

## OPTIMALITY ON HOMOGENEOUS SPACES, AND THE ANGLE SYSTEM ASSOCIATED WITH A BILINEAR CONTROL SYSTEM\*

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**Abstract.** Let  $G$  be a Lie group. In order to study optimal control problems on a homogeneous space  $G/H$ , we identify its cotangent bundle  $T^*G/H$  as a subbundle of the cotangent bundle of  $G$ . Next, this identification is used to describe the Hamiltonian lifting of vector fields on  $G/H$  induced by elements in the Lie algebra  $\mathfrak{g}$  of  $G$ . As an application, we consider a bilinear control system  $\Sigma$  in  $\mathbb{R}^2$  whose matrices generate  $\mathfrak{sl}(2)$ . Through the Pontryagin maximum principle, we analyze the time-optimal problem for the *angle system*  $\mathbb{P}\Sigma$  defined by the projection of  $\Sigma$  onto the projective line  $\mathbb{P}^1$ . We compute some examples, and in particular we show that the bang-bang principle does not need to be true.

**Key words.** optimal time, Pontryagin maximum principle, bilinear control systems, Cartan–Killing form, real projective line

**AMS subject classification.** 35B50

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**1. Introduction.** Let  $G$  be a Lie group and  $H$  a closed subgroup. In order to study optimal control problems on a homogeneous space  $G/H$ , we identify the cotangent bundle of  $G/H$  as a subbundle of the cotangent bundle of  $G$ . Next, we use the mentioned identification to describe the Hamiltonian lifting of vector fields on  $G/H$  induced by elements in the Lie algebra  $\mathfrak{g}$  of  $G$ .

As an application of the construction we consider a bilinear control system  $\Sigma$  in  $\mathbb{R}^2$  given by the family of differential equations:

$$(1) \quad \dot{x}(t) = (A + uB)x(t), \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{R}^2,$$

where  $A, B \in \mathfrak{sl}(2)$  are real  $2 \times 2$  matrices of trace zero and

$$u \in \mathcal{U} = \{u : \mathbb{R} \rightarrow [-1, 1], u \text{ locally integrable}\}$$

is the set of the restricted admissible controls. We assume that  $\Sigma$  satisfies the Lie algebra rank condition, LARC, which means that the Lie algebra generated by  $A$  and  $B$  coincides with  $\mathfrak{sl}(2)$ .

The *angle system*  $\mathbb{P}\Sigma$  is defined by projection of  $\Sigma$  onto the projective line

$$(2) \quad \mathbb{P}\Sigma : \quad \dot{s}(t) = h(A, s(t)) + u(t)h(B, s(t)), \quad s \in \mathbb{P}^1,$$

where  $h(X, s) = (X - s^T X s I)s$ , with  $I$  the identity matrix and  $u \in \mathcal{U}$ .

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For this class of control systems we consider the following time-optimal control problem. If  $q_b$  and  $q_e$  denote the initial and final points on the projective line, respectively, then

*find a trajectory of  $\mathbb{P}\Sigma$  steering  $q_b$  to  $q_e$  at minimum time.*

Through the Pontryagin maximum principle (PMP), the time-optimal problem for single input control systems has been studied by many authors; see, for instance, [2], [3], [6], [12], [13], and [14].

Following [1] we identify through the Cartan–Killing form, the cotangent bundle of the projective line as a cone in  $\mathbb{R}^3$ ; see section 2. Next, we describe on the cone the trajectories of the lifted vector fields corresponding to the dynamics of the control system on  $\mathbb{P}^1$ . We analyze the extremals as solutions of a pseudo-Hamiltonian system. They are trajectories of the adjoint system staying on the cone  $\mathcal{C}^+$ , which corresponds to the elements of  $\mathcal{C}$  with positive three coordinates. Therefore, every time-optimal trajectory of the projected system (2) on the real projective line  $\mathbb{P}^1$  is obtained by radial projection.

**1.1. Generalities on the cotangent bundle  $T^*(G/H)$ .** Let  $G$  be a Lie group and  $H \subset G$  a closed Lie subgroup. It is well known that  $G/H = \{zH : z \in G\}$  is an analytic differentiable manifold. The elements in the cotangent bundle

$$T^*(G/H) = \bigcup_{z \in G} T_{zH}^*(G/H)$$

can be represented as elements in the cotangent bundle  $T^*(G)$  of  $G$  as follows.

Consider the canonical projection  $\pi : G \rightarrow G/H$  defined by  $\pi(z) = zH$ . Then, for any  $z \in G$ , its differential

$$d\pi_z : T_z G \rightarrow T_{\pi(z)}(G/H) \text{ has the transpose}$$

$$(d\pi_z)^* : T_{\pi(z)}^*(G/H) \rightarrow T_z^* G, \text{ given by } (d\pi_z)^*(\alpha) = \alpha \circ d\pi_z.$$

Since the linear map  $d\pi_z$  is surjective, it follows that  $(d\pi_z)^*$  is injective. In particular, for each  $z \in G$  the cotangent space  $T_{zH}^*(G/H)$  is isomorphic to its image under  $(d\pi_z)^*$ . That is,

$$T_{zH}^*(G/H) \cong (d\pi_z)^* T_{zH}^*(G/H) \subset T_z^* G.$$

Therefore, the elements of the cotangent bundle  $T^*(G/H)$  can be represented by elements of  $T^*G$ .

Recall that if  $V$  and  $W$  are two finite-dimensional vector spaces and  $T : V \rightarrow W$  is an onto linear map with  $T^* : W^* \rightarrow V^*$  its transpose. Then, the image of  $T^*$  is the annihilator of  $\ker T$ ; that is,  $\text{Im } T^* = (\ker T)^0$ , where

$$(\ker T)^0 = \{\lambda \in V^* : \forall v \in \ker T, \lambda(v) = 0\}.$$

In the sequel we identify  $T^*G$  with  $\mathfrak{g}^* \times G$  where  $\mathfrak{g} = T_1 G$  is the Lie algebra of  $G$  and  $\mathfrak{g}^*$  its dual. The identification is made through the  $G$ -translations. More precisely, to the pair  $(\lambda, z) \in \mathfrak{g}^* \times G$  we associated the element of  $T_z^* G$  defined by  $\lambda \circ dR_{z^{-1}} \in T_z^* G$ , where  $R_z$  denotes the right translation by  $z \in G$ . Its inverse is given by  $\mu \in T_z^* G \mapsto (d(R_z)_1^*(\mu), z) \in \mathfrak{g}^* \times G$ .

PROPOSITION 1.1. *Through the previous identification, the annihilator  $(\ker d\pi_z)^0$  is identified to the annihilator of  $\text{Ad}(z)\mathfrak{h}$ , where  $\mathfrak{h}$  denotes the Lie algebra of the Lie subgroup  $H$  of  $G$ .*

*Proof.* In fact, the tangent space of  $H$  at the point  $z$  is  $(dL_z)_1\mathfrak{h}$ . Therefore,

$$(\ker d\pi_z)^0 = ((dL_z)_1\mathfrak{h})^0.$$

In particular, the right translation identification shows that the annihilator is

$$(dL_z)_1\mathfrak{h} = (dR_{z^{-1}})_z(dL_z)_1\mathfrak{h}, \quad \text{i.e., } (\ker d\pi_z)^0 = (\text{Ad}(z)\mathfrak{h})^0,$$

ending the proof.  $\square$

A particular case that will be exploited below is when  $G$  is a semisimple Lie group. In this case it is well known that the Cartan–Killing form  $\mathcal{K}$  of its Lie algebra  $\mathfrak{g}$  is nondegenerate and defines an isomorphism between  $\mathfrak{g}$  and its dual  $\mathfrak{g}^*$ ,  $X \in \mathfrak{g} \mapsto \mathcal{K}(X, \cdot) \in \mathfrak{g}^*$  (see [9]). Through this isomorphism the annihilator of a subspace  $V \subset \mathfrak{g}$  is identified to the orthogonal

$$V^\perp = \{X \in \mathfrak{g} : \forall Y \in V, \mathcal{K}(X, Y) = 0\}$$

of  $V$  with respect to the Cartan–Killing form. In particular, Proposition 1.1 admits the following version.

PROPOSITION 1.2. *Suppose  $\mathfrak{g}$  is semisimple. Let us denote by  $\mathfrak{h}$  the Lie algebra of  $H$ . Through the previous identifications we have*

- (i)  $(\ker d\pi_z)^0 \cong (\text{Ad}(z)\mathfrak{h})^\perp$ ,
- (ii)  $(\text{Ad}(z)\mathfrak{h})^\perp \cong \text{Ad}(z)\mathfrak{h}^\perp$ .

**1.2. Hamiltonian lifting on homogeneous spaces.** Let  $X$  be a vector field on a differentiable manifold  $N$ . There exists a lifting  $X^*$  of  $X$  on the cotangent bundle  $T^*N$ , with flow  $X_t^*$  given by

$$(3) \quad X_t^*(\alpha) = \alpha \circ (dX_{-t})_{X_t(x)} = (dX_{-t})_{X_t(x)}^*(\alpha),$$

where  $X_t$  denotes the flow of  $X$  and  $\alpha \in T_x^*N$ .

It is well known that  $X^*$  is a Hamiltonian vector field. The corresponding Hamiltonian function  $H_{X^*} : T^*N \rightarrow \mathbb{R}$  is defined by

$$H_{X^*}(\alpha) = \alpha(X(p(\alpha))), \quad \alpha \in T^*N,$$

where  $p : T^*N \rightarrow N$ ,  $\alpha \mapsto p(\alpha)$  is the fiber bundle projection.  $H_{X^*}$  is a fiberwise linear functional on  $T_x^*N$ .

Next we consider the homogeneous space  $G/H$ , and we denote by  $\mathfrak{g}$  and  $\mathfrak{h}$ , the Lie algebras of  $G$  and  $H$ , respectively. Each  $X \in \mathfrak{g}$  induces a vector field on  $G/H$ , denoted by  $\tilde{X}$ . Its flow is given by  $\tilde{X}_t(x) = \exp(tX)x$ ,  $x \in G/H$ .

In the following we use the previous identifications to describe the lifting  $\tilde{X}^*$  of the vector fields  $\tilde{X}$  on the homogeneous space  $G/H$  induced by  $X \in \mathfrak{g}$ .

PROPOSITION 1.3. *Let  $\mathfrak{h} \subset \mathfrak{g}$  be the Lie algebra of  $H$  and  $X \in \mathfrak{g}$ . Through the previous identifications, the vector field  $\tilde{X}^*$  on the cotangent bundle  $T^*(G/H)$  is given by*

$$\tilde{X}^*(\lambda) = -\text{ad}(X)^*(\lambda) = -\lambda \circ \text{ad}(X)$$

and its flow is

$$\tilde{X}_t^* (\lambda) = \text{Ad} (\exp (-tX))^* \lambda.$$

*Proof.* First, we consider the identification of  $T^*G/H$  with its image via  $d\pi^*$ . Formula (3) yields

$$\tilde{X}_t^* (\alpha) = \left( d\tilde{X}_{-t} \right)_{\tilde{X}_t(x)}^* (\alpha).$$

Applying the transpose  $d\pi_{X_t(z)}^*$  to this equality we get

$$d\pi_{X_t(z)}^* \circ \tilde{X}_t^* = d\pi_{X_t(z)}^* \circ \left( d\tilde{X}_{-t} \right)_{\tilde{X}_t(x)}^* = d \left( \tilde{X}_{-t} \circ \pi \right)_{X_t(z)}^* .$$

For each  $g \in G$  we denote by  $L_g$  the left translation on  $G$ . Since the projection  $\pi$  commutes with left translations, i.e.,  $\pi \circ L_g = L_g \circ \pi$ , we obtain

$$d\pi_{(\exp tX)(z)}^* \circ \tilde{X}_t^* = d \left( \pi \circ L_{\exp(-tX)} \right)_{(\exp tX)z}^* = d \left( L_{\exp(-tX)} \right)_{(\exp tX)z}^* \circ d\pi_z^* .$$

Therefore, if  $\alpha \in T_{zH}^* (G/H)$  is identified with  $\mu \in T_z^*G$ , the vector

$$X_t^* (\alpha) \in T_{\exp(tX)zH}^* (G/H)$$

is also identified with

$$d \left( L_{\exp(-tX)} \right)_{(\exp tX)z}^* (\mu) .$$

By applying the linear map  $d \left( R_{\exp(tX)} \right)_1^*$  we have

$$\begin{aligned} & d \left( R_{\exp(tX)} \right)_1^* \circ d \left( L_{\exp(-tX)} \right)_{\exp(tX)z}^* \\ &= d \left( L_{\exp(-tX)} \circ R_{\exp(tX)} \right)_1^* \circ (dR_z)_1^* = \text{Ad} (\exp (-tX))^* \circ (dR_z)_1^* . \end{aligned}$$

On the other hand, Proposition 1.1 shows that

$$(d\pi)_z^* (T_{zH}^* (G/H)) = (\ker d\pi_z)^0 .$$

So, if  $\alpha \in T_{zH}^* (G/H)$  is identified with  $\lambda \in \mathfrak{g}^*$ , then  $X_t^* \alpha \in T_{\exp(tX)zH}^* (G/H)$  is identified with  $\text{Ad} (\exp (-tX))^* \lambda \in \mathfrak{g}^*$ .  $\square$

In the semisimple case we identify  $\mathfrak{g}$  with its dual  $\mathfrak{g}^*$ , to get the following statement.

**PROPOSITION 1.4.** *Let  $\mathfrak{g}$  be a semisimple Lie algebra and  $\mathfrak{h} \subset \mathfrak{g}$  the Lie algebra of  $H$ . Then, the vector field  $\tilde{X}^*$  on  $T^* (G/H)$  induced by  $X \in \mathfrak{g}$  is given by*

$$\tilde{X}^* (Y) = \text{ad} (X) (Y) \text{ with flow } \tilde{X}_t^* (Y) = \text{Ad} (\exp tX) Y .$$

*Proof.* We know that  $Y \in \mathfrak{g}$  corresponds to  $\mathcal{K}(Y, \cdot) \in \mathfrak{g}^*$ . Since  $\mathcal{K}(\text{Ad} (g) Y, Z) = \mathcal{K}(Y, \text{Ad} (g^{-1}) Z)$ , the result follows.  $\square$

**2. The cotangent bundle  $T^*(\mathbb{P}^1)$  as a cone.** As usual we denote by  $\mathrm{SL}(2)$  the Lie group of the Lie algebra  $\mathfrak{sl}(2)$ , the set of  $2 \times 2$  real matrices with determinant 1. In this particular case, the Cartan–Killing form  $\mathcal{K}$  is a multiple of the trace form and is given by

$$\mathcal{K}(X, Y) = k \operatorname{tr}(XY) = k \langle X, Y \rangle,$$

where  $k \neq 0$  is a constant. Furthermore, consider the corresponding quadratic form  $Q(X) = \langle X, X \rangle = \operatorname{tr}(X^2)$  and denote the zero set of  $Q$  by

$$\mathcal{C} = \{X \in \mathfrak{sl}(2) : Q(X) = 0\}.$$

In particular,  $\mathcal{C}$  is invariant under conjugation by invertible matrices; i.e.,  $gZg^{-1} \in \mathcal{C}$  if  $Z \in \mathcal{C}$ . The set  $\mathcal{C} - \{0\}$  has two connected components. We distinguish them by putting  $\mathcal{C}^+$  and  $\mathcal{C}^-$  for the one which contains the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

respectively. Each of these components is an orbit under conjugation of  $\mathrm{SL}(2)$ . For instance, to see that  $\mathrm{SL}(2)$  acts transitively on  $\mathcal{C}^+$ , note that the rotation group turns around  $\mathcal{C}^+$  while the group of diagonal matrices is transitive along the ray of upper triangular matrices in  $\mathcal{C}^+$ . Furthermore, we note that  $\mathcal{C}$  is the set of nilpotent matrices in  $\mathfrak{sl}(2)$  while  $\mathcal{C} - \{0\}$  is the set of rank one  $2 \times 2$  matrices having trace zero.

We recall that if  $u$  and  $v$  are orthogonal vectors in  $\mathbb{R}^2$ , taking them as  $2 \times 1$  matrices we have that  $u^t v$  belongs to  $\mathcal{C}$ . Moreover,  $u^t v \in \mathcal{C}^+$  if  $\{u, v\}$  is positively oriented with respect to the standard basis of  $\mathbb{R}^2$ . Thus, the map

$$\phi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} -y & x \end{pmatrix}$$

induces a map from  $\mathbb{P}^1$  into the set  $[\mathcal{C}^+]$  of rays of  $\mathcal{C}^+$  as  $x \in \mathbb{P}^1 \rightarrow \mathbb{R}^+ x$ . It turns out that the action of  $\mathrm{SL}(2)$  on  $[\mathcal{C}^+]$  is equivalent to the action of  $\mathrm{SL}(2)$  in the projective line  $\mathbb{P}^1$ , through the map  $\phi$ . Next, an algebraic picture of the cone is given.

**PROPOSITION 2.1.** *The cone  $\mathcal{C}$  is identified with the cotangent bundle  $T^*(\mathbb{P}^1)$ .*

*Proof.* Consider the parabolic subgroup  $H \subset G = \mathrm{SL}(2)$  defined by the matrices of the form

$$\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}$$

with  $a \in \mathbb{R} - \{0\}$ . Obviously the  $G$ -action on  $\mathbb{P}^1$  is transitive and the stabilizer of the line  $l = [e_1] \in \mathbb{P}^1$  is  $H$ . In particular, the homogeneous space  $G/H$  is isomorphic with  $\mathbb{P}^1$ . The Lie algebra  $\mathfrak{h}$  of  $H$  is given by the matrices of the form

$$\begin{pmatrix} \alpha & * \\ 0 & -\alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}.$$

That is,

$$\mathfrak{h} = \operatorname{Im} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

Related to the Cartan–Killing form, the orthogonal space  $\mathfrak{h}^\perp$  is the nilpotent Lie algebra

$$\mathfrak{h}^\perp = \{X \in \mathfrak{g} : \forall Y \in \mathfrak{h}, \mathcal{K}(X, Y) = 0\} = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\}.$$

According to Proposition 1.2, for any  $z \in G$ , the cotangent space  $T_{zH}^*(G/H)$  is identified with

$$\text{Ad}(z)(\mathfrak{h}^\perp) = \text{Ad}(z)(\mathfrak{n}).$$

This identification gives to the cotangent space  $T_{zH}^*(\mathbb{P}^1)$  a very simple geometric description. In fact, the nilpotent algebra  $\mathfrak{n}$ , which is the line generated by the matrix

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

is contained in  $\mathcal{C}$ . Since the action of  $\text{SL}(2)$  on  $\mathfrak{sl}(2)$  is given by conjugation, we obtain that  $\text{Ad}(z)\mathfrak{n}$ ,  $z \in G$ , is also a line contained in  $\mathcal{C}$ . It follows that the fiber bundle  $T^*(\mathbb{P}^1)$  is identified to the entire cone  $\mathcal{C}$ .  $\square$

**2.1. Trajectories on the cone.** Any  $Z \in \mathfrak{sl}(2)$  defines, through the linear application  $\text{ad } Z$ , a linear differential equation in  $\mathfrak{sl}(2)$ , explicitly given by  $\dot{P} = [Z, P]$ ,  $P \in \mathfrak{sl}(2)$ , whose trajectories are

$$\exp(t \text{ ad } Z)P = \text{Ad}(\exp tZ)(P), \text{ with } P \in \mathfrak{sl}(2) \text{ and } t \in \mathbb{R}.$$

The tangent plane  $T_P \mathcal{C}^+$  of  $\mathcal{C}^+$  at  $P \in \mathcal{C}^+$  is determined by

$$T_P \mathcal{C}^+ = \{W \in \mathfrak{sl}(2) : \langle P, W \rangle = 0\}.$$

In particular, each  $Z \in \mathfrak{sl}(2)$  induces a differential equation on  $\mathcal{C}^+$ . Just observe that  $\langle P, [Z, P] \rangle = 0$ . We will describe these trajectories in  $\mathcal{C}^+$  as the intersections of  $\mathcal{C}^+$  with the planes

$$Z^\perp = \{P \in \mathcal{C}^+ : \langle P, Z \rangle = c\}$$

orthogonal to  $Z$  with respect to the trace form. As a matter of fact, for any  $Z \in \mathfrak{sl}(2)$ , the map  $\text{Ad}(\exp Z) = \exp(\text{ad } Z)$  is an isometry; i.e., for every  $P, W \in \mathfrak{sl}(2)$ ,  $\langle \exp(\text{ad } Z)P, \exp(\text{ad } Z)W \rangle = \langle P, W \rangle$ .

In practice, with respect to the trace form, the plane orthogonal to any matrix of trace zero can be seen with the aid of the inner product  $(\cdot, \cdot)$  in  $\mathfrak{sl}(2)$  which is defined by

$$(P, W) = \langle P, W^t \rangle = \text{tr}(PW^t).$$

Actually, if we denote by  $Z^{(\perp)}$  the orthogonal plane to  $Z$  with respect to  $(\cdot, \cdot)$ , then  $Z^\perp = (Z^{(\perp)})^t$ . On the other hand, transposition is obtained by a reflection through the plane  $\mathfrak{s}$  of symmetric matrices. Therefore,  $Z^\perp$  is the reflection through  $\mathfrak{s}$  of the plane orthogonal to  $Z$  with respect to  $(\cdot, \cdot)$ .

In [1] the authors give the following description of the trajectories; see Figure 1 and Figure 2. Let us denote by  $\mathcal{C}_{int}^+$ ,  $\mathcal{C}_{ext}^+$  the regions of  $\mathbb{R}^3$  inside and outside the cone. We observe that the map  $\det$  also characterizes these regions. In fact,  $\det(W) = 0$  if  $W \in \mathcal{C}^+$ .

**PROPOSITION 2.2.** *According to the location of  $Z$  with respect to the cone, the trajectories of  $\dot{P} = [Z, P]$  in  $\mathcal{C}^+$  are*

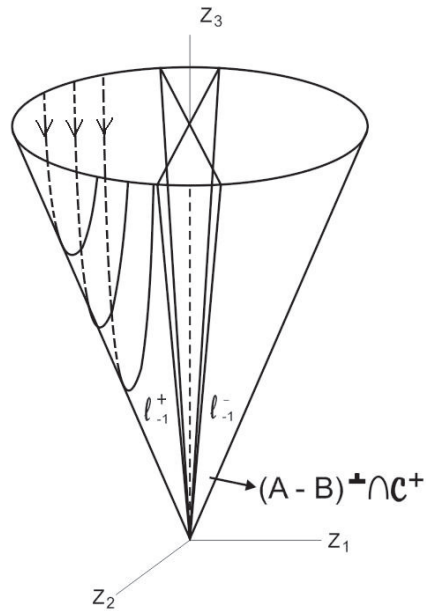


FIG. 1. Points in the ray of  $C^+$  and parabolas on  $C^+$ .

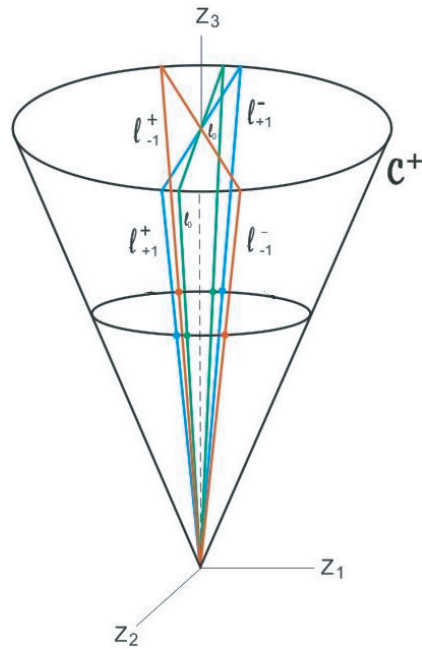


FIG. 2. Intersection of orthogonal planes with  $C^+$ .

1. Ellipses around  $C^+$ , if  $Z \in C_{\text{int}}^+$ .
2. Points in the ray of  $C^+$  orthogonal to  $Z$  or the parabolas  $\{\langle Z, P \rangle = c\} \cap C^+$ , if  $Z \in C^+$ .
3. The two rays in  $Z^\perp \cap C^+ \setminus \{0\}$  or the hyperbolas  $\{\langle Z, P \rangle = c\} \cap C^+$ , if  $Z \in C_{\text{ext}}^+$ .

**3. The time-optimal problem on  $\mathbb{P}^1$ .** In this section, we first establish the PMP for the angle system  $\mathbb{P}\Sigma$  on  $\mathbb{P}^1$  determined by a bilinear control system  $\Sigma$  in the plane. Next, we formulate the problem through the dynamics on the cone. We analyze the trajectories, in particular the singular trajectories and the time-optimal synthesis.

For our purposes it is enough to use the version of PMP stated and proved in [6]. Recall that the principle gives a first order necessary conditions for continuous-time optimal problems. Its geometric formulation takes place on the cotangent bundle of the state space.

We consider the following bilinear control system  $\Sigma$  in  $\mathbb{R}^2$ ,

$$\dot{x}(t) = (A + uB)x(t), \quad t \in \mathbb{R}, x(t) \in \mathbb{R}^2,$$

where  $A, B \in \mathfrak{sl}(2)$ ,  $u \in \mathcal{U}$ .

The *angle system*  $\mathbb{P}\Sigma$  is defined by the projection of  $\Sigma$  onto the projective line

$$\mathbb{P}\Sigma : \quad \dot{s}(t) = h(A, s(t)) + u(t)h(B, s(t)), \quad s \in \mathbb{P}^1,$$

where  $h(X, s) = (X - s^T X s I)s$ , with  $I$  the identity matrix and  $u \in \mathcal{U}$ .

In this particular case, the principle reads:

*If  $u_*$  is an optimal control and  $s_*$  is the associated trajectory of  $\mathbb{P}\Sigma$  on  $\mathbb{P}^1$ , there exists a constant  $\lambda_0 \geq 0$  and an absolutely continuous function  $\lambda : [0, T] \rightarrow T^*\mathbb{P}^1$  such that for almost every  $t \in \text{Dom}(s_*)$  the adjoint equation,*

$$\dot{\lambda} = -\lambda(A + u_*B),$$

*is satisfied. Furthermore, the optimal control  $u_*$  minimizes the Hamiltonian*

$$\mathcal{H} = \lambda(A + uB)s + \lambda_0$$

*on  $[-1, 1]$ , through the curve  $(\lambda(t), s_*(t))$ .*

The pair  $(u, s)$  is called an *extremal pair* if  $u$  is an admissible control and if the corresponding trajectory  $s$  satisfies the PMP conditions.

In our case the application of the PMP leads to two kinds of controls, the singular and bang-bang one. The difference depends on the fact that the associated solutions belong to the interior or to the boundary of a control set, respectively.

Denote by  $\text{cl}M$  the closure of a set  $M$  and by  $\mathbb{S}(x)$  the action of the semigroup of the angle system on the state  $x \in \mathbb{P}^1$ . We also denote by  $\varphi(t, x, u)$  the solution of the system with initial condition  $x$  and control  $u$ . Of course the projected system may be controllable or not, see [8]. Since we concentrate the analysis in the uncontrollable case, we need the notion of the control set in particular (see, for instance, [5], [10], [11]).

**DEFINITION 3.1.** *A set  $D \subset \mathbb{P}^1$  is called a control set of  $\mathbb{P}\Sigma$  if*

- (i) *for all  $x \in D$  we have  $D \subset \text{cl}\mathbb{S}(x)$ ,*
- (ii) *for all  $x \in D$  there exists a control  $u \in \mathcal{U}$  with  $\varphi(t, x, u) \in D$  for all  $t \geq 0$ ,*  
*and*

- (iii) *with respect to the set inclusion,  $D$  is maximal with the properties (i) and (ii).*

*When a control set has nonvoid interior it is called a main control set.*

**3.1. The problem.** Let  $G$  be the Lie group  $\text{SL}(2)$  and consider as before its action on the real projective line  $\mathbb{P}^1$ . The bilinear control system (1) induces control systems on  $\mathbb{P}^1$ .

We assume that the bilinear control system (1) satisfies the Lie algebra rank condition; that is,  $\text{Span}_{\mathcal{L}\mathcal{A}}\{A, B\} = \mathfrak{sl}(2)$ .



The cotangent bundle  $T^*\mathbb{P}^1$  is identified with the cone  $\mathcal{C}$ . Therefore, each curve  $[0, T] \rightarrow T^*_{zH}\mathbb{P}^1$  given by the PMP is identified with a curve in  $\mathcal{C}^+$ .

An admissible piecewise constant control  $u$  induces the vector field  $X^u = A + uB \in \mathfrak{g}$ . Through the previous identifications, the Hamiltonian function for the time optimal control problem takes the form

$$\mathcal{H}_{\tilde{X}^u} : \mathcal{C}^+ \rightarrow \mathbb{R},$$

$$\mathcal{H}_{\tilde{X}^u}(P) = \langle P, A \rangle + u \langle P, B \rangle + \lambda_0, \quad P \in \mathcal{C}^+,$$

where  $\tilde{X}^u$  is the lifted vector field on  $\mathcal{C}^+$  induced by  $X^u$  on  $\mathbb{P}^1$  and  $\lambda_0 \in \mathbb{R}$ .

The Hamiltonian to be minimized is the map

$$\mathcal{H} : T^*\mathbb{P}^1 \rightarrow \mathbb{R}, \quad \mathcal{H}(P) = \min_{v \in [-1, 1]} \mathcal{H}(P).$$

By Proposition 2.1 and the form of the Hamiltonian  $\mathcal{H}$ , it follows that the equation for  $\lambda(\cdot)$  can be rewritten as

$$(4) \quad \frac{dP(t)}{dt} = [A + uB, P(t)], \quad P(t) \in \mathcal{C}^+.$$

In fact, each  $P \in \mathfrak{sl}(2)$  is associated with  $\lambda \in \mathfrak{sl}(2)^*$  through the Cartan–Killing form  $\mathcal{K}$  by  $\lambda = \mathcal{K}(\cdot, P)$  and for any arbitrary  $V$

$$\mathcal{K} \left( V, \frac{dP(t)}{dt} \right) = \frac{d\lambda}{dt}(V) = -\lambda(t) [V, A + uB] = \mathcal{K}([A + uB, P(t)], V).$$

**3.2. Characterization of the extremal controls.** In order to obtain necessary conditions to determine the optimal control, we apply the PMP. An admissible control  $u : [a, b] \rightarrow [-1, 1]$  is said to be bang-bang if  $u(t) \in \{-1, 1\}$  a.e. in  $[a, b]$ . Moreover, if  $u(t) \in \{-1, 1\}$  and  $u(t)$  is constant for almost every  $t \in [a, b]$ , then  $u$  is called a bang control.

As pointed out by Sussmann, the bang-bang theorem for bilinear control systems does not need to be true as in the linear case. We recall Theorem 8.3.1 in [12].

**THEOREM 3.2.** *Consider a real analytic system*

$$\dot{x} = f(x) + ug(x), \quad x \in M, \quad |u| \leq 1$$

*on a real analytical manifold  $M$ . Suppose that, for each  $p \in M$  and each integer  $m > 0$ , there exists a neighborhood  $U$  of  $p$  such that, on  $U$ ,  $[g, (\text{ad}f)^m(g)]$  is a linear combination*

$$\sum_{i=0}^{m+1} \alpha_i (\text{ad}f)^i(g)$$

*with analytic coefficients  $\alpha_i$ , such that  $|\alpha_{m+1}| < 1$ , for every  $q \in U$ . Then, the system has the bang-bang property with a bounded number of switchings (b-condition).*

We use these results to narrow the class of possible optimal trajectories. According to Theorem 3.2, to apply the bang-bang theorem in our special situation it is sufficient condition to have

$$\text{ad}_A(B) = \text{Span} \{B, [A, B], [A, [A, B]]\} = \mathfrak{sl}(2)$$

and  $|\alpha_2| < 1$ .

Let us denote by  $u_*$  the optimal control minimizing the Hamiltonian; then

$$\langle P, A \rangle + u_* \langle P, B \rangle + \lambda_0 = \min_{u(\cdot) \in [-1,1]} (\langle P, A \rangle + u \langle P, B \rangle + \lambda_0).$$

Equivalently,

$$u_* \langle P, B \rangle = \min_{-1 \leq u \leq 1} u \langle P, B \rangle.$$

If we write  $\phi(P) = \langle P, B \rangle = \text{tr}(PB)$ , we get

$$\phi(P)u_*(t) = \min_{-1 \leq u \leq 1} \phi(P)u.$$

It follows that

$$u_*(t) = \begin{cases} 1 & \text{if } \phi(P) > 0, \\ -1 & \text{if } \phi(P) < 0. \end{cases}$$

Thus, the switching function  $\phi$  determines regions on the cone where the optimal path needs to move.

**3.3. Analysis of trajectories and singular controls.** We observe that if  $\phi(t) = \langle P(t), B \rangle = 0$ , the control is not determined by the optimality condition of the PMP. On any interval where  $\phi$  has no zeros, the corresponding control is bang, and if it has finitely many zeros, the control is bang-bang. We recall the following.

**DEFINITION 3.3.** *A control  $u$  is said to be singular on an open interval  $I \subset [0, T]$ , if it is computed from the solution of the resultant equation  $\phi^{(r)} \equiv 0$  for some  $r$ , for which the control variable  $u$  appears explicitly in the derivative  $\phi^{(r)}(t)$ .*

If a control  $u$  is singular on  $I$ , then the corresponding function  $\phi(t)$  and all its derivatives must vanish on  $I$ . Therefore, to find the corresponding control to a singular trajectory, one should compute the derivatives of  $\phi(t) = \langle P(t), B \rangle$ .

On the other hand, we need to analyze when the singular controls are optimal. In the following we use the term singular trajectory on an open interval  $I$  if the switching function  $\phi$  vanishes.

**DEFINITION 3.4.** *Consider a single-input control system which is linear in the control. The order of a singular trajectory on an interval  $I \subset [0, T]$  is the integer  $k$  such that  $r = 2k$ ,  $\phi^{(r)}(t) \equiv 0$  on  $I$ , and  $r$  is the smallest natural number for which the control variable  $u$  appears explicitly in  $\phi^{(r)}(t)$ .*

A  $k$ -order necessary condition for optimality of a singular trajectory of order  $k$ , is the generalized Legendre–Clebsch condition in [7], given by the following theorem.

**THEOREM 3.5.** *Assume that  $u$  is a singular control and  $(\lambda(t), y(t), u(t))$ ,  $t \in [0, T]$  is the associated extremal pair (singular lift). A  $k$ -order necessary condition for the optimality of the singular trajectory  $y(\cdot)$  is*

$$(5) \quad (-1)^k \left[ \frac{\partial}{\partial u} \frac{d^{2k}}{dt^{2k}} \frac{\partial \mathcal{H}}{\partial u} (\lambda(t), y(t), u(t)) \right] \geq 0,$$

where  $\mathcal{H}$  is the corresponding Hamiltonian.

Suppose that the control  $u$  is singular on a nonempty open interval  $I$ . Then  $\phi(t)$  is constant on  $I$  and we get  $\dot{\phi}(t) = -\langle P(t), [A, B] \rangle \equiv 0$ . A further derivation yields  $\ddot{\phi}(t) = \langle P(t), [A + uB, [A, B]] \rangle$ .

Since in  $\ddot{\phi}(t) \equiv 0$  the variable of control appears explicitly, it follows that the singular control  $u$  is of order  $k = 1$  on the interval  $I$ . Furthermore, we get

$$\langle P(t), [A, [A, B]] \rangle + u \langle P(t), [B, [A, B]] \rangle \equiv 0.$$

A direct computation implies that  $\frac{\partial \mathcal{H}}{\partial u}$  is  $\phi(t)$ . So, the second-order necessary condition of Legendre–Clebsch in  $\mathcal{C}^+$  reads

$$(-1) \frac{\partial}{\partial u} \frac{d^2}{dt^2} \frac{\partial \mathcal{H}}{\partial u} = - \langle P(t), [B, [A, B]] \rangle$$

evaluated along the extremal lift.

The singular control is of order 1 on the interval  $I$ . Hence, we can solve the control  $u(t)$  in feedback form. If the corresponding control value is admissible, i.e., has a value in  $(-1, 1)$ , then the singular control is well defined. Otherwise the singular control is not admissible.

**PROPOSITION 3.6.** *The singular trajectories for the time-optimal problem on  $\mathcal{C}^+$  is the locus where  $B$  and  $[A, B]$  are linearly dependent.*

*Proof.* Consider the adjoint system

$$\dot{P}(t) = [A + u(t)B, P(t)], u \in [-1, 1].$$

The singular trajectories need to satisfy  $\langle P(t), B \rangle \equiv 0$ . Hence,

$$(6) \quad \langle P(t), [A, B] \rangle \equiv 0.$$

On the other hand, from the PMP, we get  $\mathcal{H} = 0$ ; thus  $\langle P(t), A \rangle = -1$ . By differentiating (6) it follows that  $u \langle P(t), [B, A] \rangle \equiv 0$ . In particular, the singular trajectories are associated with any control  $u \in (-1, 1)$ .

Therefore, the singular trajectories live in the locus where  $B$  and  $[A, B]$  are linearly dependent.  $\square$

Notice that the singular trajectories on the real projective line  $\mathbb{P}^1$  are given by the projection of the intersection of  $\mathcal{C}^+$  with the orthogonal plane to  $B$  with respect to the trace form. We use the Legendre–Clebsch condition to analyze the optimality of singular trajectories. Summarizing, we have the following.

**PROPOSITION 3.7.** *Let  $(y(t), u_*(t))$  be an extremal pair. If the control  $u_*(\cdot)$  is singular on an open interval, then it is of order 1. Furthermore, if the second-order of Legendre–Clebsch is satisfied, the singular control is given in feedback form as*

$$u_*(t) = - \frac{\langle P(t), [A, [A, B]] \rangle}{\langle P(t), [B, [A, B]] \rangle}.$$

The set of extremal trajectories can be described by a finite-dimensional reduction. More precisely, every extremal trajectory is a finite concatenation of trajectories that are either bang (corresponding to a constant control equal to  $\pm 1$ ) or singular (corresponding to the singular control).

**4. Abnormal extremals.** Now we discuss the behavior of abnormal extremals. Recall that an extremal corresponding to  $\lambda_0 = 0$  is said to be abnormal extremal; otherwise we call it a *normal extremal*.

The following propositions describe the switching behavior of abnormal extremals.

**PROPOSITION 4.1.** *An abnormal extremal cannot contain a singular trajectory.*

*Proof.* From the maximality condition of the PMP, we get

$$0 = \mathcal{H}_u(P) = \langle P, A \rangle + u\langle P, B \rangle + \lambda_0, \quad P \in \mathcal{C}^+.$$

Since  $\lambda_0 = 0$ , we have

$$(7) \quad \langle P, A \rangle = -u\langle P, B \rangle.$$

If the extremal trajectory is singular on an open interval  $I$ , then the switching function  $\phi(t) = \langle P(t), B \rangle$  vanishes on  $I$ . From (7) we obtain  $\langle P, A \rangle = 0$  on  $I$ . Since  $\dot{\phi}(t) = \langle P(t), [A, B] \rangle \equiv 0$  on  $I$  and the matrices  $A, B$  and  $[A, B]$  generate  $\mathfrak{sl}(2)$  we get  $P(t) \equiv 0$  on  $I$ . According to PMP this is a contradiction because the pair  $(P(t), \lambda_0)$  should be nonzero. So, there are no singular trajectories which correspond to abnormal extremal trajectories.  $\square$

**PROPOSITION 4.2.** *Suppose that the pair  $(P(s), s(t))$  is a regular abnormal extremal of the time-optimal problem on  $\mathbb{P}^1$ . Then, this pair corresponds to the lifting vector field associated with the Hamiltonian  $\mathcal{H} = \langle P, A \rangle$ , which is given by the differential equation  $\dot{P} = [A, P]$ .*

*Proof.* Let  $(P(s), s(t))$  be a regular abnormal extremal. From the PMP we know that  $\frac{\partial \mathcal{H}}{\partial u} = 0$  along this extremal; i.e.,  $\langle P(t), B \rangle = 0$ . Hence, from the Hamiltonian equation

$$\mathcal{H}_u(P) = \langle P, A \rangle + u\langle P, B \rangle, \quad P \in \mathcal{C}^+,$$

the proof follows.  $\square$

**5. The time-optimal synthesis on  $\mathbb{P}^1$ .** Let  $q_b$  and  $q_e$  be two arbitrary points on  $\mathbb{P}^1$ . Our goal is to find a time-optimal trajectory connecting  $q_b$  to  $q_e$ . We identify  $\mathbb{P}^1$  with the clockwise oriented circle  $\{e^{-i\theta} : -\pi/2 < \theta \leq \pi/2\}$ . As is well known, if there exists a trajectory connecting  $q_b$  with  $q_e$ , then there exists a time-optimal trajectory connecting these two points. Our interest here is to show our construction with some examples. In particular, we look at optimal bang-bang paths, that is, optimal trajectories corresponding to bang-bang controls.

In the  $(z_1, z_2, z_3)$  coordinates, the equation of  $\mathcal{C}^+$  with respect to the basis

$$(8) \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of the Lie algebra  $\mathfrak{sl}(2)$  is given by  $z_1^2 + z_2^2 = z_3^2$ . Since we restrict the analysis to the cone  $\mathcal{C}^+$ , the condition  $z_3 < 0$  remains excluded.

From Corollary 5.2 in [4], we know that the LARC condition of (1) at the  $SL(2)$ -level is equivalent to  $\det[A, B] \neq 0$ , where  $[A, B] = AB - BA$ .

On the other hand, Theorem 5.2 in [1] gives a characterization of the controllability property of (1); namely it is controllable if and only if the segment

$$(9) \quad A + uB : -1 \leq u \leq 1 \text{ intersects } \mathcal{C}_{int}^+.$$

In the uncontrollable case there are exactly two control sets in  $\mathbb{P}^1$ , denoted by  $I^+$  and  $I^-$  (see [1], [4], [10], and [11] for more general results). They satisfy the following properties:

- (i)  $I^- \cap I^+ = \emptyset$ ,  $I^-$  is closed, and  $I^+$  is open.
- (ii)  $I^-$  is invariant; i.e.,  $\mathbb{S}x \subset I^-$  for all  $x \in I^-$ . On the other hand,  $I^+$  is  $\mathbb{S}^{-1}$ -invariant.

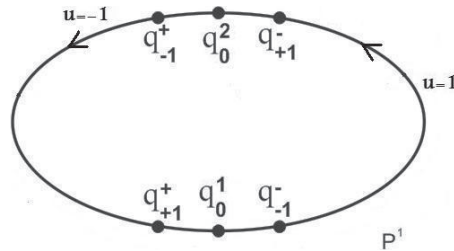


FIG. 3. Dynamics on  $\mathbb{P}^1$ .

(iii) If  $g \in \mathbb{S}$  is diagonalizable, then its attractor belongs to  $I^-$  and its repeller to the closure of  $I^+$ .

Now we take a noncontrollable system with  $A + B$ ,  $A - B$  and  $B \in \mathcal{C}_{ext}^+$ . The trajectories of  $A + B$  and  $A - B$  are either rays in  $Z^\perp \cap \mathcal{C}^+ \setminus \{0\}$  or the hyperbolas  $\{(Z, P) = c\} \cap \mathcal{C}^+$ . We denote by  $l_{+1}^+$  and  $l_{+1}^-$  the rays associated to  $A + B$  and by  $l_{-1}^+$  and  $l_{-1}^-$  those for  $A - B$ .

Recall that these rays correspond to eigenvectors of the matrices. We denote the projections of these rays into  $\mathbb{P}^1$  by

- $q_{+1}^-$  the attractor and  $q_{+1}^+$  the repeller of  $A + B$ .
- $q_{-1}^-$  the attractor and  $q_{-1}^+$  the repeller of  $A - B$ .

The intersection between  $\mathcal{C}^+$  and the orthogonal plane to  $B$  are the two lines denoted for  $l_0^1$  and  $l_0^2$ . We let  $q_0^1$  and  $q_0^2$  be the projections into  $\mathbb{P}^1$  of  $l_0^1$  and  $l_0^2$ , respectively. See Figure 3. Without loss of generality we assume that  $q_{+1}^- < q_{-1}^-$ . The other case  $q_{-1}^- < q_{+1}^-$  is treated analogously.

In this case the control sets are

$$I^- = [q_{+1}^-, q_{-1}^-] \quad I^+ = (q_{+1}^+, q_{-1}^+).$$

We write

$$R_1 = (q_{-1}^-, q_{+1}^+), \quad R_2 = (q_{-1}^+, q_{+1}^-),$$

for their respective complementary subsets.

We assume that the plane  $\pi_B$  determined by  $\langle P, B \rangle = 0$  gives rise to two regions on the cone and also does not intersect a control set. Without loss of generality, assume that  $I^-$  is inside of the region determined by  $\langle P, B \rangle > 0$  (otherwise we take  $-B$  instead of  $B$ ). In order to construct optimal bang-bang path we apply  $u = 1$  inside of  $\langle P, B \rangle > 0$  and  $u = -1$  when  $\langle P, B \rangle < 0$ . In terms of the regions  $I^-$ ,  $R_1$ ,  $I^+$  and  $R_2$  we have the following synthesis:

- (i) Restricted to  $I^-$  the system is controllable. According to the dynamics on the circle (and in the cone), there exists an optimal bang-bang path connecting  $q_b, q_e \in I^-$  when  $q_e < q_b$ . However, this is not the case in the other direction. In fact, if  $q_b < q_e$ , then, starting on  $q_b$  under the influence of  $u = 1$  the trajectory converges to the attractor  $q_{+1}^-$ . So, an optimal curve associated with a bang control from  $q_b$  will never reach  $q_e$ .
- (ii) If  $q_b \in R_1$ , there exists an optimal bang-bang path connecting  $q_b$  with  $q_e \in R_1$  when  $q_e < q_b$ . However, when  $q_b < q_e$ , starting on  $q_b \in R_1$  the trajectory converges to the attractor  $q_{+1}^-$ . Hence, when  $q_b < q_e$  an optimal curve associated with a bang control from  $q_b$  will never reach  $q_e$ .

- (iii) Let  $q_b \in I^+$ . Then the orbit  $\mathbb{S}(q)$  is  $\mathbb{P}^1 \setminus \{q_{-1}^+\}$ . Starting on  $q_b$ , there exists an optimal bang-bang path connecting any point in the interval  $[q_{+1}^-, q_b)$ . However, this is not the case when the ending point belongs to the interval  $(q_b, q_{+1}^-)$ .
- (iv) Let  $q_b \in R_2$ . Then  $\mathbb{S}(q_b) = (q_b, q_{-1}^-)$ . There exists an optimal bang-bang path connecting  $q_b$  with any point  $q_e \in (q_b, q_{-1}^-)$ .

As a concrete example consider  $\Sigma$  with the basis vectors  $A = S$  and  $B = H$ ; see (8). The system  $\Sigma$  satisfies LARC but is uncontrollable. The plane  $B^\perp$  splits  $\mathcal{C}^+$  in two regions

$$\mathcal{C}_+^+ = \text{cone} \cap \{z_1 > 0\}, \quad \mathcal{C}_-^+ = \text{cone} \cap \{z_1 < 0\}.$$

Furthermore,  $q_0^1 \in \mathcal{C}_+^+$  and  $q_0^2 \in \mathcal{C}_-^+$ . In coordinates, the differential equation  $\dot{Z} = [A + B, Z]$ ,  $Z \in \mathcal{C}^+$ , reads  $\dot{z}_1 = 2z_3$ ,  $\dot{z}_2 = -2z_3$ ,  $\dot{z}_3 = 2z_1 - 2z_2$ .

So, the direction of the trajectories in  $\mathcal{C}_+^+$  goes from  $\mathbb{R}_{(-++)}^3$  towards  $\mathbb{R}_{(-++)}^3$ . (Where the subscript with signals in  $\mathbb{R}^3$  refers to the orthants determined by the basis (8).) Analogously, under  $u \equiv -1$ , the direction of the dynamics on  $\mathcal{C}_-^+$  goes from  $\mathbb{R}_{(-++)}^3$  to  $\mathbb{R}_{(-++)}^3$ . Finally, the control sets are given by

$$I^- = \left[ \frac{(-1 - \sqrt{2}, -1)}{\|(-1 - \sqrt{2}, -1)\|}, \frac{(1 - \sqrt{2}, -1)}{\|(1 - \sqrt{2}, -1)\|} \right]$$

$$I^+ = \left( \frac{(-1 + \sqrt{2}, -1)}{\|(-1 + \sqrt{2}, -1)\|}, \frac{(1 + \sqrt{2}, -1)}{\|(1 + \sqrt{2}, -1)\|} \right).$$

Despite the fact that  $q_b = \frac{1}{2} \frac{(-1 - \sqrt{2}, -1)}{\|(-1 - \sqrt{2}, -1)\|}$  reach, for instance, the middle point  $q_e \in \text{int}(I^-)$ , there is not an optimal bang-bang path connecting  $q_b$  with  $q_e$ .

The singular extremals satisfy

$$\{(z_1, z_2, z_3) \in \mathbb{R}^3 : (z_2 - z_3)(z_2 + z_3) = 0\}$$

Thus, the only singular trajectories correspond to the lines  $l_0^1$  and  $l_0^2$  on  $\mathcal{C}^+$ . Since

$$[B, [A, B]] = -4A, \quad [A, [A, B]] = -4B, \quad \text{and} \quad \langle P(t), A \rangle = -1,$$

there exists a singular control  $u$ . Since  $u = 0$ , the singular trajectories on  $\mathcal{C}^+$  are the integral curves of  $Z = [A, Z]$ ,  $Z \in \mathcal{C}^+$  which are projected on  $\mathbb{P}^1$  onto the points  $q_0^1$  and  $q_0^2$ . However, the admissible singular control  $u$  is not optimal. In fact, in this case  $\frac{\partial}{\partial u} \frac{d^2}{dt^2} \frac{\partial \mathcal{H}}{\partial u} > 0$ , violating the Legendre-Clebsch condition.

Finally we give an exemple with a controllable system.

*Example.* Let  $\Sigma$  be determined by  $A = R$  and  $B = H + 2R$ , see (8). Since,  $A \in \text{int}\mathcal{C}^+$  the system is controllable, see (9). The vector  $[B, [A, B]]$  belongs to  $\text{ad}_A(B)$ , but  $\alpha_3 = \frac{3}{2} > 1$ . So,  $\Sigma$  does not satisfy the conditions of Theorem 3.2. However,  $\mathbb{P}\Sigma$  has the bang-bang property. In fact,  $\langle P, B \rangle = 0$  divided the entire cone in  $\mathcal{C}^+$  and  $\mathcal{C}^-$ . According to PMP we must use  $u = 1$  on  $\mathcal{C}^+$ . Since the trajectories of  $A + B$  on  $\mathcal{C}^+$  are ellipses, the system has the bang-bang property.

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