Dynamics of endomorphisms of Lie groups

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Abstract For a given endomorphism φ of a connected Lie group G this paper studies subgroups of G that are intrinsically connected with the dynamical properties of φ.

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1 Introduction

In [1] was shown that associated to a given continuous flow of automorphisms of a connected Lie group G there are subgroups of G that are intrinsically connected with the dynamics of the flow. The author shows there that only by looking at such subgroups one can get information about the controllability of any control system whose drift generates 1-parameter flow of automorphisms. In the present paper we extend such results by showing that for any endomorphism of G, one can also define such subgroups and they still shares many of the properties of the continuous case. Moreover, such decompositions generalizes the one for linear maps on Euclidean spaces.

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The paper is structured as follows: In Section 2 we introduce the subalgebras induced by an arbitrary endomorphism ϕ of a given Lie algebra $\mathfrak{g}$ and show that it decomposes $\mathfrak{g}$ in a dynamical way. In Section 3 we show that the decompositions on the Lie algebra can be carried on to a connected Lie group and a given endomorphism φ of G. Then we establish the main properties of them.

2 Endomorphisms of Lie algebra

The aim of this section is to introduce Lie subalgebras induced by an endomorphism on a given Lie algebra and to show their main properties.

Let $\mathfrak{g}$ be a Lie algebra of dimension d and assume that $\phi : \mathfrak{g} \to \mathfrak{g}$ is an endomorphism of $\mathfrak{g}$, that is, ϕ is a linear map satisfying $\phi[X, Y] = [\phi X, \phi Y]$ for any $X, Y \in \mathfrak{g}$.

2.1 Proposition: *Let $\mathfrak{g}$ be a Lie algebra over a closed field and $\phi : \mathfrak{g} \to \mathfrak{g}$ be a endomorphism. For any eigenvalue α of ϕ consider the generalized eigenspace of ϕ given by*

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g};\ (\phi - \alpha)^n X = 0, \quad \textit{for some}\ \ n \geq 1\}.$$

If β is also an eigenvalue of ϕ then

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha\beta}, \tag{1}$$

where $\mathfrak{g}_{\alpha\beta} = \{0\}$ if $\alpha\beta$ is not an eigenvalue of ϕ.

Proof: Decompose any eigenspace $\mathfrak{g}_\lambda$ of ϕ in its Jordan components, that is, consider linearly independent vectors $Z_1, \ldots Z_r \in \mathfrak{g}_\lambda$ such that

$$\phi(Z_j) = \lambda Z_j + Z_{j-1}, \ \ j = 1, \ldots r \quad \text{with} \quad Z_0 = 0.$$

In order to prove the proposition is then enough to show that if

$$\{X_1, \ldots, X_n\} \subset \mathfrak{g}_\alpha \quad \text{and } \{Y_1, \ldots, Y_m\} \subset \mathfrak{g}_\beta$$

are linearly independ sets as above, we have

$$[X_i, Y_j] \subset \mathfrak{g}_{\alpha\beta}, \quad i = 1, \ldots, n;\ j = 1, \ldots, m.$$

The prove is done by induction on the sum $i + j$. Since

$$\phi[X_i, Y_j] = [\phi X_i, \phi Y_j] = [\alpha X_i + X_{i-1}, \beta Y_j + Y_{j-1}]$$

$$= \alpha\beta[X_i, Y_j] + \alpha[X_i, Y_{j-1}] + \beta[X_{i-1}, Y_j] + [X_{i-1}, Y_{j-1}]$$

we have

$$(\phi - \alpha\beta)[X_i, Y_j] = \alpha[X_i, Y_{j-1}] + \beta[X_{i-1}, Y_j] + [X_{i-1}, Y_{j-1}]. \tag{2}$$

If $i = j = 1$ we have that $(\phi - \alpha\beta)[X_1, Y_1] = 0$ which implies $[X_1, Y_1] \in \mathfrak{g}_{\alpha\beta}$. Let us assume that the result holds for $i + j < n$ and let $i + j = n$. By the induction hypothesis, every term in the right-side of equation (2) are in $\mathfrak{g}_{\alpha\beta}$ which implies that $(\phi - \alpha\beta)[X_i, Y_j] \in \ker((\phi - \alpha\beta)^n)$ for some $n \geq 1$. Consequently

$$(\phi - \alpha\beta)^{n+1}[X_i, Y_j] = 0$$

showing that $[X_i, Y_j] \in \mathfrak{g}_{\alpha\beta}$ and concluding the proof. □

2.2 Corollary: *The above proposition remains valid if $\mathfrak{g}$ is a real Lie algebra.*

Proof: Consider the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. Since the elements of $\mathfrak{g}_{\mathbb{C}}$ are of the form $\sum_j a_j X_j$ with $a_j \in \mathbb{C}$, $X_j \in \mathfrak{g}$ we can define the homomorphism $\phi_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ given by

$$\phi_{\mathbb{C}}\left(\sum_j a_j X_j\right) := \sum_j a_j \phi(X_j).$$

Using the fact that $X + iY = 0$ if and only if $X = Y = 0$ we have that the eigenspaces of ϕ and $\phi_{\mathbb{C}}$ are related by $(\mathfrak{g}_\alpha)_{\mathbb{C}} = (\mathfrak{g}_{\mathbb{C}})_\alpha$. Moreover, since $[(\mathfrak{g}_\alpha)_{\mathbb{C}}, (\mathfrak{g}_\beta)_{\mathbb{C}}] = [\mathfrak{g}_\alpha, \mathfrak{g}_\beta]_{\mathbb{C}}$ the result follows from Proposition 2.1 above. □

We also have the following primary decomposition.

2.3 Proposition: *Let ϕ be an automorphism of $\mathfrak{g}$ and consider its Jordan decomposition*

$$\phi = \phi^S \phi^N = \phi^N \phi^S$$

with ϕ^S semisimple and ϕ^N unipotent. Then ϕ^S and ϕ^N are automorphisms.

Proof: We can assume w.l.o.g. that the field of the scalars is algebraically closed. Then, in order to prove that ϕ^S is an automorphism, it is enough to show that $\phi^S([X, Y]) = [\phi^S(X), \phi^S(Y)]$ for every couple of basis elements. Being that $\mathfrak{g}$ is decomposed in generalized eigenspaces of ϕ it is enough to show that ϕ^S satisfies the property of automorphisms for $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$ and α, β eigenvalues of ϕ. Moreover, from Proposition 1 $[X, Y] \in \mathfrak{g}_{\alpha\cdot\beta}$ and since the eigenspaces of ϕ and of ϕ^S coincide, we get that

$$\phi^S([X, Y]) = \alpha \cdot \beta[X, Y] \text{ and } [\phi^S(X), \phi^S(Y)] = [\alpha X, \beta Y] = \alpha \cdot \beta[X, Y]$$

showing that ϕ^S is in fact an automorphism. Therefore $\phi^N = \left(\phi^S\right)^{-1} \phi$ is also an automorphism concluding the proof. □

The above results allow to associate to any endomorphism ϕ several Lie subalgebras that are intrinsically connected with its dynamics, defined as:

$$\mathfrak{g}_\phi = \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha, \quad \mathfrak{k}_\phi = \ker(\phi^d)$$

$$\mathfrak{g}^+ := \bigoplus_{|\alpha|>1} \mathfrak{g}_\alpha, \quad \mathfrak{g}^0 := \bigoplus_{\alpha;|\alpha|=1} \mathfrak{g}_\alpha \quad \mathfrak{g}^- := \bigoplus_{0<|\alpha|<1} \mathfrak{g}_\alpha,$$

$$\mathfrak{g}^{+,0} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \quad \text{and} \quad \mathfrak{g}^{-,0} = \mathfrak{g}^- \oplus \mathfrak{g}^0.$$

We also have that $\mathfrak{g}_\phi = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$ and $\mathfrak{g} = \mathfrak{g}_\phi \oplus \mathfrak{k}_\phi$. By the property (1) is easy to see that the above subspaces are in fact Lie subalgebras and that $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are nilpotent.

2.4 Remark: We should note that the restriction of $\phi|_{\mathfrak{g}_\phi}$ is an automorphism of $\mathfrak{g}_\phi$. Also, the restriction of ϕ to the Lie subalgebras $\mathfrak{g}^+$, $\mathfrak{g}^0$ and $\mathfrak{g}^-$ satisfies:

$$|\phi^m(X)| \geq c\mu^{-m}|X| \quad \text{for any} \quad X \in \mathfrak{g}^+, m \in \mathbb{N},$$

and

$$|\phi^m(Y)| \leq c^{-1}\mu^m|Y| \quad \text{for any} \quad Y \in \mathfrak{g}^-, m \in \mathbb{N},$$

for some $c \geq 1$ and $\mu \in (0,1)$ and, for all $a > 0$ and $Z \in \mathfrak{g}^0$ it holds that

$$|\phi^m(Z)|\, \mu^{a|m|} \to 0 \quad \text{as } m \to \pm\infty.$$

The next proposition shows that any linear map that commutes two endomorphisms preserves the above subalgebras.

2.5 Proposition: *Let $\phi_i : \mathfrak{g}_i \to \mathfrak{g}_i$ be an endomorphism of the Lie algebra $\mathfrak{g}_i$, $i = 1, 2$, and let $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a surjective linear map such that $f \circ \phi_1 = \phi_2 \circ f$. It holds that*

$$f(\mathfrak{g}_{\phi_1}) = \mathfrak{g}_{\phi_2}, \quad f(\mathfrak{k}_{\phi_1}) = \mathfrak{k}_{\phi_2},$$
$$f(\mathfrak{g}_1^+) = \mathfrak{g}_2^+, \quad f(\mathfrak{g}_1^0) = \mathfrak{g}_2^0 \quad \textit{and} \quad f(\mathfrak{g}_1^-) = \mathfrak{g}_2^-.$$

Proof: Let α be an eigenvalue of ϕ_1 and $X \in \mathfrak{g}_\alpha$. There exists then $n \geq 1$ such that $(\phi_1 - \alpha)^n X = 0$. By the commutating property, we get

$$(\phi_2 - \alpha)^n f(X) = f\left((\phi_1 - \alpha)^n X\right) = f(0) = 0.$$

Consequently $f((\mathfrak{g}_1)_\alpha) \subset (\mathfrak{g}_2)_\alpha$, where $(\mathfrak{g}_2)_\alpha = \{0\}$ when α is not an eigenvalue of ϕ_2. In particular we get

$$f(\mathfrak{g}_{\phi_1}) \subset \mathfrak{g}_{\phi_2}, \;\; f(\mathfrak{k}_{\phi_1}) \subset \mathfrak{k}_{\phi_2}, \;\; f(\mathfrak{g}_1^+) \subset \mathfrak{g}_2^+, \;\; f(\mathfrak{g}_1^0) \subset \mathfrak{g}_2^0 \;\; \text{and} \;\; f(\mathfrak{g}_1^-) \subset \mathfrak{g}_2^-.$$

Since $\mathfrak{g}_i = \mathfrak{g}_{\phi_i} \oplus \mathfrak{k}_{\phi_i}$ for $i = 1, 2$ and f is a surjective linear map, we must have the equality $f(\mathfrak{g}_{\phi_1}) = \mathfrak{g}_{\phi_2}$ and $f(\mathfrak{k}_{\phi_1}) = \mathfrak{k}_{\phi_2}$. By restricting f to $\mathfrak{g}_{\phi_1}$ we get the other equalities concluding the proof. □

3 Endomorphisms of Lie groups

For given Lie groups G, H a continuous map $\varphi : G \to H$ is said to be a **homomorphism** if it preserves the group structure, that is, $\varphi(gh) = \varphi(g)\varphi(h)$ for any $g, h \in G$. If $G = H$ such map is said to be an **endomorphism** of G. Our aim here is to show that associated with any endomorphism of a connected Lie group G there are connected Lie subgroups which contain most of the dynamical information of the endomorphism. It will always be assumed that the Lie groups and subgroups are connected.

3.1 Definition: *Let G, H be Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{h}$, respectively, and $\varphi : G \to H$ be an homomorphism. If there are constants $c \geq 1$ and $\mu \in (0, 1)$ such that*

$$|(d\varphi)_e^m X| \leq c^{-1}\mu^m |X|, \qquad \text{for any} \quad m \in \mathbb{Z}^+, X \in \mathfrak{g}$$

we say that φ is **contracting**. *On the other hand, if*

$$|(d\varphi)_e^m X| \geq c\mu^{-m}|X|, \qquad \text{for any} \quad m \in \mathbb{Z}^+, X \in \mathfrak{g}$$

the homomorphism φ is said to be **expanding**.

The next lemma characterizes some topological property on Lie subgroups that will be needed in the next sections.

3.2 Lemma: *Let G be a Lie group and H and K Lie subgroups of G with Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$, respectively such that $\mathfrak{h} \oplus \mathfrak{k} = \mathfrak{g}$. Then H and K are closed subgroups of G if and only if $H \cap K$ is a discrete subgroup of G;*

Proof: (i) If H and K are closed subgroups of G, their intersection $H \cap K$ is a closed Lie subgroup of G. Since $\dim(H \cap K) = 0$ the result follows.

Reciprocally, let us assume that $H \cap K$ is a discrete subgroup of G. By Proposition 6.7 of [6] and by the hypothesis on $\mathfrak{h}$ and $\mathfrak{k}$ there exist open neighborhoods $0 \in U \subset \mathfrak{h}$, $0 \in V \subset \mathfrak{k}$ and $e \in W \subset G$ such that the map $f : U \times V \to W$ defined by $f(X, Y) = \mathrm{e}^X \mathrm{e}^Y$ is a diffeomorphism. We can assume w.l.o.g. that W is small enough such that $W \cap (H \cap K) = \{e\}$. In particular, if $g = xy$ with $x \in \mathrm{e}^U \subset H$ and $y \in \mathrm{e}^V \subset K$ is such that $g \in W \cap H$ we obtain

$$K \ni y = x^{-1}g \in H \Rightarrow y \in W \cap (H \cap K) = \{e\}$$

showing that $H \cap W = \mathrm{e}^U = f(U \times \{0\})$. Thus $H \cap W$ is closed in W, since $U \times \{0\}$ is closed in $U \times V$. Consequently $H \cap W = \mathrm{cl}(H) \cap W$ and therefore H has nonemtpy interior in $\mathrm{cl}(H)$ which only happens if $H = \mathrm{cl}(H)$ showing that H is in fact a closed subgroup of G. Analogously we show that K is a closed subgroup of G as stated. $\square$

3.3 Definition: *Let φ be a endomorphism of a Lie group G. A Lie subgroup $H \subset G$ is said to be φ-invariant if $\varphi(H) \subset H$.*

If $H \subset G$ is a φ-invariant Lie subgroup, the restriction $\varphi|_H$ is an endomorphism of H in the induced topology.

Let us consider now a Lie group G and let $\varphi : G \to G$ be an endomorphism. In order to avoid cumbersome notations from here on we write $\phi := (d\varphi)_e$. Let us denote by G_φ, K_φ, G^+, G^0, G^-, $G^{+,0}$ and $G^{-,0}$ the Lie subgroups of G induced by φ, that is, the Lie subgroups of G associated with the Lie subalgebras $\mathfrak{g}_\phi$, $\mathfrak{k}_\phi$, $\mathfrak{g}^+$, $\mathfrak{g}^0$, $\mathfrak{g}^-$, $\mathfrak{g}^{+,0}$ and $\mathfrak{g}^{-,0}$, respectively.

3.4 Definition: *The subgroups G^+, G^0 and G^- are called, respectively, the* **unstable**, **central** *and* **stable** *subgroups of φ in G.*

The next result states the main properties of these subgroups.

3.5 Proposition: *It holds:*

(i) All the above subgroups are φ-invariant;

(ii) The subgroup K_φ is normal with $K_\varphi = \ker(\varphi^d)_0$. Moreover, $G = G_\varphi K_\varphi$ and therefore $G_\varphi = \mathrm{Im}(\varphi^d)$;

(iii) The restriction of φ to G^+ is expanding and to G^- is contracting.

(iv) If G_φ is a solvable Lie group it holds that

$$G_\varphi = G^{+,0}G^- = G^{-,0}G^+ = G^+G^0G^-; \tag{3}$$

(v) If G_φ is semisimple and G^0 is compact, then $G_\varphi = G^0$. Therefore, if G is any connected Lie group such that G^0 is compact, then G_φ has also the decomposition (3).

Proof: (i) Since they are connected Lie groups, their φ-invariance follows from the ϕ-invariance of their Lie algebras.

(ii) Since K_φ and $\ker(\varphi^d)_0$ are both connected Lie groups with the same Lie algebra given by $\mathfrak{k}_\varphi = \ker(\phi^d)$, it follows the desired equality. Moreover, since $\ker(\varphi^d)$ is a normal subgroup of G, its connected component of the identity K_φ is also normal.

Since K_φ is normal the product $G_\varphi K_\varphi$ is a connected subgroup of G with Lie algebra $\mathfrak{g}_\phi \oplus \mathfrak{k}_\phi = \mathfrak{g}$. Therefore $G = G_\varphi K_\varphi$ by unicity. By the above decomposition and the φ-invariance of G_φ we have that

$$\mathrm{Im}(\varphi^d) = \varphi^d(G) = \varphi^d(G_\varphi)\varphi^d(K_\varphi) \subset G_\varphi.$$

Moreover, since ϕ restricted to $\mathfrak{g}_\phi$ is an automorphism, we get that

$$\mathrm{e}^X = \mathrm{e}^{\phi^d\left(\phi|_{\mathfrak{g}_\phi}^{-d}(X)\right)} = \varphi^d\left(\mathrm{e}^{\phi|_{\mathfrak{g}_\phi}^{-d}(X)}\right) \in \mathrm{Im}(\varphi^d), \quad \text{for all} \quad X \in \mathfrak{g}_\phi$$

and consequently that $G_\varphi \subset \mathrm{Im}(\varphi^d)$ concluding the proof of (ii).

(iii) It follows by the definition of G^+ and G^- and Remark 2.4.

(iv) As for the decomposition $G = G_\varphi K_\varphi$ one can easily show that $G^{+,0} = G^+G^0 = G^0G^+$ and that $G^{-,0} = G^-G^0 = G^0G^-$. Thus, in order to prove the result it is enough to show that $G_\varphi = G^{+,0}G^-$.

We prove it by induction on the dimension of G_φ. If $\dim(G_\varphi) = 1$ the group G_φ is Abelian and the result is certainly true. Let us then assume that the result holds for any endomorphism φ such that G_φ is solvable with $\dim(G_\varphi) < n$. Assume that φ is an endomorphism of G such that G_φ is solvable with $\dim(G_\varphi) = n$. Knowing that G_φ is solvable there exists a nontrivial closed normal closed Lie subgroup B_φ of G_φ that is φ-invariant and Abelian (see for instance the proof in Proposition 2.9 of [1]). By considering $H_\varphi = G_\varphi/B_\varphi$ we have that H_φ is a connected solvable Lie group and $\dim(H_\varphi) = \dim(G_\varphi) - \dim(B_\varphi) < n$. Moreover, the canonical projection $\pi : G_\varphi \to H_\varphi$ induces in H_φ a well-defined surjective endomophism $\widetilde{\varphi}$ given by $\widetilde{\varphi}(\pi(g)) := \pi(\varphi(g))$.

By the induction hypothesis we have that $H_\varphi = H^{+,0}H^-$. However, by differentiation we get that $\widetilde{\phi} \circ (d\pi)_e = (d\pi)_e \circ \phi$ which by Proposition 2.5 and the fact that all the subgroups are connected give us that $\pi(G^{+,0}) = H^{+,0}$ and $\pi(G^-) = H^-$. Consequently $H_\varphi = \pi(G^{+,0}G^-)$ and so $G_\varphi = G^{+,0}G^-B_\varphi$. Since B_φ is Abelian, we obtain that $B_\varphi = B^{+,0}B^-$ with $B^{+,0} \subset G^{+,0}$ and $B^- \subset G^-$. Since B is also normal, we get

$$G = G^{+,0}G^-B_\varphi = G^{+,0}B_\varphi G^- = G^{+,0}B^{+,0}B^-G^- = G^{+,0}G^-$$

which concludes the proof of (iii).

(v) Let us first show that the second assertion is implied by the first one. Since R_φ is φ-invariant, like before we obtain an induced surjective endomorphism $\widetilde{\varphi}$ on G_φ/R_φ such that $(G_\varphi/R_\varphi)^0 = \pi(G^0)$, where $\pi : G_\varphi \to G_\varphi/R_\varphi$ is the canonical projection. On the other hand, since G_φ/R_φ is semisimple and $\pi(G^0)$ is compact, the first assertion would imply that $\pi(G^0) = (G_\varphi/R_\varphi)^0 = G_\varphi/R_\varphi$ and so $G_\varphi = G^0R_\varphi$. Moreover, R_φ is a solvable subgroup which by item (v) we get $R_\varphi = R^{+,0}R^-$ and so

$$G_\varphi = G^0R_\varphi = G^0R^{+,0}R^- \subset G^{+,0}G^- \subset G_\varphi$$

as stated.

Assume that G_φ is semisimple and that G^0 is a compact subgroup. Since $\phi|_{\mathfrak{g}_\phi}$ is an automorphism, we have by Theorem 5.4 of [2] that there exists $k \in \mathbb{N}$ such that $\phi|_{\mathfrak{g}_\phi}^k = \mathrm{Ad}(g)$ for some $g \in G_\varphi$. It follows that

$$\mathfrak{g}^+_{\mathrm{Ad}(g)} = \mathfrak{g}^+, \quad \mathfrak{g}^0_{\mathrm{Ad}(g)} = \mathfrak{g}^0 \quad \text{and} \quad \mathfrak{g}^-_{\mathrm{Ad}(g)} := \mathfrak{g}^-.$$

Since G_φ is semisimple, there exists an Iwasawa decomposition $G_\varphi = KAN$ and elements $a \in A$, $u \in K$ and $n \in N$ such that

$$\mathrm{Ad}(g) = \mathrm{Ad}(u)\,\mathrm{Ad}(a)\,\mathrm{Ad}(n)$$

with $\mathrm{Ad}(a)$ hyperbolic, $\mathrm{Ad}(n)$ unipotent and $\mathrm{Ad}(u)$ elliptic are commutating matrices (see Chapter IX, Lemma 7.1 of [2]). Therefore, $\mathfrak{g}^+ = \mathfrak{g}^+_{\mathrm{Ad}(g)}$ is the sum of eigenspaces associated with the positive eigenvalues of $\mathrm{Ad}(a)$, $\mathfrak{g}^- = \mathfrak{g}^-_{\mathrm{Ad}(g)}$ the sum of eigenspaces associated with the negative eigenvalues of $\mathrm{Ad}(a)$ and $\mathfrak{g}^0 = \mathfrak{g}^0_{\mathrm{Ad}(g)} = \ker(\mathrm{Ad}(a))$. Furthermore, the subgroup A is a simply connected Abelian Lie group and $A \subset G^0$. By the hypothesis on the compacity of G^0 we must have $a = e$ and so $\mathfrak{g}^+ = \mathfrak{g}^- = \{0\}$ implying that $G^0 = G$ as desired. □

3.6 Definition: *If φ is an endomorphism of the Lie group G such that G_φ has the decomposition in (3), we say φ* **decomposes** *G.*

Let us assume now that φ is in fact an automorphism when restricted to G_φ. By Remark 2.4 we have that, for any right (left) invariant Riemannian metric ϱ, it holds

$$\varrho(\varphi^n(x), e) \leq c^{-1}\mu^n \varrho(x, e), \quad \text{for any } \; x \in G^-, n \in \mathbb{N} \tag{4}$$

$$\varrho(\varphi^n(y), e) \geq c\mu^{-n} \varrho(y, e), \quad \text{for any } \; y \in G^+, n \in \mathbb{N} \tag{5}$$

and for any $a > 0$,

$$\varrho(\varphi^n(z), e)\, \mu^{a|n|} \to 0, \; n \to \pm\infty \quad \text{for any } \; z \in G^0. \tag{6}$$

That implies the following topological properties of the induced subgroups.

3.7 Proposition: *If φ restricted to G_φ is an automorphism in the induced topology of G then:*

(i) $G^{+,0} \cap G^- = G^+ \cap G^- = G^0 \cap G^- = G^{-,0} \cap G^+ = G^+ \cap G^0 = \{e\}$;

(ii) All the subgroups induced by φ are closed in G;

(iii) For $n \geq d$ it holds that $\ker(\varphi^n) = K_\varphi$. In particular, $\ker(\varphi^n)$ is connected.

Proof: Let us show that $G^{-,0} \cap G^+ = \{e\}$ since the other cases are analogous. Let $y \in G^{-,0} \cap G^+$ and consider $x \in G^-$ and $z \in G^0$ such that $y = xz$. By the right invariance of the metric we get

$$\varrho(\varphi^n(y), e) = \varrho(\varphi^n(x)\varphi^n(z), e) \leq \varrho(\varphi^n(x), e) + \varrho(\varphi^n(z), e).$$

Since $y \in G^+$ and $x \in G^-$ we get (5) and (4) that

$$c\mu^{-n}\varrho(y, e) \leq \varrho(\varphi^n(z), e) + c^{-1}\mu^n \varrho(x, e)$$

and so

$$\varrho(y, e) \leq c^{-1}\varrho(\varphi^n(z), e)\mu^n + c^{-2}\mu^{2n}\varrho(x, e).$$

However, since $z \in G^0$, equation (6) implies that all the terms on the right hand term of the above inequality goes to zero as $n \to +\infty$. Therefore, $\varrho(y, e) = 0$ implying that $G^{-,0} \cap G^+ = \{e\}$ as desired.

(ii) Since for $n \in \mathbb{N}$ we have that

$$G_\varphi \cap \ker(\varphi^n) = \ker(\varphi|_{G_\varphi}^n), \tag{7}$$

and by assumption, $\varphi|_{G_\varphi}$ is an automorphism, we obtain that $G_\varphi \cap K_\varphi = \{e\}$ which by Proposition 3.2 implies that G_φ is closed in G. Using again Proposition 3.2 and item (i) above, we get also that $G^+, G^0, G^-, G^{+,0}$ and $G^{-,0}$ are closed subgroups of G_φ and consequently are also closed subgroups of G.

(iii) Let $x \in \ker(\varphi^n)$, $n \geq d$ and consider its decomposition $x = gk$ with $g \in G_\varphi$ and $k \in K_\varphi$ given by item (ii) of Proposition 3.5. Then

$$G_\varphi \ni g = xk^{-1} \in \ker(\varphi^n) K_\varphi \subset \ker(\varphi^n)$$

which by (7) implies that $x = k \in K_\varphi$ and concludes the proof. □

3.8 Proposition: *Let φ be an endomorphism of a simply connected Lie group G. Then, G_φ and K_φ are simply connected. Moreover, the restriction of φ to G_φ is an automorphism.*

Proof: By Proposition III.3.17 of [3] both, the subgroup $\ker(\varphi^d)$ and the quotient $G/\ker(\varphi^d)$ are simply connected. Since the map $G/K_\varphi \to G/\ker(\varphi^d)$ is a covering map, Proposition 6.12 of [5] implies that $K_\varphi = \ker(\varphi^d)$. Moreover, by the decomposition $G = G_\varphi K_\varphi$ we obtain that $\varphi^d : G \to G_\varphi$ is a surjective continuous homomorphism and so, by the Isomorphism Theorem we have that G_φ and $G/\ker\left(\varphi^d\right)$ are isomorphic, showing that G_φ is simply connected. Knowing that ϕ restricted to $\mathfrak{g}_\phi$ is an automorphism and G_φ is simply connected we must have that φ restricted to G_φ is an automorphism concluding the proof. □

3.9 Corollary: *If G is a simply connected Lie group, then all the subgroups induced by an endomorphism φ of G are closed.*

The next result shows that the unstable/stable subgroup of a compact φ-invariant subgroup of G_φ is contained in its center.

3.10 Proposition: *Let G be a Lie group and φ an endomorphism of G. If $H \subset G_\varphi$ is a φ-invariant compact subgroup, then $H^+, H^- \subset Z_H$. In particular, if G_φ is compact G is decomposable.*

Proof: Since H is a compact subgroup it is in particular reductible and so $H = Z_H H'$, where Z_H is the center of H and H' the the derivated subgroup. Since both, H' and Z_H are φ-invariant subgroups and H' is semisimple, item (v) of Proposition 3.5 implies that $H' \subset G^0$ and consequently $H^+, H^- \subset Z_H$ by φ-invariance.

If G_φ is compact, we have that $G'_\varphi \subset G^0$ and so $G_\varphi = Z_{G_\varphi} G^0$. Since Z_{G_φ} is in particular a solvable subgroup, item (iv) of Proposition 3.5 implies that $Z_{G_\varphi} \subset G^{+,0}G^-$ which gives us the result. □

For solvable Lie groups, the next result says that the fixed points of any automorphism φ has to be contained in the subgroup G^0.

3.11 Theorem: *Let G be a solvable Lie group and φ an endomorphism of G. If $\varphi|_{G_\varphi}$ is an automorphism, then any fixed point of φ is contained in G^0.*

Proof: Since $\varphi|_{G_\varphi}$ is an automorphism we have that $G_\varphi \cap K_\varphi = \{e\}$. Therefore, the decomposition of $x \in G$ as $x = gk$ with $g \in G_\varphi$ and $k \in K_\varphi$ is unique. Then, $x = gk$ is fixed point of φ if and only if g and k are fixed points of φ. Since $\varphi^d(k) = e$ we must have that $k = e$ and we only have to analyze the case where $g \in G_\varphi$ is a fixed point.

By Proposition 3.5 item (iv), we have that $g = g_1g_2g_3$ with $g_1 \in G^+$, $g_2 \in G^0$ and $g_3 \in G^-$. Moreover, by Proposition 3.7 item (i) and the φ-invariance of the subgroups we obtain that g is a fixed point of φ if and only if g_i is a fixed point of φ for $i = 1, 2, 3$. However, since $g_1 \in G^+$, we get by equation (5) that

$$\varrho(g_1, e) = \varrho(\varphi^n(g_1), e) \geq c\mu^{-n}\varrho(g_1, e), \quad \text{for any} \quad n \in \mathbb{N}$$

which happens if and only if $g_1 = e$. In the same way, using that $g_3 \in G^-$ is a fixed point and equation (4) we get that $g_2 = e$ showing that $x = g_2 \in G^0$ as desired. □

Examples

3.12 Example: When $G = \mathbb{R}^d$ and $A \in \mathrm{gl}(d)$ is a matrix we have the endomorphism φ_A of G given by $\varphi_A(x) := Ax$. In this case, the subgroups induced by φ_A are given as sums of the eigenspaces of A.

3.13 Example: Consider $G = Sl(n)$ to be the group of the invertible matrices with determinant equal to one. If $A = \mathrm{diag}(a_1 > \ldots > a_d)$ is a matrix with trace equal to zero we can define the automorphism $\varphi_A : G \to G$ defined by $\varphi_A(B) = \mathrm{e}^A B \mathrm{e}^{-A}$, where e^A is the matrix exponencial of A. An easy calculation shows that in this case

$$G^+ = \{B \in G;\ B \text{ is upper triangular with 1's in the main diagonal}\},$$

$$G^- = \{B \in G;\ B \text{ is lower triangular with 1's in the main diagonal}\}$$

and $G^0 = \{B \in G;\ B \text{ is diagonal}\}$.

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