3, \mathbb{C})$. In turn these simple components are isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. (See Helgason [7, Chapter X, section 6.4.])

We continue to denote by $\mathfrak{sl}(2, \mathbb{C})$ its own realification and check if the Lie algebra generated by (A, B) over the reals is full.

Let $\mathcal{L}_{\mathbb{C}}$ and $\mathcal{L}_{\mathbb{R}}$ be the Lie algebras generated by the matrices

$$A = \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -\gamma - 2x\gamma & 2(\gamma + x\gamma) \\ -2x\gamma & \gamma + 2x\gamma \end{pmatrix}$$

over $\mathbb{C}$ and $\mathbb{R}$, respectively. That is, $\mathcal{L}_\mathbb{C}$ is the complex subspace spanned over $\mathbb{C}$ by the successive brackets between A and B, while $\mathcal{L}_\mathbb{R}$ is the real subspace spanned over $\mathbb{R}$ by the same brackets. Hence, $\mathcal{L}_\mathbb{C} = \mathbb{C} \cdot \mathcal{L}_\mathbb{R}$ so that either $\mathcal{L}_\mathbb{R} = \mathcal{L}_\mathbb{C}$ or $\mathcal{L}_\mathbb{R}$ is a real form of $\mathcal{L}_\mathbb{C}$, that is, a real subalgebra whose complexification is $\mathcal{L}_\mathbb{C}$.

If $\alpha \neq 0 \neq \gamma$ and $x \neq 0, -1$ then an easy computation shows that A, B and $[A, B]$ are linearly independent, so that for a pair in standard form $\mathcal{L}_\mathbb{C} = \mathfrak{sl}(2,\mathbb{C})$. It follows that either $\mathcal{L}_\mathbb{R} = \mathfrak{sl}(2,\mathbb{C})$ and the pair satisfies the Lie algebra rank condition or $\mathcal{L}_\mathbb{R}$ is a real form of $\mathfrak{sl}(2,\mathbb{C})$. Up to conjugation the real forms of $\mathfrak{sl}(2,\mathbb{C})$ are $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$. Hence, if the Lie algebra rank condition is not satisfied by a pair (A, B), then $\mathcal{L}_\mathbb{R}$ is conjugate to $\mathfrak{su}(2)$ or $\mathfrak{sl}(2,\mathbb{R})$.

This discussion is helped by the trace form on $\mathfrak{sl}(2,\mathbb{C})$ which is defined by

$$\beta(C, D) = \mathrm{tr}(CD), \qquad C, D \in \mathfrak{sl}(2,\mathbb{C}).$$

It assumes real values on any conjugate of $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$. It is indeed negative definite on a compact real form conjugate to $\mathfrak{su}(2)$.

Now a direct computation on a pair (A, B) in standard form shows that

$$\beta(A, A) = 2\alpha^2, \qquad \beta(B, B) = 2\gamma^2.$$

Since for a complex number z we have that z^2 is real if and only if z is real or purely imaginary, we get at once the following sufficient condition to have $\mathcal{L}_\mathbb{R} = \mathfrak{sl}(2,\mathbb{C})$.

PROPOSITION 5.5. *The pair (A, B) in standard form satisfies the Lie algebra rank condition if α or γ is not real or purely imaginary.*

When both α and γ are real we have $\beta(A, A)$, $\beta(B, B) > 0$ and the possibilities are $\mathcal{L}_\mathbb{R} = \mathfrak{sl}(2,\mathbb{C})$ or $\mathcal{L}_\mathbb{R}$ is conjugate to $\mathfrak{sl}(2,\mathbb{R})$. In this case we get the following necessary and sufficient condition for x.

PROPOSITION 5.6. *Let (A, B) be a pair in standard form with $\alpha, \gamma \in \mathbb{R}$. Then $\mathcal{L}_\mathbb{R}$ is conjugate to $\mathfrak{sl}(2,\mathbb{R})$ if and only if $x \in \mathbb{R}$. In this case $\mathcal{L}_\mathbb{R} = \mathfrak{sl}(2,\mathbb{R})$.*

Proof. If $x \in \mathbb{R}$, then A, $B \in \mathfrak{sl}(2,\mathbb{R})$, and since A, B and $[A, B]$ are linearly independent (because of the conditions $\alpha \neq 0 \neq \gamma$ and $x \neq 0, -1$) we have that $\mathcal{L}_\mathbb{R} = \mathfrak{sl}(2,\mathbb{R})$.

Conversely, if $\mathcal{L}_\mathbb{R} = \mathrm{Ad}(g)(\mathfrak{sl}(2,\mathbb{R}))$, then g maps the real line into a circle $C \subset \mathbb{C} \cup \{\infty\}$ which is invariant by $G = \langle \exp \mathcal{L}_\mathbb{R} \rangle$. This circle must contain the singularities of A, $B \in \mathcal{L}_\mathbb{R}$ and hence contains 0, 1, and ∞. Therefore C is the real line and $x \in \mathbb{R}$. □

As an example let C and D be Hermitian matrices with C diagonal. Then $\{C, D, [C, D]\}$ spans $\mathcal{L}_\mathbb{R}$. In fact, an easy computation shows that $[C, [C, D]]$ as well as $[D, [C, D]]$ are linear combinations with real coefficients of D and C. Hence if D is not diagonal, then $[C, D] \neq 0$ and $\mathcal{L}_\mathbb{R}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Let $(a, 1)$ and $(b, 1)$ with $a = -b/|b|^2 \neq 0$ be orthogonal eigenvectors of the nondiagonal D. The correponding singularities on $\mathbb{C} = \mathbb{C}P^1$ are b and $a = -b/|b|^2$. Hence the matrix $g = t_1 s_x$ that conjugates D to its standard form maps the line through b, 0 and $-b/|b|^2$ to the real line. For this same matrix g we have $g\mathcal{L}_\mathbb{R} g^{-1} = \mathfrak{sl}(2,\mathbb{R})$.

6. Invariant control set. In this section we look at properties of the control sets of the control semigroup S generated by a pair (A, B) in standard form. Since we assume the Lie algebra rank condition $\mathcal{L}_\mathbb{R} = \mathfrak{sl}(2,\mathbb{C})$ we have int$S \neq \emptyset$ and, as mentioned, there are two control sets in S^2: An invariant one denoted by D^+, which is closed and a S^{-1} invariant denoted by D^-, which is open. The invariant control set has nonempty interior and both control sets are connected because the semigroup S is connected.

Let

$$\mathcal{L}(S) = \{X \in \mathfrak{sl}(2,\mathbb{C}) : \exp tX \in \mathrm{cl}S, t \geq 0\}$$

be the Lie cone of S. By invariance of D^+ we have $\exp tX \cdot z \in D^+$ when $t \geq 0$ and $z \in D^+$.

The next statement of a general nature will be useful to describe the control sets of a pair (A, B).

PROPOSITION 6.1. *Let γ be a simply closed curve whose image is contained in the invariant control set D^+. By the Jordan curve theorem, γ separates the plane $\mathbb{C}$ into two regions: the interior and the exterior ones, which we denote by* $\mathrm{rint}\gamma$ *and* $\mathrm{rext}\gamma$*, respectively. Take $X \in \mathcal{L}(S)$ and assume that* $\mathrm{rint}\gamma$ *does not contain a singularity or a periodic orbit of X. See Figure* 3.

Then, $\mathrm{rint}\gamma \subset D^+$*. The same statement holds for* $\mathrm{rext}\gamma$.

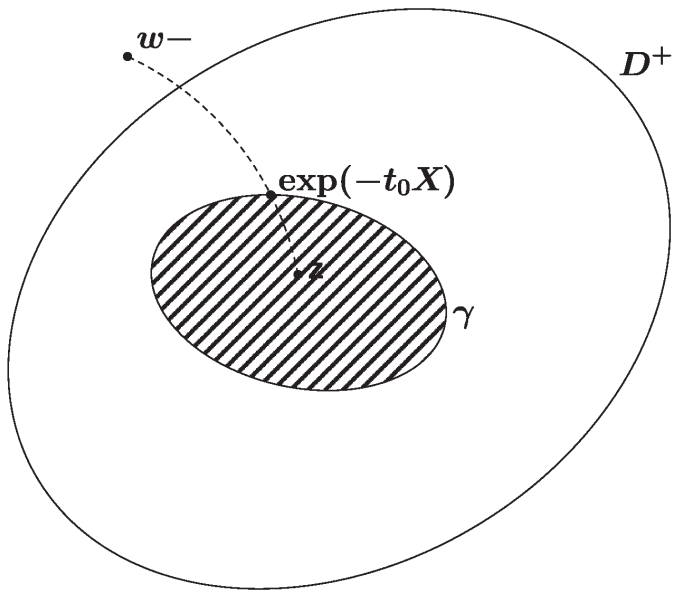

FIG. 3. *Proposition* 6.1.

Proof. By the Poincaré–Bendixon theorem, if $z \in \mathbb{C} \cup \{\infty\}$, then the limit points of $\exp tX \cdot z$, when $t \to \pm\infty$ are singularities or periodic orbits of X. Hence by assumption if $z \in \mathrm{rint}\gamma$, then limit points of $\exp tX \cdot z$, when $t \to -\infty$ are not contained to $\mathrm{rint}\gamma$. It turns out that any trajectory of X with initial condition $z \in \mathrm{rint}\gamma$ crosses the image of γ for some $-t_0 \leq 0$. Since the image of γ is contained in D^+ and D^+ is invariant by $\exp tX$, $t \geq 0$, we obtain

$$z = (\exp t_0 X)(\exp(-t_0 X)) \cdot z \in D^+$$

showing that $\mathrm{rint}\gamma \subset D^+$. ∎

Now, suppose that $X \in \mathcal{L}(S)$ is diagonalizable with eigenvalues $\pm\alpha$ with $\mathrm{Re}\,\alpha > 0$. Denote its singularities in $\mathbb{C}P^1$ by w^+ and w^- with w^+ the attractor and w^- the repeller. We have $w^+ \in D^+$ because $\lim_{t\to+\infty} e^{tA} z = w^+$ if $z \neq w^-$ and D^+ is closed and invariant. In the same way $w^- \in \mathrm{cl}D^-$.

PROPOSITION 6.2. *Let $X \in \mathcal{L}(S)$ be diagonalizable with eigenvalues $\pm\alpha$. Assume that* $\mathrm{Re}\,\alpha \neq 0$ *and* $\mathrm{Im}\,\alpha \neq 0$*. Then, its attractor w^+ belongs to* $\mathrm{int}D^+$.

Proof. We use a geometric argument based on the last proposition. Take $Y \in \mathcal{L}(S)$ such that $Y(w^+) \neq 0$. It exists by the Lie algebra rank condition. Let $\sigma = \{e^{tY}w^+ : t \in \mathbb{R}\}$ be the orbit of Y through w^+. It is not reduced to w^+. On the other hand, the trajectories of X around w^+ are spirals converging to w^+. This means that there are a point $z = e^{t_1 Y} \cdot w^+ \in \sigma$, $t_1 > 0$, and finite times $t_2 > 0$ and $t_3 > 0$ such that

$$e^{t_2 X} \cdot z = e^{t_2 X} e^{t_1 Y} \cdot w^+ = e^{t_3 Y} \cdot w^+,$$

i.e., the trajectory of X starting on z returns to $e^{t_3 Y} \cdot w^+$, $t_3 > 0$. Furthermore, from the shape of the spiral through z we get $t_3 < t_1$. Now let δ be the closed curve that starts at $z = e^{t_1 Y} \cdot w^+$, follows e^{tX} up to $t = t_2$, reaching $e^{t_3 Y} \cdot w^+$, and finally closes following σ from $e^{t_3 Y} \cdot w^+$ until $e^{t_1 Y} \cdot w^+$. Then w^+ belongs to $\mathrm{rint}\delta$, the interior region of δ. Moreover, if t_1 is small enough, then $\mathrm{rint}\delta$ does not contain a singularity or a periodic orbit of Y. Hence by Proposition 6.1, $\mathrm{rint}\delta \subset D^+$ showing that $w^+ \in \mathrm{int}D^+$. ∎

Reversing the time, that is, by taking the matrices $-A$ and $-B$, we obtain the same results for the open control set D^-, because $\mathrm{cl}D^-$ is the invariant control set for the pair $-A$, $-B$.

COROLLARY 6.3. *With the same conditions of Proposition* 6.2 *we obtain* $w^- \in D^-$.

Proof. Since w^- is an attractor of $-A$, it follows that w^- belongs to the interior invariant control set for the reversed control system. By the mentioned proposition, w^- belongs to the interior of this invariant control set which is D^-. ∎

7. Real and imaginary eigenvalues. In this section we look at controllability when the matrices have real or purely imaginary eigenvalues. First we have noncontrollability when the eigenvalues are real.

PROPOSITION 7.1. *If A and B have real eigenvalues, then the pair* (A, B) *is not controllable.*

Proof. Assume without loss of generality that the pair (A, B) is in standard form. In this case A is a real matrix. On the other hand B is real if and only if x and hence its repeller $w = 1 + 1/x$ is real, because γ is assumed to be real. Therefore if $w \in \mathbb{R}$, then the pair is not controllable because it does not satisfy the Lie algebra rank condition.

If w is not real, then there exists a unique circle, say, C, through 0, 1, and w. Let R be the region in $\mathbb{C}$ bounded by the arc C_1 of C between 0 and 1 not containing w, and the segment $[0, 1]$ in the real line. We claim that R is forward invariant by A and B.

In fact, R is A invariant because its trajectories are the rays starting at 0 converging to 0 in forward time. Since R is convex these trajectories stay in R.

To see B-invariance recall that the trajectories of B are contained in the circles through 1 (the attractor) and w (the repeller). Then C_1 is a piece of such trajectory of B converging to 1 as $t \to +\infty$. Hence C_1 is forward invariant by B. On the other hand any circle through w and 1 crosses the real line in two points, say, x and 1, and the arc between x and 1 is contained in R. Hence if a B-trajectory starts in the segment $[0, 1]$ of the real line it does not meet the segment $[0, 1]$ or C_1 in positive time and stays in R. Hence a B-trajectory starting at the boundary of R stays in R, showing its B-invariance.

Therefore R is forward invariant by A and B, showing that the pair (A, B) is not controllable. ∎

If one of the matrices, say, A, has imaginary eigenvalues, then the invariant control set is easily obtained.

PROPOSITION 7.2. *Let (A, B) be a pair in standard form with $\alpha \in i\mathbb{R}$. Then the invariant control set is an annulus*

$$D^+ = \{z \in \mathbb{C} \cup \{\infty\} : \rho_1 \leq |z| \leq \rho_2\},$$

where $0 \leq \rho_1 < \rho_2 \leq \infty$. (The inequality $\rho_1 < \rho_2$ is strict because $\mathrm{int} D^+ \neq \emptyset$*.)*

Proof. Since α is imaginary, the orbits of A are their level circles that are centered at the origin. So, D^+ is a union of such circles. Hence D^+ is a closed annulus because it is closed and connected. ∎

According to this proposition, to obtain controllability is enough to show that B does not leave invariant a proper annulus with center at the origin.

Before proceeding we note that if both A and B have imaginary eigenvalues, then the pair is controllable if and only if it satisfies the Lie algebra rank condition. This is because the 1-parameter groups of A and B are periodic and hence the semigroup is generated by $e^{\pm tA}$ and $e^{\pm tB}$, $t \geq 0$. Alternatively, if A and B have imaginary eigenvalues, then their orbits are their level circles, and the Lie algebra rank condition prevents that a level circle of A coincides with a level circle of B. Hence no annulus $\{z \in \mathbb{C} \cup \{\infty\} : \rho_1 \leq |z| \leq \rho_2\}$ is B-invariant, so the pair is controllable.

In general if the eigenvalues are imaginary for only one of the matrices controllability depends on the relative positions of the singularities. The next statement clarifies the invariant control set in this case.

PROPOSITION 7.3. *Let (A, B) be a pair in standard form and assume the eigenvalues $\pm\alpha$ of A are imaginary,* $\mathrm{Im}\,\gamma \neq 0$ *and* $\mathrm{Re}\,\gamma > 0$*. The attractor of B is* 1 *and we write $w = 1 + 1/x$ for its repeller. Then the following hold:*

1. *If $|w| = 1$, then the pair is controllable.*
2. *If $|w| > 1$ the invariant control set is a closed ball $B[0, \rho]$ with center at the origin and radius $\infty \geq \rho > 1$.*
3. *If $|w| < 1$ the control set is the complement of an open ball $B(0, \rho)$ with center at origin and radius $0 \leq \rho < 1$.*

Proof. Since 1 is the attractor of B, Proposition 6.2 ensures that $1 \in \mathrm{int} D^+$. On the other hand $\mathrm{int} D^+$ is a union of circles, thus $C(0, 1) \subset \mathrm{int} D^+$. So, if $|w| = 1$ we get $w \in \mathrm{int} D^+$. By Corollary 6.3, $w \in D^-$ so that $D^+ \cap D^- \neq \emptyset$, which implies controllability.

Now D^+ is an annulus

$$D^+ = \{z \in \mathbb{C} \cup \{\infty\} : \rho_1 \leq |z| \leq \rho_2\}.$$

If $|w| > 1$ we obtain $\rho_1 = 0$. In fact, there exists a circle $C(0, r)$ with radius $\rho_1 < r < 1$ which is contained in D^+ because $1 \in \mathrm{int} D^+$. If $|w| > 1$, then a ball $B(0, r)$ inside $C(0, r)$ does not contain singularities of B. So, by Proposition 6.1 $B(0, r) \subset D^+$. Therefore, $\rho_1 = 0$. The case $|w| < 1$ follows by a similar application of Proposition 6.1, considering the exterior of a circle. ∎

As an immediately consequence, the pair (A, B) is not controllable if and only if B leaves invariant a ball or a complement of a ball depending on $|w| > 1$ or $|w| > 1$.

COROLLARY 7.4. *Assume that A has imaginary eigenvalues. Then, the pair (A, B) given in the standard form is not controllable if and only if $(\exp tB)C \subset C$, $t \geq 0$, where*

1. *$C = B[0, \rho]$ with $\rho > 1$ if $|w| > 1$; in this case $\rho \leq |w|$;*
2. *C is the complement of $B[0, \rho]$ with $\rho < 1$ if $|w| < 1$.*

Proof. We just need to observe that if $|w| > 1$ there exits an invariant ball when $\rho \leq |w|$. In fact, if $\rho > |w|$ the repeller w of B belongs to the ball $B[0,\rho]$ and this ball cannot be invariant. □

Remark. If A has imaginary eigenvalues, then its 1-parameter semigroup is periodic and hence the semigroup is generated by $e^{\pm tA}$ and e^{tB}, $t \geq 0$, as happens to a control system with unrestricted controls considered in [9] and [10]. The conditions of these papers do not apply here since they would require that the eigenvalues of A to have nonzero real part.

8. Balls invariant by diagonalizable elements. Let X be a diagonalizable matrix with eigenvalues $\pm\delta$ and $\operatorname{Re}\delta > 0$. Denote by w^+ the attractor of X and by w^- the repeller. The problem here is to find the balls $B[c,r]$ which are invariant by X, in positive time, i.e.,

$$(\exp tX)B[c,r] \subset B[c,r] \qquad \text{if} \qquad t \geq 0.$$

Among these balls we include

1. the halfspaces bounded by a line,
2. the complement of the open balls in $\mathbb{C}$ which are seen as closed balls with center $c = \infty$.

We observe that a halfspace and complement of balls are images of compact balls in $\mathbb{C}$ by Möbius transformations.

Proposition 8.1. *In the following cases, the ball $B[c,r]$ is not invariant by X:*

1. $w^- \in B(c,r) = \operatorname{int}B[c,r]$.
2. $w^+ \notin B[c,r]$.

Proof. For the first case we have

$$\lim_{t\to-\infty}(\exp tX)z = w^-$$

if $z \neq w^+$. Hence there are $z \notin B[c,r]$ and $t > 0$ such that $z' = \exp(-tX)\cdot z \in B(c,r)$. It follows that $(\exp tX)z' = z \notin B[c,r]$, despite the fact $z' \in B[c,r]$. The proof of the second case is similar. The trajectories in $B[c,r]$ approach w^+ in positive time. □

The following example shows that it is possible to have invariance even when both fixed points $w^\pm$ belong to the boundary.

Example. Consider the matrix

$$X = \begin{pmatrix} -\delta & 0 \\ 0 & \delta \end{pmatrix}, \qquad \delta \in \mathbb{R}.$$

Thus, $w^+ = 0$, $w^- = \infty$ and the trajectories of X remain inside lines through the origin. Therefore, the halfspace $\{(x,y) : y \geq x\}$ is invariant (in positive and negative time). The fixed points $w^\pm$ belong to the boundary $\{y = x\}$. Furthermore, any halfspace

$$\{(x,y) : y \leq c\}, \qquad c > 0,$$

is positively invariant and the repeller ∞ belongs to the boundary.

If the eigenvalues are real we obtain a simple criteria for invariance.

Proposition 8.2. *Let us assume that X has real eigenvalues and let $B[c,r]$ a closed ball. Then $(\exp tX)B[c,r] \subset B[c,r]$, $t \geq 0$, if and only if $w^+ \in B[c,r]$ and $w^- \notin B(c,r)$. Furthermore, we have the following:*

1. *$B[c,r]$ is forward and backward invariant if and only if $w^\pm$ belongs to the boundary $C(c,r)$ of the ball.*
2. *If $|w^+ - c| < r$ and $|w^- - c| > r$, then $(\exp tX)B[c,r] \subset B(c,r)$ for each $t > 0$.*

Proof. If the eigenvalues of X are real, then its orbits are contained in the circles through w^+ and w^- (e.g., the lines through 0 if X is diagonal). Inside any of these circles there are four orbits: the two singularities $w^\pm$ and the two arcs determined by them.

If $w^\pm \in C(c,r)$ no trajectory crosses $C(c,r)$, which means that the interior and the exterior of $B[c,r]$ are invariant regions for any $t \in \mathbb{R}$.

If $|w^+ - c| \leq r$ and $|w^- - c| > r$, then the circles through w^+ and w^- cross $C(c,r)$ in exactly two points and hence every trajectory of X cross $C(c,r)$ just once, since these trajectories start in w^- and end in w^+. Therefore, any trajectory starting in $B[c,r]$ remains there. In case of strict inequality $|w^+ - c| < r$, the trajectories of X starting in $C(c,r)$ go inside $B(c,r)$, showing item 2. ∎

We now apply the case of real eigenvalues to look at an arbitrary diagonalizable X with complex eigenvalues. When the eigenvalues of X are not real, the trajectories are spirals around w^+ and w^-. Thus, $B[c,r]$ is not invariant by X if w^+ or w^- belong to the boundary $C(c,r)$. In other words, the only possibility for invariance corresponds to the case $w^+ \in B(c,r)$ and $w^- \notin B[c,r]$.

Assume the eigenvalues $\pm\delta$ of X are not real with

$$w^+ \in B(c,r) \qquad \text{and} \qquad w^- \notin B[c,r].$$

The analysis of the invariance property is based on the following considerations:

1. The matrix X has the form
$$X = g \begin{pmatrix} -\delta & 0 \\ 0 & \delta \end{pmatrix} g^{-1} = \delta g \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g^{-1} = \delta Y,$$
where Y has eigenvalues ± 1. The fixed points of X and Y are the same and if $\operatorname{Re}\delta > 0$, then 0 is the attractor of Y. The corresponding vector fields on $\mathbb{C}P^1$ (also denoted by X and Y) satisfy $X = \delta Y$. Write $\delta = \rho e^{i\phi}$ with $\rho = |\delta|$ and $\phi = \arg\delta$ which we take in the interval $(-\pi,\pi)$. The vector fields induced by X and Y are related by $X = e^{i\phi}(\rho Y)$. Now, for any z, the vector $\rho Y(z)$ has the same direction as $Y(z)$, and the direction of $X(z)$ is obtained by rotating $Y(z)$ by the angle $\arg\delta$.
2. Since Y has real eigenvalues, and we take $w^+ \in B(c,r)$ and $w^- \notin B[c,r]$, we have that for any $z \in C(c,r)$ the vector $Y(z)$ points to the interior of $B[c,r]$. For each $z \in C(c,r)$ we let $\theta(z)$ with $0 < \theta(z) < \pi$ be the angle between $Y(z)$ and the tangent line to $C(c,r)$ at z, measured counterclockwise. By continuity and compactness the angles $\theta(z)$ run through an interval $[\theta_{\min}, \theta_{\max}] \subset (0,\pi)$ as z runs through $C(c,r)$.
These angles $\theta_{\min}$ and $\theta_{\max}$ are called *critical angles* of Y with respect to $C(c,r)$. (We show below that $\theta_{\min} + \theta_{\max} = \pi$.)
Now $X = e^{i\phi}(\rho Y)$ so that a vector $X(z)$, $z \in C(c,r)$, points to the interior of $B[c,r]$ if and only if $0 < \arg\delta + \theta(z) < \pi$. Hence, X points to the interior of $B[c,r]$ at all points in the boundary if and only if

$$-\theta_{\max} < \arg \delta < \pi - \theta_{\min}.$$

This is the condition ensuring that $B[c, r]$ is forward X-invariant.

The angles $\theta_{\min}$ and $\theta_{\max}$ are obtained from basic facts in plane geometry. Let C be a circle and l a line intersecting C at P and Q. These points divide C into the circular arcs C_1 and C_2. The arc C_1 defines the inscribed angle $\theta = \widehat{PSQ}$, $S \in C_1$. The inscribed angle defined by C_2 is $\pi - \theta$.

Now let R be a point in the interior of the circle C. Then any line l through R crosses C in P and Q determining inscribed angles $\theta(l)$ and $\pi - \theta(l)$. We choose $\theta(l) \geq \pi/2$ and $\pi - \theta(l) \leq \pi/2$.

LEMMA 8.3. *Fix a point R in the interior of C. Then among the lines l containing R, $\theta(l)$ is maximum when l is orthogonal to the diameter of C containing R. See Figure* 4.

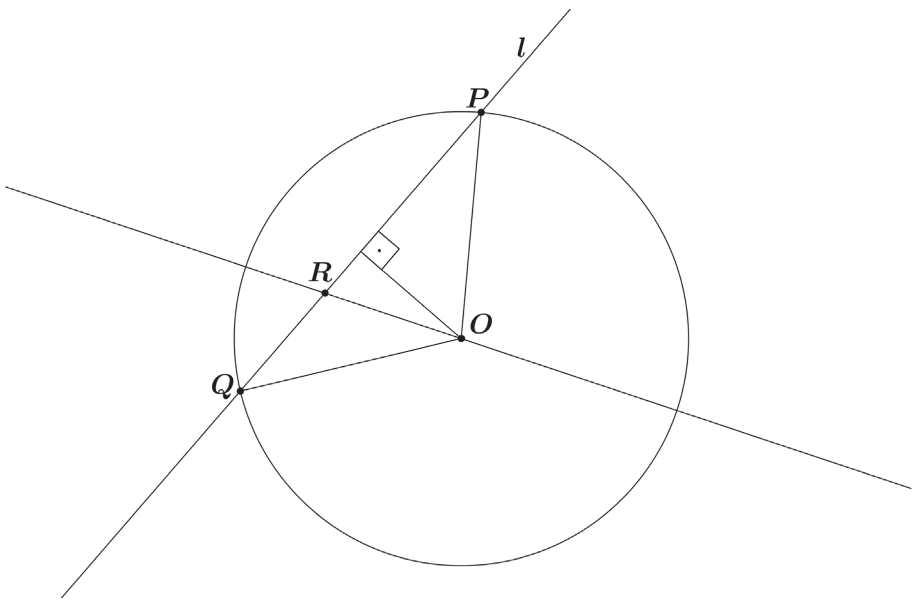

FIG. 4. *Lemma* 8.3.

Proof. If R is the center O of C, then $\theta(l) = \pi/2$ for any line l and there is nothing to prove. For $R \neq O$, $\theta(l)$ is maximum when the central angle $\widehat{POQ} = 2(\pi - \theta(l))$ is minimum. The triangle POQ is isosceles. Hence the angle $2(\pi - \theta(l)) - \widehat{POQ}$ when the height with respect to the side PQ is maximum. This happens precisely when RO is the height of the triangle PQO, that is, when PQ is orthogonal to RO. ∎

Now we can find the angles $\theta_{\min}$ and $\theta_{\max}$ and relate them. We consider first the case where Y is diagonal.

PROPOSITION 8.4. *Let $C = C(c, r)$ be a circle containing 0 in its interior (that is $|c| < r$) and take*

$$Y = \begin{pmatrix} -\rho & 0 \\ 0 & \rho \end{pmatrix}, \qquad 0 < \rho \in \mathbb{R}.$$

Then, $\theta_{\min} + \theta_{\max} = \pi$ and they are given as follows: Let l_1 be the line through 0 orthogonal to the diameter of $C(c, r)$ and denote by P and Q its intersections with

$C(c,r)$. Let l_2 be the tangent to $C(c,r)$ through P (or Q). Then $\theta_{\min}$ is the angle $\leq \pi/2$ formed by l_1 and l_2. See Figure 5.

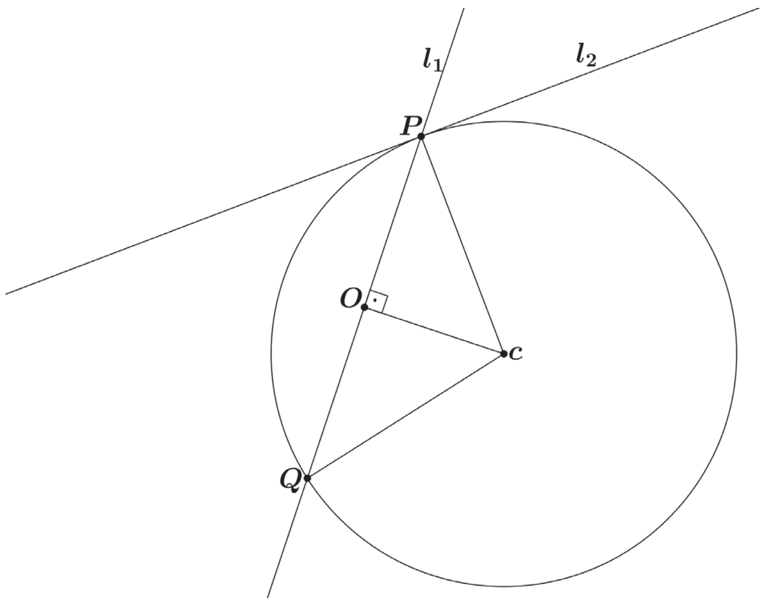

FIG. 5. *Proposition* 8.4.

Proof. The trajectories of Y follow straight lines through 0. Hence the proposition follows directly from the previous lemma by taking R to be $0 \in \mathbb{C}$. In fact, among the lines l through 0 the one which has the largest angle $\theta(l) \geq \pi/2$ with the tangent at its intersection is the line l_1 orthogonal to the diameter containing 0. Its complement is the smallest such angle and hence is $\theta_{\min}$ and $\theta_{\min} + \theta_{\max} = \pi$, as claimed. ∎

Now we can take conjugations and look at the angles $\theta_{\min}$ and $\theta_{\max}$ for an arbitrary Y with real eigenvalues. To this purpose we recall that a Möbius function is conformal, that is, preserves angles.

PROPOSITION 8.5. *Let Y be diagonalizable with eigenvalues $\pm\rho$, $\rho > 0$, and denote by w^+ and w^- its attractor and repeller singularities on $\mathbb{C} \cup \{\infty\}$, respectively. Suppose that $w^+ \in B(c,r)$ and $w^- \notin B[c,r]$. Then $\theta_{\min} + \theta_{\max} = \pi$ and these angles are given as follows: Let C_1 be the only circle through w^+, w^-, and c (which is a line if the points are collinear). Also, let C_2 be the circle through w^+, w^- and be orthogonal to C_1 at w^+. Denote by P and Q the intersections of C_1 with $C(c,r)$. Then $\theta_{\min}$ and $\theta_{\max}$ are the angles ($\leq \pi/2$ and $\geq \pi/2$, respectively) between C_1 and the diameter of $C(c,r)$ through P (or Q).*

Proof. Take $g \in \mathrm{Sl}(2,\mathbb{C})$ such that $Y = gY_0g^{-1}$ with

$$Y_0 = \begin{pmatrix} -\rho & 0 \\ 0 & \rho \end{pmatrix}.$$

Then $g(0) = w^+$, $g(\infty) = w^-$, and $gC(c',r') = C(c,r)$ for a circle $C(c',r')$ having 0 in its interior. Since g is conformal the critical angles of Y with respect to $C(c,r)$ are equal to the critical angles of Y_0 with respect to $C(c',r')$. Moreover, under g the lines l_1 and l_2 of Proposition 8.4 correspond to the circles C_1 and C_2 in the statement. Hence the result is a consequence of Proposition 8.4. ∎

Summarizing the above discussion, the following steps are used to decide whether a diagonalizable X, with eigenvalues $\pm\delta$, leaves invariant a ball $B[c,r]$:

1. Check if $w^+ \in B(c,r)$ and $w^- \notin B(c,r)$, where w^+ and w^- are the attractor and repeller singularities of X, respectively.
2. Write the matrix Y having the same eigenvectors as X but with eigenvalues $\pm|\delta|$.
3. Find the critical angles $\theta_{\min}$ and $\theta_{\max}$ of Y with respect to $B[c,r]$ according to the recipe given in Proposition 8.5 (or in Proposition 8.4 if X is already diagonal).
4. If

$$-\theta_{\max} < \arg\delta < \pi - \theta_{\min} = \theta_{\max},$$

then $B(c,r)$ is forward invariant by X.

We write θ_{crit} for the critical angle $\theta_{\min}$.

9. An open set of noncontrollable pairs in $\mathfrak{sl}(2,\mathbb{C})$. In the classical papers by Jurdjevic and Kupka [9], [10], they show that for the class of unrestricted admissible controls, the controllable pairs in $\mathfrak{sl}(2,\mathbb{C})$ are dense (see also [17]). In this section we show that for the restricted case this situation in no longer true.

As usual $\text{Re}(\alpha)$, $\text{Im}(\alpha)$ denote the real and the imaginary parts of $\alpha \in \mathbb{C}$ and $\arg(\alpha)$ its argument, i.e.,

$$\alpha = \text{Re}(\alpha) + \text{Im}(\alpha)i,$$
$$\arg(\alpha) = \tan^{-1}\left(\frac{\text{Im}(\alpha)}{\text{Re}(\alpha)}\right).$$

Given α, γ, $x \in \mathbb{C}\backslash\{0\}$ with $\text{Re}(\alpha) > 0$, $\text{Re}(\gamma) > 0$, and $x \neq \pm 1$, consider the pair in standard form:

$$A(\alpha) = \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad B(x,\gamma) = \begin{pmatrix} -(\gamma + 2x\gamma) & 2(\gamma + x\gamma) \\ -2x\gamma & \gamma + 2x\gamma \end{pmatrix}.$$

THEOREM 9.1. *There exits an open subset of* $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ *of noncontrollable pairs of matrices in standard form* $A(\alpha)$, $B(x,\gamma)$. *Precisely, there are an open set* $\Lambda \subset \mathfrak{sl}(2,\mathbb{C})$ *in the space of the diagonal matrices and an open set* $\Theta \subset \mathfrak{sl}(2,\mathbb{C})$ *whose elements have* 1 *as the attractor and their repellers are different from* ∞, *such that any pair*

$$(A(\alpha), B(x,\gamma)) \in \Lambda \times \Theta$$

is not controllable

Proof. We first prove the existence of open complex sets U, V, W such that for any $(x,\gamma) \in V \times W$ there are $c \in \mathbb{C}$, $\rho > 0$ and a level circle $C(c,\rho)$ of $B(x,\gamma)$ which is invariant for $A(\alpha)$. In other words,

$$\exp(tA(\alpha)B_c(\rho) \subset B_c(\rho) \text{ for each } t > 0 \text{ and } \alpha \in U.$$

We start to compute the angle θ_{crit} of $A(\alpha)$ for a level circle $C(c,\rho)$ of $B(x,\gamma)$. Let

$$l_1 = \{tic : t \in \mathbb{R}\}$$

be the line through the origin orthogonal to c, i.e., l_1 is orthogonal to the diameter through the origin. The intersection points between the circle $C(c,\rho)$ and the line l_1 are given by the different values of t that satisfy

$$|tic - c| = \rho \Leftrightarrow t = \pm\sqrt{\left(\frac{\rho}{|c|}\right)^2 - 1}.$$

Next we consider $t_0 = \sqrt{\left(\frac{\rho}{|c|}\right)^2 - 1}$, $P = t_0 ic$ and

$$l_2 = \{c + t(t_0 ic - c) : t \in \mathbb{R}\}$$

the line through the points c and P, i.e., the normal line to the circle $C(c,\rho)$ at P. The angle θ_{crit} is the smallest angle between the lines l_1 and l_2. We have

$$\begin{aligned}\cos(\theta_{\text{crit}}) &= \frac{\langle ic, t_0 ic - c\rangle}{|ic|\ |t_0 ic - c|} = \frac{t_0}{\sqrt{(t_0)^2+1}} \\ &= \sqrt{\left(\frac{\rho}{|c|}\right)^2 - 1}\sqrt{\left(\frac{|c|}{\rho}\right)^2}.\end{aligned}$$

Therefore,

$$\theta_{\text{crit}} = \arccos\left(\sqrt{1 - \left(\frac{|c|}{\rho}\right)^2}\right).$$

Now the center c and the radius ρ of a level circle of $C(c,\rho)$ are given by two parameters x and r as

$$c = 1 + \frac{\overline{x}}{|x|^2 - r^2} \text{ and } \rho = \frac{r}{||x|^2 - r^2|}.$$

From this we get

$$\theta_{\text{crit}} = \arccos\left(\sqrt{\frac{r^2 - ||x|^2 - r^2 + \overline{x}|^2}{r^2}}\right).$$

Consider $r = \frac{1}{2}$. It follows that θ_{crit} is a continuous function of $x \in \mathbb{C}$. On the other hand, by fixing $r = \frac{1}{4}$, any $x \in \mathbb{C}$ determine one level circle of $B(x,\gamma)$. In what follows we prove that θ_{crit} is well defined on the ball $B_{\frac{1}{2}}(0)$. In fact, letting $x \in B_{\frac{1}{2}}(0)$, we have

$$\frac{1}{2} - \left(|\overline{x}| - \frac{1}{2}\right)^2 = \frac{1}{4} - |\overline{x}|^2 + |\overline{x}| = \left||x|^2 - \frac{1}{4}\right| + |\overline{x}| \geq \left||x|^2 - \frac{1}{4} + \overline{x}\right|.$$

Therefore, $\left||x|^2 - \frac{1}{4} + \overline{x}\right| < \frac{1}{2}$.

Let $x_0 \in B_{\frac{1}{2}}(0)$ and $\epsilon_0 = \frac{1}{4}(\frac{\pi}{2} - \theta_{\text{crit}}(x_0))$. By continuity of $\theta_{\text{crit}} : B_{\frac{1}{2}}(0) \to (0, \frac{\pi}{2})$ there exits $\delta_0 > 0$ such that

$$\theta_{\text{crit}}(B_{\delta_0}(0) \cap B_{\frac{1}{2}}(0)) \subset (\theta_{\text{crit}}(x_0) - \epsilon_0, \theta_{\text{crit}}(x_0) + \epsilon_0).$$

Let $V = B_{\delta_0}(0) \cap B_{\frac{1}{2}}(0)$, $W = \mathbb{C}^+$ be the set of the complex numbers γ such that $\mathrm{Re}(\gamma) > 0$ and

$$U = \left\{ \alpha \in \mathbb{C} : \arg(\alpha) \in \left(\frac{\epsilon_0}{4}, \frac{\epsilon_0}{2} \right) \right\}.$$

Then

$$\theta_{\mathrm{crit}}(x) + \arg(\alpha) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \text{ for each } x \in V \text{ and } \alpha \in U.$$

It follows by Proposition 8.4 that the open ball determined by the level circle of the matrix $B(x,\gamma)$ of radius $\rho = \frac{2}{1-4|x|^2}$ and center $c = 1 + \frac{4\overline{x}}{4|x|^2-1}$ is invariant by $A(\alpha)$ for any $\alpha \in U$, $x \in V$, and $\gamma \in W$, concluding the proof. ∎

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